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PPF-Dependent Fixed Point Results for Multi-Valued $\phi - F$ -Contractions in Banach Spaces and Applications

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Received: 27 September 2019; Accepted: 21 October 2019; Published: 6 November 2019



Abstract: The solutions for many real life problems is obtained by interpreting the given problem mathematically in the form of $f(x) = x$. One of such examples is that of the famous Borsuk–Ulam theorem, in which using some fixed point argument, it can be guaranteed that at any given time we can find two diametrically opposite places in a planet with same temperature. Thus, the correlation of symmetry is inherent in the study of fixed point theory. In this paper, we initiate $\phi - F$ -contractions and study the existence of PPF-dependent fixed points (fixed points for mappings having variant domains and ranges) for these related mappings in the Razumikhin class. Our theorems extend and improve the results of Hammad and De La Sen [Mathematics, 2019, 7, 52]. As applications of our PPF dependent fixed point results, we study the existence of solutions for delay differential equations (DDEs) which have numerous applications in population dynamics, bioscience problems and control engineering.

Keywords: PPF-dependent fixed point; $\phi - F$ -contractions; the Razumikhin class; Banach space

MSC: Primary 47H10; Secondary 54H25

1. Introduction

Bernfeld et al. [1] initiated the notion of fixed points for mappings having variant domains and ranges. These elements are called PPF-dependent fixed points (or fixed points with the PPF-dependence). They [1] also established the existence of PPF-dependent fixed point theorems in the Razumikhin class for a Banach type contraction non-self mapping. On the other hand, Sintunavarat and Kumam [2], Ćirić et al. [3], Agarwal et al. [4] and Hussain et al. [5] investigated the existence and uniqueness of a PPF-dependent fixed point for variant types of contraction mappings, where the main result of Bernfeld et al. [1] has been generalized (see also [6]). For results on PPF-dependent fixed point for hybrid rational and Suzuki-Edelstein type contractions in Banach spaces, please see Parvaneh et al. [7].

From now on, we denote by \mathbb{N} , \mathbb{R} and \mathbb{R}^+ the set of all natural numbers, real numbers and positive real numbers, respectively. \mathcal{F} represents the collection of all functions $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ so that

- (F1) F is strictly increasing;
 (F2) For each positive sequence $\{\alpha_n\}$, $\lim_{n \rightarrow \infty} \alpha_n = 0$ iff $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$;
 (F3) There is $\rho \in (0, 1)$ such that $\lim_{\delta \rightarrow 0^+} \delta^\rho F(\delta) = 0$.

Definition 1. Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is said an F -contraction if there are $\tau > 0$ and $F \in \mathcal{F}$ such that for all $x, y \in X$,

$$d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y)). \quad (1)$$

Example 1. The functions $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ given as

- (1) $F(\mu) = \ln \mu$,
- (2) $F(\mu) = \ln \mu + \mu$,
- (3) $F(\mu) = \frac{-1}{\sqrt{\mu}}$,
- (4) $F(\mu) = \ln(\mu^2 + \mu)$,

belong to \mathcal{F} .

For results dealing with F -contractions, see [8–14]. Now, assume that $(E, \|\cdot\|_E)$ is a Banach space, I denotes a closed interval $[a, b]$ in \mathbb{R} and $E_0 = C(I, E)$ denotes the set of all continuous E -valued functions on I equipped with the supremum norm $\|\cdot\|_{E_0}$ defined by

$$\|\phi\|_{E_0} = \sup_{t \in I} \|\phi(t)\|_E.$$

For a fixed element $c \in I$, the Razumikhin or minimal class of functions in E_0 is defined by

$$\mathcal{R}_c = \{\phi \in E_0 : \|\phi\|_{E_0} = \|\phi(c)\|_E\}.$$

Clearly, every constant function from I to E belongs to \mathcal{R}_c .

Definition 2. Let A be a subset of E_0 . Then

- (i) A is called algebraically closed with respect to difference, that is, $\phi - \zeta \in A$ when $\phi, \zeta \in A$;
- (ii) A is called topologically closed if it is closed with respect to the topology on E_0 generated by the norm $\|\cdot\|_{E_0}$.

Definition 3 ([1]). A mapping $\zeta \in E_0$ is said a PPF-dependent fixed point or a fixed point with PPF-dependence of mapping $T : E_0 \rightarrow E$ if $T\zeta = \zeta(c)$ for some $c \in I$.

Definition 4 ([2]). Let $S : E_0 \rightarrow E_0$ and $T : E_0 \rightarrow E$. A point $\zeta \in E_0$ is said a PPF-dependent coincidence point or a coincidence point with PPF-dependence of S and T if $T\zeta = (S\zeta)(c)$ for some $c \in I$.

Definition 5 ([15]). A mapping $\zeta \in E_0$ is said a PPF-dependent fixed point or a fixed point with PPF-dependence of a multi-valued mapping $T : E_0 \rightarrow 2^E$ if $\zeta(c) \in T\zeta$ for some $c \in I$.

Definition 6 ([15]). Let $S : E_0 \rightarrow E_0$ and $T : E_0 \rightarrow 2^E$. A point $\zeta \in E_0$ is said a PPF-dependent coincidence point or a coincidence point with PPF-dependence of S and T if $(S\zeta)(c) \in T\zeta$ for some $c \in I$.

Definition 7 ([1]). The mapping $T : E_0 \rightarrow E$ is called a Banach type contraction if there is $k \in [0, 1)$ so that

$$\|T\phi - T\zeta\|_E \leq k\|\phi - \zeta\|_{E_0}$$

for all $\phi, \zeta \in E_0$.

$CB(E)$ stands for the family of all non-empty closed bounded subsets of E . Let $H_G(\cdot, \cdot)$ be the Hausdorff $\|\cdot\|_E$ metric on $CB(E)$, that is, for $U, V \in CB(E)$ we have

$$H_E(U, V) = \max \left\{ \sup_{\xi \in U} d(\xi, V), \sup_{\zeta \in V} d(U, \zeta) \right\}$$

where

$$d(\xi, V) = \inf_{\zeta \in V} \|\xi - \zeta\|.$$

In 2019, Hammad and De La Sen [15] introduced the following.

Definition 8 ([15]). A mapping $T : E_0 \rightarrow CB(E)$ is called a multi-valued generalized F -contraction if there are $\tau > 0$ and $F \in \mathcal{F}$ so that

$$H_E(T\zeta, T\tilde{\zeta}) > 0 \implies \tau + F(H_E(T\zeta, T\tilde{\zeta})) \leq F(\|\zeta - \tilde{\zeta}\|_{E_0}) \quad (2)$$

for all $\zeta, \tilde{\zeta} \in E_0$.

Hammad and De La Sen [15] proved that a multi-valued generalized F -contraction has a PPF-dependent fixed point in \mathcal{R}_c . In this paper, we introduce $\phi - F$ -contractions and investigate the existence of PPF-dependent fixed point for such mappings in the Razumikhin class. As an application of our PPF dependent fixed point results, we deduce corresponding PPF-dependent coincidence point results in the Razumikhin class. These results extend and generalize some known results in the literature.

2. Main Results

In this section we introduce new concepts called Multi-Valued generalized $\phi - F$ -contraction ($\alpha - \phi - F$ -contraction) and we present some important results for such contractions in the setting of Banach space.

