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Parallel LQP alternating direction method for solving variational inequality problems with separable structure

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Dedicated to Professor Shih-Sen Chang on the occasion of his 80th birthday.

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Abstract

In this paper, we propose a logarithmic-quadratic proximal alternating direction method for structured variational inequalities. The predictor is obtained by solving series of related systems of nonlinear equations, and the new iterate is obtained by a convex combination of the previous point and the one generated by a projection-type method along a new descent direction. Global convergence of the new method is proved under certain assumptions. Preliminary numerical experiments are included to verify the theoretical assertions of the proposed method.

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1 Introduction

The problem we are concerned with in this paper is for the following variational inequalities: find $u \in \Omega$ such that

$$(u' - u)^T F(u) \geq 0, \quad \forall u' \in \Omega, \tag{1.1}$$

with

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad F(u) = \begin{pmatrix} f(x) \\ g(y) \end{pmatrix} \quad \text{and}$$
$$\Omega = \{(x, y) \mid x \in \mathcal{R}^n_+, y \in \mathcal{R}^m_+, Ax + By = b\}, \tag{1.2}$$

where $A \in \mathcal{R}^{l\times n}$, $B \in \mathcal{R}^{l\times m}$ are given matrices, $b \in \mathcal{R}^l$ is a given vector, and $f : \mathcal{R}^n_+ \to \mathcal{R}^n$, $g : \mathcal{R}^m_+ \to \mathcal{R}^m$ are given monotone operators. Studies and applications of such problems can be found in [1–11]. By attaching a Lagrange multiplier vector $\lambda \in \mathcal{R}^l$ to the linear constraint $Ax + By = b$, problem (1.1)-(1.2) can be explained in terms of finding $w \in \mathcal{W}$ such that

$$(w' - w)^T Q(w) \geq 0, \quad \forall w' \in \mathcal{W}, \tag{1.3}$$

where

$$w = \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix}, \qquad Q(w) = \begin{pmatrix} f(x) - A^T\lambda \\ g(y) - B^T\lambda \\ Ax + By - b \end{pmatrix}, \qquad \mathcal{W} = \mathcal{R}^n_+ \times \mathcal{R}^m_+ \times \mathcal{R}^l. \tag{1.4}$$

Problem (1.3)-(1.4) is referred to as SVI (*structured variational inequalities*).

The alternating direction method (ADM) is a powerful method for solving the structured problem (1.3)-(1.4), since it decomposes the original problems into a series of subproblems with a lower scale, originally proposed by Gabay and Mercier [4] and Gabay [3]. The classical proximal alternating direction method (PADM) [12–15] is an effective numerical approach for solving variational inequalities with separable structure. In [16], He proposed splitting augmented Lagrangian methods for structured monotone variational inequalities whose operator is composed by two separable operators. For given $(x^k, y^k, \lambda^k) \in \mathcal{R}^n_{++} \times \mathcal{R}^m_{++} \times \mathcal{R}^l$, the predictor $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ is obtained via the following steps:

Step 1. Solve the following variational inequality to obtain $\tilde{x}^k$:

$$\left(x' - \tilde{x}^k\right)^T \left\{f\left(\tilde{x}^k\right) - A^T\left[\lambda^k - H\left(A\tilde{x}^k + By^k - b\right)\right]\right\} \geq 0, \quad \forall x' \in \mathcal{R}^n_{++}. \tag{1.5}$$

Step 2. Solve the following variational inequality to obtain $\tilde{y}^k$:

$$\left(y' - \tilde{y}^k\right)^T \left\{g\left(\tilde{y}^k\right) - B^T\left[\lambda^k - H\left(Ax^k + B\tilde{y}^k - b\right)\right]\right\} \geq 0, \quad \forall y' \in