Let Φ denote the set of all functions $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ satisfying:

- (φ_1) $\lim_{n \rightarrow \infty} \frac{\varphi^n(t)}{n} < 0$ for each $t > 0$;
- (φ_2) $\varphi(t) < t$ for each $t \in \mathbb{R}$;
- (φ_3) φ is strictly increasing and upper semi-continuous from right.

Example 2. The functions $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ given as

- (1) $\varphi_1(t) = t - \tau$ with $\tau > 0$;
- (2) $\varphi_2(t) = \begin{cases} t^3 - 1, & t < 1 \\ \sqrt{t} - 1, & t > 1; \end{cases}$
- (3) $\varphi_3(t) = \begin{cases} 3t - 4, & t < 1 \\ t - 1, & t \geq 1. \end{cases}$

belong to Φ .

Note that any function φ satisfying (φ_1) implies $\lim_{n \rightarrow \infty} \varphi^n(t) = -\infty$ for any $t \geq 0$. Now we give a generalized of definition (8) by using (φ_1) as follows.

Definition 9. A mapping $T : E_0 \rightarrow CB(E)$ is called a multi-valued generalized $\phi - F$ -contraction if there are $F \in \mathcal{F}$ and $\varphi \in \Phi$ so that

$$H_E(T\zeta, T\tilde{\zeta}) > 0 \implies F(H_E(T\zeta, T\tilde{\zeta})) \leq \varphi(F(\|\zeta - \tilde{\zeta}\|_{E_0})) \quad (3)$$

for all $\zeta, \xi \in E_0$.

Now, we state and prove the first result concerning PPF dependent fixed point for multi-valued generalized $\varphi - F$ -contraction.

Theorem 1. Let $T : E_0 \rightarrow CB(E)$ be a multi-valued generalized $\varphi - F$ -contraction. Assume that \mathcal{R}_c is topologically closed and algebraically closed with respect to difference. Assume also that F has the additional condition

$$(F_4) \quad F(\inf B) = \inf(F(B)) \text{ for each } B \subseteq (0, \infty) \text{ with } \inf(B) > 0.$$

Then T has a PPF dependent fixed point $\zeta \in \mathcal{R}_c$.

Proof. Let $\zeta_0 \in \mathcal{R}_c$. Since $T\zeta_0 \subset E$, there is $x_1 \in E$ so that $x_1 \in T\zeta_0$. Choose $\zeta_1 \in \mathcal{R}_c$ such that

$$\zeta_1(c) = x_1 \in T\zeta_0.$$

If $\zeta_1(c) \in T\zeta_1$, then ζ_1 is a PPF dependent fixed point of T . Let $\zeta_1(c) \notin T\zeta_1$. Thus, $H_E(T\zeta_0, T\zeta_1) \geq d(\zeta_1(c), T\zeta_1) > 0$. Using (3), we have

$$\begin{aligned} F(d(\zeta_1(c), T\zeta_1)) &\leq F(H_E(T\zeta_0, T\zeta_1)) \\ &\leq \varphi(F(\|\zeta_0 - \zeta_1\|_{E_0})) \\ &< F(\|\zeta_0 - \zeta_1\|_{E_0}). \end{aligned} \quad (4)$$

By property (F_4) , we have

$$F(d(\zeta_1(c), T\zeta_1)) = F(\inf_{x \in T\zeta_1} \|\zeta_1(c) - x\|) = \inf_{x \in T\zeta_1} F(\|\zeta_1(c) - x\|).$$

From (4) and above equation, there is $x_2 \in T\zeta_1$ so that $F(\|\zeta_1(c) - x_2\|_E) < F(\|\zeta_0 - \zeta_1\|_{E_0})$. Choose $\zeta_2 \in \mathcal{R}_c$ so that

$$\zeta_2(c) = x_2 \in T\zeta_1.$$

Now, $F(\|\zeta_1(c) - \zeta_2(c)\|_E) < F(\|\zeta_0 - \zeta_1\|_{E_0})$. If $\zeta_2(c) \in T\zeta_2$, then ζ_2 is a PPF dependent fixed point of T . Let $\zeta_2(c) \notin T\zeta_2$. Hence, $H_E(T\zeta_1, T\zeta_2) \geq d(\zeta_2(c), T\zeta_2) > 0$. Using (3), we have

$$\begin{aligned} F(d(\zeta_2(c), T\zeta_2)) &\leq F(H_E(T\zeta_1, T\zeta_2)) \\ &\leq \varphi(F(\|\zeta_1 - \zeta_2\|_{E_0})) \\ &< \varphi(F(\|\zeta_0 - \zeta_1\|_{E_0})) \end{aligned} \quad (5)$$

From (5) and similar to the last statement, there is $x_3 \in T\zeta_2$ such that $F(\|\zeta_2(c) - x_3\|_E) < \varphi(F(\|\zeta_1 - \zeta_2\|_{E_0}))$. Choose $\zeta_3 \in \mathcal{R}_c$ such that,

$$\zeta_3(c) = x_3 \in T\zeta_2.$$

Continuing this process we obtain a sequence $\{\zeta_n\}$ in $\mathcal{R}_c \subseteq E_0$ such that, $\zeta_n(c) \in T\zeta_{n-1}$, for all $n \in \mathbb{N}$ and

$$F(\|\zeta_n(c) - \zeta_{n+1}(c)\|_E) < \varphi^{n-1}(F(\|\zeta_0 - \zeta_1\|_{E_0})). \quad (6)$$

Now put $\alpha_n = \|\zeta_n(c) - \zeta_{n+1}(c)\|_E$. Then, from (6) we have

$$F(\alpha_n) < \varphi^{n-1}(F(\alpha_0)), \quad \text{for all } n \in \mathbb{N} \quad (7)$$

Taking limit in both sides of (7), we get $\lim F(\alpha_n) = -\infty$ and so $\lim_{n \rightarrow \infty} \alpha_n = 0$. From (F3), there is $k \in (0, 1)$ so that $\lim_{n \rightarrow \infty} \alpha_n^k F(\alpha_n) = 0$. From (7), we get

$$\alpha_n^k F(\alpha_n) < \alpha_n^k \varphi^{n-1}(F(\alpha_0)).$$

Taking limit in both sides of the above equation we obtain $\lim_{n \rightarrow \infty} \alpha_n^k \varphi^{n-1}(F(\alpha_0)) = 0$. Also from (φ_1) there exists $\alpha > 0$ such that $|\frac{\varphi^{n-1}(F(\alpha_0))}{n-1}| > \alpha$. Now we have

$$n\alpha_n^k \alpha \leq n\alpha_n^k |\frac{\varphi^{n-1}(F(\alpha_0))}{n-1}|$$

Taking limit in both sides of the above equation we obtain $\lim n\alpha_n^k \alpha = 0$. So, $\lim n\alpha_n^k = 0$. Thus, there exists $N \in \mathbb{N}$ such that $\alpha_n \leq \frac{1}{n^k}$ for all $n \geq N$. Now for any $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} \|\zeta_n(c) - \zeta_m(c)\|_E &\leq \sum_{i=n}^{m-1} \|\zeta_i(c) - \zeta_{i+1}(c)\|_E \\ &= \sum_{i=n}^{m-1} \alpha_i \leq \sum_{i=n}^{m-1} \frac{1}{i^k} \end{aligned}$$

Since the last term of the above inequality tends to zero as $m, n \rightarrow \infty$, so we have $\|\zeta_n(c) - \zeta_m(c)\|_E \rightarrow 0$ as $m, n \rightarrow \infty$. This means that $\{\zeta_n\}$ is a Cauchy sequence. Since E_0 is complete there exists $\zeta \in E_0$ such that $\|\zeta_n - \zeta\|_{E_0} \rightarrow 0$ as $n \rightarrow \infty$. Since \mathcal{R}_c is topologically closed, we get $\zeta \in \mathcal{R}_c$. Also since \mathcal{R}_c is algebraically closed with respect to difference, we have $\zeta_n - \zeta \in \mathcal{R}_c$. Now $\|\zeta_n(c) - \zeta(c)\|_E = \|\zeta_n - \zeta\|_{E_0} \rightarrow 0$. Then, we shall show that ζ is a PPF dependent fixed point of T . First note that from (3) we can conclude that $H_E(T\zeta, T\zeta) \leq \|\zeta - \zeta\|_{E_0}$ for all $\zeta, \zeta \in \mathcal{R}_c$. Now, we have

$$\begin{aligned} d(\zeta(c), T\zeta) &\leq \|\zeta(c) - \zeta_{n+1}(c)\|_E + d(\zeta_{n+1}(c), T\zeta) \\ &\leq \|\zeta(c) - \zeta_{n+1}(c)\|_E + H_E(T\zeta_n, T\zeta) \\ &\leq \|\zeta(c) - \zeta_{n+1}(c)\|_E + \|\zeta_n - \zeta\|_{E_0} \end{aligned} \tag{8}$$

Passing to limit in (8) yields that $d(\zeta(c), T\zeta) = 0$ and so $\zeta(c) \in T\zeta$, that is, ζ is a PPF dependent fixed point of T . \square

One can notice that the above theorem is a generalized version of the main result of Hammad and De La Sen [15]. In fact by taking $\varphi(t) = t - \tau$, we obtain Theorem 3 of [15].

Corollary 1. Let $T : E_0 \rightarrow CB(E)$ be a multi-valued mapping such that there are $F \in \mathcal{F}$ and $\tau > 0$ such that

$$H_E(T\zeta, T\zeta) > 0 \implies \tau + F(H_E(T\zeta, T\zeta)) \leq F(M(\zeta, \zeta)) \tag{9}$$

for all $\zeta, \zeta \in E_0$. Assume that \mathcal{R}_c is topologically and algebraically closed with respect to difference. Suppose that F has the following additional condition.

$$(F_4) \ F(\inf B) = \inf F(B) \text{ for all } B \subseteq (0, \infty) \text{ with } \inf B > 0.$$

Then T has a PPF dependent fixed point $\zeta \in \mathcal{R}_c$.

For a single-valued mapping $T : E_0 \rightarrow E$, defining $S : E_0 \rightarrow CB(E)$ by $S(\zeta) = \{T\zeta\}$ one can result the following corollary from Theorem 1.

Corollary 2. Let $T : E_0 \rightarrow E$ is a single-valued mapping. Assume that there exists $F \in \mathcal{F}$ and $\varphi \in \Phi$ such that

$$d(T\zeta, T\tilde{\zeta}) > 0 \implies F(d(T\zeta, T\tilde{\zeta})) \leq \varphi(F(\|\zeta - \tilde{\zeta}\|_{E_0})) \quad (10)$$

for all $\zeta, \tilde{\zeta} \in E_0$. Assume, \mathcal{R}_c is topologically closed and algebraically closed with respect to difference. Then, T has a PPF dependent fixed point $\zeta \in \mathcal{R}_c$.

Proof. Define $S : E_0 \rightarrow CB(E)$ by $S(\zeta) = \{T\zeta\}$. By (10), the mapping S satisfies (10). Therefore by Theorem 1, S has a PPF dependent fixed point $\zeta \in \mathcal{R}_c$, that is $\zeta(c) \in S(\zeta) = \{T(\zeta)\}$. Therefore $\zeta(c) = T(\zeta)$. \square

Now, we will introduce the concept of α -admissible and multi-valued generalized $\alpha - (\varphi - F)$ -contraction in the setting of Banach Space.

Definition 10. A mapping $T : E_0 \rightarrow E$ is called α -admissible, if there exists a function $\alpha : E_0 \times E_0 \rightarrow [0, \infty)$ such that for any $\zeta, \tilde{\zeta}, \eta, \varsigma \in E_0$

$$\alpha(\zeta, \tilde{\zeta}) \geq 1, \eta(c) = T\zeta, \varsigma(c) = T\tilde{\zeta} \implies \alpha(\eta, \varsigma) \geq 1$$

Definition 11. A mapping $T : E_0 \rightarrow 2^E$ is called α -admissible, if there exists a function $\alpha : E_0 \times E_0 \rightarrow [0, \infty)$ such that for any $\zeta, \tilde{\zeta} \in E_0$ with $\tilde{\zeta}(c) \in T\zeta$ and $\alpha(\zeta, \tilde{\zeta}) \geq 1$, then $\alpha(\tilde{\zeta}, \eta) \geq 1$ for all $\eta \in E_0$ with $\eta(c) \in T\tilde{\zeta}$.

Definition 12. A mapping $T : E_0 \rightarrow CB(E)$ is called multi-valued generalized $\alpha - (\varphi - F)$ -contraction if there exist a function $\alpha : E_0 \times E_0 \rightarrow [0, \infty)$, $F \in \mathcal{F}$ and $\varphi \in \Phi$ such that

$$H_E(T\zeta, T\tilde{\zeta}) > 0 \implies F(H_E(T\zeta, T\tilde{\zeta})) \leq \varphi(F(\|\zeta - \tilde{\zeta}\|_{E_0})) \quad (11)$$

for all $\zeta, \tilde{\zeta} \in E_0$ with $\alpha(\zeta, \tilde{\zeta}) \geq 1$.

Now, we prove the existence of PPF dependent fixed point for multi-valued generalized $\alpha - \varphi - F$ contraction.

Theorem 2. Let $T : E_0 \rightarrow CB(E)$ be a multi-valued generalized $\alpha - (\varphi - F)$ -contraction. Assume, \mathcal{R}_c is topologically closed and algebraically closed with respect to difference. Assume also that F has the additional condition

$$(F_4) \quad F(\inf B) = \inf(F(B)) \text{ for all } B \subseteq (0, \infty) \text{ with } \inf(B) > 0.$$

Moreover, assume that

- (i) there are $\zeta_0, \zeta_1 \in \mathcal{R}_c$ with $\zeta_1(c) \in T\zeta_0$ and $\alpha(\zeta_0, \zeta_1) \geq 1$;
- (ii) T is α -admissible;
- (iii) for any sequence $\{\zeta_n\}$ in \mathcal{R}_c with $\alpha(\zeta_n, \zeta_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\zeta_n \rightarrow \zeta$, then $\alpha(\zeta_n, \zeta) \geq 1$ for all $n \in \mathbb{N}$.

If T or F be continuous, then T has a PPF dependent fixed point $\zeta \in \mathcal{R}_c$.

Proof. Let $\zeta_0, \zeta_1 \in \mathcal{R}_c$ be such that $\zeta_1(c) \in T\zeta_0$ and $\alpha(\zeta_0, \zeta_1) \geq 1$. If $\zeta_1(c) \in T\zeta_1$, then ζ_1 is a PPF dependent fixed point of T . Let $\zeta_1(c) \notin T\zeta_1$. Thus $H_E(T\zeta_0, T\zeta_1) \geq d(\zeta_1(c), T\zeta_1) > 0$. Using (11) we have

$$\begin{aligned} F(d(\zeta_1(c), T\zeta_1)) &\leq F(H_E(T\zeta_0, T\zeta_1)) \\ &\leq \varphi(F(\|\zeta_0 - \zeta_1\|_{E_0})) \\ &< F(\|\zeta_0 - \zeta_1\|_{E_0}) \end{aligned}$$

Thus there exists $x_2 \in T\zeta_1$ such that $F(\|\zeta_1(c) - x_2\|_E) < F(\|\zeta_0 - \zeta_1\|_{E_0})$. Choose $\zeta_2 \in \mathcal{R}_c$ such that,

$$\zeta_2(c) = x_2 \in T\zeta_1.$$

Since T is α -admissible, we get $\alpha(\zeta_1, \zeta_2) \geq 1$. If $\zeta_2(c) \in T\zeta_2$, then ζ_2 is a PPF dependent fixed point of T . Let $\zeta_2(c) \notin T\zeta_2$. Thus $H_E(T\zeta_1, T\zeta_2) \geq d(\zeta_2(c), T\zeta_2) > 0$. Using (11), we have

$$\begin{aligned} F(d(\zeta_2(c), T\zeta_2)) &\leq F(H_E(T\zeta_1, T\zeta_2)) \\ &\leq \varphi(F(\|\zeta_1 - \zeta_2\|_{E_0})) \\ &= \varphi(F(\|\zeta_1(c) - \zeta_2(c)\|_E)) \\ &< \varphi(F(\|\zeta_0 - \zeta_1\|_{E_0})) \end{aligned}$$

Thus there exists $x_3 \in T\zeta_2$ such that $F(\|\zeta_2(c) - x_3\|_E) < \varphi(F(\|\zeta_0 - \zeta_1\|_{E_0}))$. Choose $\zeta_3 \in \mathcal{R}_c$ such that,

$$\zeta_3(c) = x_3 \in T\zeta_2.$$

Since T is α -admissible, we get $\alpha(\zeta_1, \zeta_2) \geq 1$. Continuing this process we obtain a sequence $\{\zeta_n\}$ in $\mathcal{R}_c \subseteq E_0$ such that, $\zeta_n(c) \in T\zeta_{n-1}$, for all $n \in \mathbb{N}$ and

$$F(\|\zeta_n(c) - \zeta_{n+1}(c)\|_E) < \varphi^{n-1}(F(\|\zeta_0 - \zeta_1\|_{E_0})). \quad (12)$$

Now, put $\alpha_n = \|\zeta_n(c) - \zeta_{n+1}(c)\|_E$. Then, from (12) we have

$$F(\alpha_n) < \varphi^{n-1}(F(\alpha_0)), \quad \text{for all } n \in \mathbb{N}. \quad (13)$$

Similar to Theorem 1, $\{\zeta_n\}$ is Cauchy, so there is $\zeta \in \mathcal{R}_c$ such that $\|\zeta_n(c) - \zeta(c)\|_E \rightarrow 0$. From (iii), we deduce $\alpha(\zeta_n, \zeta) \geq 1$. We shall show that ζ is a PPF dependent fixed point of T . From (11), we may conclude that $F(H_E(T\zeta, T\zeta)) \leq F(\|\zeta - \zeta\|_{E_0})$, and so $H_E(T\zeta, T\zeta) \leq \|\zeta - \zeta\|_{E_0}$ for all $\zeta, \zeta \in \mathcal{R}_c$ with $\alpha(\zeta, \zeta) \geq 1$. Now since $\alpha(\zeta_n, \zeta) \geq 1$ we have

$$\begin{aligned} d(\zeta(c), T\zeta) &\leq \|\zeta(c) - \zeta_{n+1}(c)\|_E + d(\zeta_{n+1}(c), T\zeta) \\ &\leq \|\zeta(c) - \zeta_{n+1}(c)\|_E + H_E(T\zeta_n, T\zeta) \\ &\leq \|\zeta(c) - \zeta_{n+1}(c)\|_E + \|\zeta_n - \zeta\|_{E_0}. \end{aligned}$$

Taking limit in both sides of the above inequality, we get $d(\zeta(c), T\zeta) = 0$. This yields that $\zeta(c) \in T\zeta$, that is, ζ is a PPF dependent fixed point of T . \square

Let $T : E_0 \rightarrow 2^E$ and $S : E_0 \rightarrow E_0$. Then $\zeta \in E_0$ is called a PPF-dependent coincidence point, if $S\zeta(c) \in T\zeta$ for some $c \in I$. Using Theorem 1, we deduce the following PPF-dependent coincidence point result for single and multi-valued mappings.

Theorem 3. Let $T : E_0 \rightarrow CB(E)$ and $S : E_0 \rightarrow E_0$. Assume that

$$H_E(T\zeta, T\zeta) > 0 \implies F(H_E(T\zeta, T\zeta)) \leq \varphi(F(\|S\zeta - S\zeta\|_{E_0})) \quad (14)$$

for all $\zeta, \zeta \in E_0$. Let $S(\mathcal{R}_c) \subseteq \mathcal{R}_c$. Suppose that $S(\mathcal{R}_c)$ is topologically closed and algebraically closed with respect to difference. Then T and S have a PPF dependent coincidence point.

Proof. As $S : E_0 \rightarrow E_0$, there exists $F_0 \subseteq E_0$ such that $S(F_0) = S(E_0)$ and $S|_{F_0}$ is one-to-one. Since $T(F_0) \subseteq T(E_0) \subseteq E$, we can define the mapping $\mathcal{A} : S(F_0) \rightarrow E$ by $\mathcal{A}(S\phi) = T\phi$ for all $\phi \in F_0$. Again, $S|_{F_0}$ is one-to-one, then \mathcal{A} is well-defined. By (14), we have

$$\begin{aligned} F(\|\mathcal{A}(S\zeta) - \mathcal{A}(S\xi)\|_E) &= F(\|T(\zeta) - T(\xi)\|_E) \\ &\leq \varphi(F(\|S\zeta - S\xi\|_{E_0})) \end{aligned}$$

for all $\zeta, \xi \in F_0$.

This shows that \mathcal{A} is a $\varphi - F$ -contraction and all conditions of Theorem 1 hold. Then there is a PPF dependent fixed point $\zeta \in S(F_0)$ of \mathcal{A} , i.e., $\zeta(c) \in \mathcal{A}\zeta$. Since $\zeta \in S(F_0)$, there is $\omega \in F_0$ such that, $\zeta = S(\omega)$. Now,

$$(S\omega)(c) = \zeta(c) \in \mathcal{A}\zeta = \mathcal{A}(S\omega) = T\omega.$$

That is, ω is a PPF dependent coincidence point of S and T . \square

3. Multi-Valued Generalized Weakly $\varphi - F$ -Contractions

In this section we introduce new concepts called Multi-Valued generalized weakly $\varphi - F$ -contraction ($\alpha - \varphi - F$ -contraction) and we present some important results for such contractions in the setting of Banach space.

Definition 13. A mapping $T : E_0 \rightarrow CB(E)$ is called a multi-valued generalized weakly $\varphi - F$ -contraction if there are $F \in \mathcal{F}$ and $\varphi \in \Phi$ such that

$$H_E(T\zeta, T\xi) > 0 \implies F(H_E(T\zeta, T\xi)) \leq \varphi(F(M(\zeta, \xi))) \quad (15)$$

for all $\zeta, \xi \in E_0$, where

$$\begin{aligned} M(\zeta, \xi) &= \max\{\|\zeta - \xi\|_{E_0}, d(\zeta(c), T\zeta), \\ &\quad d(\xi(c), T\xi), \frac{1}{2}[d(\zeta(c), T\xi) + d(\xi(c), T\zeta)]\}. \end{aligned}$$

Theorem 4. Let $T : E_0 \rightarrow CB(E)$ be a multi-valued generalized weakly $\varphi - F$ -contraction. Assume that \mathcal{R}_c is topologically and algebraically closed with respect to difference. Assume also that F has the additional condition

$$(F_4) \quad F(\inf B) = \inf F(B) \text{ for all } B \subseteq (0, \infty) \text{ with } \inf(B) > 0.$$

If T or F be continuous, then T has a PPF dependent fixed point $\zeta \in \mathcal{R}_c$.

Proof. Let $\zeta_0 \in \mathcal{R}_c$. Since $T\zeta_0 \subset E$, there exists $x_1 \in E$ such that $x_1 \in T\zeta_0$. Choose $\zeta_1 \in \mathcal{R}_c$ such that

$$\zeta_1(c) = x_1 \in T\zeta_0.$$

If $\zeta_1(c) \in T\zeta_1$, then ζ_1 is a PPF dependent fixed point of T . Let $\zeta_1(c) \notin T\zeta_1$. Thus $H_E(T\zeta_0, T\zeta_1) \geq d(\zeta_1(c), T\zeta_1) > 0$. Using (15) we have

$$F(d(\zeta_1(c), T\zeta_1)) \leq F(H_E(T\zeta_0, T\zeta_1)) \leq \varphi(F(M(\zeta_0, \zeta_1))). \quad (16)$$

On the other hand,

$$\begin{aligned} M(\zeta_0, \zeta_1) &= \max\{\|\zeta_0 - \zeta_1\|_{E_0}, d(\zeta_0(c), T\zeta_0), d(\zeta_1(c), T\zeta_1), \\ &\quad \frac{1}{2}[d(\zeta_0(c), T\zeta_1) + d(\zeta_1(c), T\zeta_0)]\} \\ &\leq \max\{\|\zeta_0 - \zeta_1\|_{E_0}, d(\zeta_1(c), T\zeta_1)\}. \end{aligned}$$

If $\max\{\|\zeta_0 - \zeta_1\|_{E_0}, d(\zeta_1(c), T\zeta_1)\} = d(\zeta_1(c), T\zeta_1)$, then from (16), we get

$$F(d(\zeta_1(c), T\zeta_1)) \leq \varphi(F(d(\zeta_1(c), T\zeta_1))) < F(d(\zeta_1(c), T\zeta_1))$$

which is a contradiction. Thus, $\max\{\|\zeta_0 - \zeta_1\|_{E_0}, d(\zeta_1(c), T\zeta_1)\} = \|\zeta_0 - \zeta_1\|_{E_0}$. From (16), we get

$$F(d(\zeta_1(c), T\zeta_1)) \leq \varphi(F(\|\zeta_0 - \zeta_1\|_{E_0})) < F(\|\zeta_0 - \zeta_1\|_{E_0}).$$

Thus there is $x_2 \in T\zeta_1$ such that $F(\|\zeta_1(c) - x_2\|_E) < F(\|\zeta_0 - \zeta_1\|_{E_0})$. Choose $\zeta_2 \in \mathcal{R}_c$ such that

$$\zeta_2(c) = x_2 \in T\zeta_1.$$

Now,

$$F(\|\zeta_1(c) - \zeta_2(c)\|_E) < F(\|\zeta_0 - \zeta_1\|_{E_0}). \quad (17)$$

If $\zeta_2(c) \in T\zeta_2$, then ζ_2 is a PPF dependent fixed point of T . Let $\zeta_2(c) \notin T\zeta_2$. Thus $H_E(T\zeta_1, T\zeta_2) \geq d(\zeta_2(c), T\zeta_2) > 0$. Using (15), we have

$$F(d(\zeta_2(c), T\zeta_2)) \leq F(H_E(T\zeta_1, T\zeta_2)) \leq \varphi(F(M(\zeta_1, \zeta_2))). \quad (18)$$

Similar to the above step, we can conclude from Equation (18) that

$$F(d(\zeta_2(c), T\zeta_2)) \leq \varphi(F(\|\zeta_1 - \zeta_2\|_{E_0})). \quad (19)$$

Now, from (17)–(19), we obtain

$$F(d(\zeta_2(c), T\zeta_2)) < \varphi(F(\|\zeta_0 - \zeta_1\|_{E_0})). \quad (20)$$

Thus there is $x_3 \in T\zeta_2$ such that $F(\|\zeta_2(c) - x_3\|_E) < \varphi(F(\|\zeta_0 - \zeta_1\|_{E_0}))$. Choose $\zeta_3 \in \mathcal{R}_c$ so that

$$\zeta_3(c) = x_3 \in T\zeta_2.$$

Continuing this process, we obtain a sequence $\{\zeta_n\}$ in $\mathcal{R}_c \subseteq E$ such that $\zeta_n(c) \in T\zeta_{n-1}$ for all $n \in \mathbb{N}$ and

$$F(\|\zeta_n(c) - \zeta_{n+1}(c)\|_E) < \varphi^{n-1}(F(\|\zeta_0 - \zeta_1\|_{E_0})). \quad (21)$$

Let $\alpha_n = \|\zeta_n(c) - \zeta_{n+1}(c)\|_E$. Then, from (21) we have

$$F(\alpha_n) < \varphi^{n-1}(F(\alpha_0)), \quad \text{for all } n \in \mathbb{N}. \quad (22)$$

Similar to Theorem 1, we get $\{\zeta_n\}$ is Cauchy. Since E_0 is complete, there is $\zeta \in E_0$ such that $\|\zeta_n - \zeta\|_{E_0} \rightarrow 0$ as $n \rightarrow \infty$. Since \mathcal{R}_c is topologically closed, we get $\zeta \in \mathcal{R}_c$. Also, since \mathcal{R}_c is algebraically closed with respect to difference, we have $\zeta_n - \zeta \in \mathcal{R}_c$. Now, $\|\zeta_n(c) - \zeta(c)\|_E = \|\zeta_n - \zeta\|_{E_0} \rightarrow 0$. We shall show that ζ is a PPF dependent fixed point of T . If T is continuous, then

$$\begin{aligned} d(\zeta(c), T\zeta) &= \lim_{n \rightarrow \infty} d(\zeta_{n+1}(c), T\zeta) \\ &\leq \lim_{n \rightarrow \infty} H_E(T\zeta_n, T\zeta) = 0. \end{aligned}$$

Thus, $d(\zeta(c), T\zeta) = 0$ which gives us $\zeta(c) \in T\zeta$. In the case that F is continuous, we consider two cases:

Case 1: For any $i \in \mathbb{N}$, there exists $n_i > i$ such that $\zeta_{n_i+1}(c) \in T\zeta$. In this case we have

$$d(\zeta(c), T\zeta) = \lim_{n \rightarrow \infty} d(\zeta_{n_i+1}(c), T\zeta) = 0.$$

Thus, $d(\zeta(c), T\zeta) = 0$, that is, $\zeta(c) \in T\zeta$.

Case 2: There is $N \in \mathbb{N}$ such that $\zeta_{n+1}(c) \notin T\zeta$ for each $n \geq N$. Here,

$$\begin{aligned} F(d(\zeta(c), T\zeta)) &= \lim_{n \rightarrow \infty} F(d(\zeta_{n+1}(c), T\zeta)) \\ &\leq \lim_{n \rightarrow \infty} F(H(T\zeta_n, T\zeta)) \\ &\leq \lim_{n \rightarrow \infty} \varphi(F(M(\zeta_n, \zeta))). \end{aligned} \tag{23}$$

On the other hand,

$$\begin{aligned} d(\zeta(c), T\zeta) &\leq M(\zeta_n, \zeta) \\ &= \max\{\|\zeta_n - \zeta\|_{E_0}, d(\zeta_n(c), T\zeta_n), d(\zeta(c), T\zeta), \\ &\quad \frac{1}{2}[d(\zeta_n(c), T\zeta) + d(\zeta(c), T\zeta_n)]\} \\ &\leq \max\{\|\zeta_n - \zeta\|_{E_0}, d(\zeta_n(c), T\zeta_n), d(\zeta(c), T\zeta), \\ &\quad \frac{1}{2}[d(\zeta_n(c), \zeta(c)) + d(\zeta(c), T\zeta) + d(\zeta(c), \zeta_n(c)) + d(\zeta_n(c), T\zeta_n)]\}. \end{aligned}$$

Taking the limit in both sides of the above equation, we get

$$\lim_{n \rightarrow \infty} M(\zeta_n, \zeta) = d(\zeta(c), T\zeta).$$

Suppose to the contradiction that $d(\zeta(c), T\zeta) > 0$. Taking the limit in (23) yields that $F(d(\zeta(c), T\zeta)) \leq \varphi(F(d(\zeta(c), T\zeta)))$, which is a contradiction. Thus, $d(\zeta(c), T\zeta) = 0$, and so $\zeta(c) \in T\zeta$, that is, ζ is a PPF dependent fixed point of T . \square

One can notice that in the above theorem by taking $\varphi(t) = t - \tau$, we obtain Theorem 5 of [15] in case $S = T$ and taking $M(\zeta, \xi)$ is either (i) or (ii) or (iv) or (v) or (vi) or (x) or (xiii) that listed after Theorem 5 in [15].

Definition 14. A mapping $T : E_0 \rightarrow CB(E)$ is called a multi-valued generalized weakly $\alpha - (\varphi - F)$ -contraction if there are $\alpha : E_0 \times E_0 \rightarrow [0, \infty)$, $F \in \mathcal{F}$ and $\varphi \in \Phi$ so that

$$H_E(T\zeta, T\xi) > 0 \implies F(H_E(T\zeta, T\xi)) \leq \varphi(F(M(\zeta, \xi))) \tag{24}$$

for all $\zeta, \xi \in E_0$ with $\alpha(\zeta, \xi) \geq 1$, where

$$\begin{aligned} M(\zeta, \xi) &= \max\{\|\zeta - \xi\|_{E_0}, d(\zeta(c), T\zeta), d(\xi(c), T\xi), \\ &\quad \frac{1}{2}[d(\zeta(c), T\xi) + d(\xi(c), T\zeta)]\}. \end{aligned}$$

Now, we prove the existence of PPF-dependent fixed point for multi-valued generalized weakly $\alpha - \varphi - F$ -contraction.

Theorem 5. Let $T : E_0 \rightarrow CB(E)$ be a multi-valued generalized weakly $\alpha - \varphi - F$ -contraction. Assume that \mathcal{R}_c is topologically and algebraically closed with respect to difference. Assume also that F has the additional condition

$$(F_4) \quad F(\inf B) = \inf F(B) \text{ for all } B \subseteq (0, \infty) \text{ with } \inf B > 0.$$

Moreover, assume that

- (i) there are $\zeta_0, \zeta_1 \in \mathcal{R}_c$ such that $\zeta_1(c) \in T\zeta_0$ and $\alpha(\zeta_0, \zeta_1) \geq 1$;
- (ii) T is α -admissible;
- (iii) either T is continuous, or F is continuous and for any sequence $\{\zeta_n\}$ in \mathcal{R}_c with $\alpha(\zeta_n, \zeta_{n+1}) \geq 1$ for each $n \in \mathbb{N}$ and $\zeta_n \rightarrow \zeta$, then $\alpha(\zeta_n, \zeta) \geq 1$ for each $n \in \mathbb{N}$.

Then T has a PPF dependent fixed point $\zeta \in \mathcal{R}_c$.

Proof. Let $\zeta_0, \zeta_1 \in \mathcal{R}_c$ be such that $\zeta_1(c) \in T\zeta_0$ and $\alpha(\zeta_0, \zeta_1) \geq 1$. If $\zeta_1(c) \in T\zeta_1$, then ζ_1 is a PPF dependent fixed point of T . Let $\zeta_1(c) \notin T\zeta_1$. Thus, $H_E(T\zeta_0, T\zeta_1) \geq d(\zeta_1(c), T\zeta_1) > 0$. Using (24), we have

$$F(d(\zeta_1(c), T\zeta_1)) \leq F(H_E(T\zeta_0, T\zeta_1)) \leq \varphi(F(M(\zeta_0, \zeta_1))). \quad (25)$$

On the other hand,

$$\begin{aligned} M(\zeta_0, \zeta_1) &= \max\{\|\zeta_0 - \zeta_1\|_{E_0} d(\zeta_0(c), T\zeta_0), d(\zeta_1(c), T\zeta_1), \\ &\quad \frac{1}{2}[d(\zeta_0(c), T\zeta_1) + d(\zeta_1(c), T\zeta_0)]\} \\ &\leq \max\{\|\zeta_0 - \zeta_1\|_{E_0}, d(\zeta_1(c), T\zeta_1)\}. \end{aligned}$$

If $\max\{\|\zeta_0 - \zeta_1\|_{E_0}, d(\zeta_1(c), T\zeta_1)\} = d(\zeta_1(c), T\zeta_1)$, then from (25), we get

$$F(d(\zeta_1(c), T\zeta_1)) \leq \varphi(F(d(\zeta_1(c), T\zeta_1))) < F(d(\zeta_1(c), T\zeta_1))$$

which is a contradiction. Thus, $\max\{\|\zeta_0 - \zeta_1\|_{E_0}, d(\zeta_1(c), T\zeta_1)\} = \|\zeta_0 - \zeta_1\|_{E_0}$. From (25), we get

$$F(d(\zeta_1(c), T\zeta_1)) \leq \varphi(F(\|\zeta_0 - \zeta_1\|_{E_0})) < F(\|\zeta_0 - \zeta_1\|_{E_0}).$$

Thus there is $x_2 \in T\zeta_1$ such that $F(\|\zeta_1(c) - x_2\|_E) < F(\|\zeta_0 - \zeta_1\|_{E_0})$. Choose $\zeta_2 \in \mathcal{R}_c$ such that

$$\zeta_2(c) = x_2 \in T\zeta_1.$$

Now,

$$F(\|\zeta_1(c) - \zeta_2(c)\|_E) < F(\|\zeta_0 - \zeta_1\|_{E_0}). \quad (26)$$

Since T is α -admissible, we get $\alpha(\zeta_1, \zeta_2) \geq 1$. If $\zeta_2(c) \in T\zeta_2$, then ζ_2 is a PPF dependent fixed point of T . Let $\zeta_2(c) \notin T\zeta_2$. Thus $H_E(T\zeta_1, T\zeta_2) \geq d(\zeta_2(c), T\zeta_2) > 0$. Using (24), we have

$$F(d(\zeta_2(c), T\zeta_2)) \leq F(H_E(T\zeta_1, T\zeta_2)) \leq \varphi(F(M(\zeta_1, \zeta_2))). \quad (27)$$

Similar to the above step, we can conclude from equation (24) that

$$F(d(\zeta_2(c), T\zeta_2)) \leq \varphi(F(\|\zeta_1 - \zeta_2\|_{E_0})). \quad (28)$$

Now, from (26)–(28), we obtain

$$F(d(\zeta_2(c), T\zeta_2)) < \varphi(F(\|\zeta_0 - \zeta_1\|_{E_0})). \quad (29)$$

Thus, there exists $x_3 \in T\zeta_2$ such that $F(\|\zeta_2(c) - x_3\|_E) < \varphi(F(\|\zeta_0 - \zeta_1\|_{E_0}))$. Choose $\zeta_3 \in \mathcal{R}_c$ such that,

$$\zeta_3(c) = x_3 \in T\zeta_2.$$

Continuing this process, we obtain a sequence $\{\zeta_n\}$ in $\mathcal{R}_c \subseteq E$ such that $\zeta_n(c) \in T\zeta_{n-1}$ for each $n \in \mathbb{N}$ and

$$F(\|\zeta_n(c) - \zeta_{n+1}(c)\|_E) < \varphi^{n-1}(F(\|\zeta_0 - \zeta_1\|_{E_0})). \quad (30)$$

Assume that $\alpha_n = \|\zeta_n(c) - \zeta_{n+1}(c)\|_E$. Then from (30) we have

$$F(\alpha_n) < \varphi^{n-1}(F(\alpha_0)), \alpha(\zeta_n, \zeta_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N}. \quad (31)$$

Similar to Theorem 1, we get $\{\zeta_n\}$ is Cauchy. Since E_0 is complete, there is $\zeta \in E_0$ such that $\|\zeta_n - \zeta\|_{E_0} \rightarrow 0$ as $n \rightarrow \infty$. Since \mathcal{R}_c is topologically closed, we get $\zeta \in \mathcal{R}_c$. Recall that \mathcal{R}_c is

algebraically closed with respect to difference, so we have $\zeta_n - \zeta \in \mathcal{R}_c$. Now, $\|\zeta_n(c) - \zeta(c)\|_E = \|\zeta_n - \zeta\|_{E_0} \rightarrow 0$. We claim that ζ is a PPF dependent fixed point of T . If T is continuous, then

$$d(\zeta(c), T\zeta) = \lim d(\zeta_{n+1}(c), T\zeta) \leq \lim H_E(T\zeta_n, T\zeta) = 0.$$

Thus, $d(\zeta(c), T\zeta) = 0$, i.e., $\zeta(c) \in T\zeta$. In the case that F is continuous and $\alpha(\zeta_n, \zeta) \geq 1$ for all $n \in \mathbb{N}$, we consider two cases:

Case 1: For any $i \in \mathbb{N}$, there is $n_i > i$ so that $\zeta_{n_i+1}(c) \in T\zeta$. Here,

$$d(\zeta(c), T\zeta) = \lim d(\zeta_{n_i+1}(c), T\zeta) = 0.$$

Thus, $d(\zeta(c), T\zeta) = 0$, which gives us that $\zeta(c) \in T\zeta$.

Case 2: There is $N \in \mathbb{N}$ so that $\zeta_{n+1}(c) \notin T\zeta$ for each $n \geq N$. Here,

$$\begin{aligned} F(d(\zeta(c), T\zeta)) &= \lim_{n \rightarrow \infty} F(d(\zeta_{n+1}(c), T\zeta)) \\ &\leq \lim_{n \rightarrow \infty} F(H(T\zeta_n, T\zeta)) \\ &\leq \lim_{n \rightarrow \infty} \varphi(F(M(\zeta_n, \zeta))). \end{aligned} \quad (32)$$

On the other hand,

$$\begin{aligned} d(\zeta(c), T\zeta) &\leq M(\zeta_n, \zeta) \\ &= \max\{\|\zeta_n - \zeta\|_{E_0}, d(\zeta_n(c), T\zeta_n), d(\zeta(c), T\zeta), \\ &\quad \frac{1}{2}[d(\zeta_n(c), T\zeta) + d(\zeta(c), T\zeta_n)]\} \\ &\leq \max\{\|\zeta_n - \zeta\|_{E_0}, d(\zeta_n(c), T\zeta_n), d(\zeta(c), T\zeta), \\ &\quad \frac{1}{2}[d(\zeta_n(c), \zeta(c)) + d(\zeta(c), T\zeta) + d(\zeta(c), \zeta_n(c)) + d(\zeta_n(c), T\zeta_n)]\}. \end{aligned}$$

Taking the limit in both sides of the above equation, we get $\lim_{n \rightarrow \infty} M(\zeta_n, \zeta) = d(\zeta(c), T\zeta)$. Suppose to the contradiction that $d(\zeta(c), T\zeta) > 0$. Passing to the limit in (32), we have $F(d(\zeta(c), T\zeta)) \leq \varphi(F(d(\zeta(c), T\zeta)))$, a contradiction. Thus, $d(\zeta(c), T\zeta) = 0$, and so $\zeta(c) \in T\zeta$, that is, ζ is a PPF dependent fixed point of T . \square

By specializing $\varphi(t)$ in the above theorem to be $t - \tau$ we obtain the following result.

Corollary 3. Let $T : E_0 \rightarrow CB(E)$ be a multi-valued mapping. Suppose that there are $F \in \mathcal{F}$ and $\tau > 0$ so that

$$H_E(T\zeta, T\zeta) > 0 \implies \tau + F(H_E(T\zeta, T\zeta)) \leq F(M(\zeta, \zeta)) \quad (33)$$

for all $\zeta, \zeta \in E_0$ with $\alpha(\zeta, \zeta) \geq 1$. Assume that \mathcal{R}_c is topologically and algebraically closed with respect to difference. Suppose that F has the following additional condition.

$$(F_4) \quad F(\inf B) = \inf F(B) \text{ for all } B \subseteq (0, \infty) \text{ with } \inf B > 0.$$

Moreover, assume that

- (i) there are $\zeta_0, \zeta_1 \in \mathcal{R}_c$ such that $\zeta_1(c) \in T\zeta_0$ and $\alpha(\zeta_0, \zeta_1) \geq 1$;
- (ii) T is α -admissible;
- (iii) either T is continuous, or F is continuous and for any sequence $\{\zeta_n\}$ in \mathcal{R}_c with $\alpha(\zeta_n, \zeta_{n+1}) \geq 1$ for each $n \in \mathbb{N}$ and $\zeta_n \rightarrow \zeta$, then $\alpha(\zeta_n, \zeta) \geq 1$ for all $n \in \mathbb{N}$.

Then, T has a PPF dependent fixed point $\zeta \in \mathcal{R}_c$.

4. Application 1

In this section we will use our results to give a solution for an integro equation. Let $I = [a, b]$ and $E_0 = C(I, \mathbb{R})$. Consider

$$x(t) - \left\| \int_a^b \kappa(t, s, x(s)) ds \right\|_{\infty} = 0 \quad (34)$$

where $t \in I$ and $\kappa : [a, b] \times [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

Theorem 6. Assume there are $F \in \mathcal{F}$ and $\varphi \in \Phi$ such that

$$F\left(\left| \left\| \int_a^b \kappa(t, s, x(s)) ds \right\|_{\infty} - \left\| \int_a^b \kappa(t, s, y(s)) ds \right\|_{\infty} \right) \leq \varphi(F(\|x - y\|_{\infty})) \quad (35)$$

for all $x, y \in C(I, \mathbb{R})$. Let there is $c \in [a, b]$ in order that \mathcal{R}_c is topologically closed and algebraically closed with respect to difference. Then there is $x \in C(I, \mathbb{R})$ so that c is a root of equation (34).

Proof. Define $T : C(I, \mathbb{R}) \rightarrow \mathbb{R}$ by $Tx = \left\| \int_a^b \kappa(t, s, x(s)) ds \right\|_{\infty}$. By (35), we get

$$F(|Tx - Ty|) \leq \varphi(F(\|x - y\|_{\infty})) \quad (36)$$

for all $x, y \in C(I, \mathbb{R})$. Using Corollary 2, there is $x \in C(I, \mathbb{R})$ so that $x(c) = Tx$, that is, $x(c) = \left\| \int_a^b \kappa(t, s, x(s)) ds \right\|_{\infty}$, i.e., c is a root of Equation (34). \square

5. Application 2

In this section, we present an application of our Theorem 1 to establish PPF-dependent solution to a periodic boundary value problem.

Consider the second-order periodic boundary value problem

$$\begin{cases} x''(t) = f(t, x(t), x_t), \\ x_0 = \phi_0 \in C([-t, 0], \mathbb{R}) = \mathcal{C}, \\ x(0) = x(1) = \phi_0(0), \end{cases} \quad (37)$$

where $t \in I = [0, 1]$, $f \in C([0, 1] \times \mathbb{R} \times \mathcal{C}, \mathbb{R})$ and $x_t(s) = x(t + s)$ with $s \in [-t, 0]$.

Problem (37) can be rewritten as

$$\begin{cases} x(t) = \phi_0(0) - \int_0^1 G(t, s) f(s, x(s), x_s) ds, \\ x_0 = \phi_0 \in C([-t, 0], \mathbb{R}) = \mathcal{C}, \\ x(0) = x(1) = \phi_0(0). \end{cases} \quad (38)$$

where the kernel is given by

$$G(t, s) = \begin{cases} s(1-t), & \text{if } s \in [0, t] \\ t(1-s), & \text{if } s \in [t, 1] \end{cases}$$

(see [16] for details.)

Let

$$\hat{E} = \{ \hat{x} = (x_t)_{t \in I} : x_t \in \mathcal{C}, x \in C([0, 1], [0, 1]), \\ x(0) = x(1) = \phi_0(0), x_0 = \phi_0 \in \mathcal{C} \}.$$

This means that $\hat{x} \in \mathcal{C}^I$. Let

$$\|\hat{x} - \hat{y}\|_{\hat{E}} = \sup_{t \in I} \max_{-t \leq s \leq 0} |x_t(s) - y_t(s)| = \sup_{t \in I} \|x_t - y_t\|_{\mathcal{C}}.$$

In [17], it has been shown that \hat{E} is complete.

Suppose that for all $x, y \in C(I, [0, 1])$ we have,

$$|f(t, x(t), x_t) - f(t, y(t), y_t)| \leq \frac{8}{3} \|x(t) - y(t)\|_{\mathcal{C}}.$$

Then the PBVP (37) has a unique solution $x \in C(I, [0, 1])$ in a Razumikhin class.

For this define operator $S : \hat{E} \rightarrow \mathbb{R}^I$ as

$$S\hat{x}(t) = \phi_0(0) - \int_0^1 G(t, s) f(s, x(s), x_s) ds.$$

Via a careful calculation, we see that

$$\int_a^1 |G(x, t)| dt = \frac{x}{2} - \frac{x^2}{2} \leq \frac{1}{8}.$$

To show that all assumptions of Theorem 1 are satisfied, it remains to prove that T is an φ -F-contraction. For each $t \in I$, we have

$$\begin{aligned} & 2 - \frac{2}{|S\hat{x}(t) - S\hat{y}(t)|} \\ &= 2 - \frac{2}{\int_0^1 G(t, s) [f(s, x(s), x_s) - f(s, y(s), y_s)] ds} \\ &= 2 - \frac{2}{\int_0^1 G(t, s) [f(s, x(s), x_s) - f(s, y(s), y_s)] ds} \\ &\leq 2 - \frac{2}{\int_0^1 G(t, s) \frac{8}{3} \|\hat{x} - \hat{y}\|_{\hat{E}} ds} \\ &\leq 2 - \frac{6}{\|\hat{x} - \hat{y}\|_{\hat{E}}} = 3 \left(2 - \frac{2}{\|\hat{x} - \hat{y}\|_{\hat{E}}} \right) - 4 = \varphi(F(\|\hat{x} - \hat{y}\|_{\hat{E}})). \end{aligned}$$

which yields that

$$F(\|S\hat{x} - S\hat{y}\|_{\infty}) \leq \varphi(F(\|\hat{x} - \hat{y}\|_{\hat{E}})), \quad (39)$$

where $F(t) = 2 - \frac{2}{t}$ and

$$\varphi(t) = \begin{cases} 3t - 4, & t < 1 \\ t - 1, & t \geq 1. \end{cases}$$

Thus, all of the assumptions of Theorem 1 are fulfilled for $c = 0$ and we deduce the existence of an $\hat{x} \in \hat{E}$ such that

$$S(\hat{x}) = \hat{x}(0) = (x_t(0))_{t \in I} = (x(t))_{t \in I}.$$

This means that the integral Equation (38) has a solution and so the second-order periodic boundary value problem (37) has a solution.

6. Conclusions

We have introduced the concept of multi-valued generalized $\varphi - F$ -contraction (weakly $\varphi - F$ -contraction) as a generalization of multi-valued generalized $\varphi - F$ -contraction. Furthermore, we introduced the concept of multi-valued generalized $\alpha - \varphi - F$ -contractions and we proved some PPF-dependent fixed point results in the setting of a Banach space. Moreover, we deduced the PPF-dependent coincidence point result for single and multi-valued mappings. Finally, we established PPF-dependent solutions to a periodic boundary value problem.

Author Contributions: All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Funding: The publication of this article was funded by the Qatar National Library.

Acknowledgments: The publication of this article was funded by the Qatar National Library. The authors are highly appreciated the referees and editor efforts of this paper who helped us to improve it in several places.

Conflicts of Interest: The authors declare no conflict of interest.

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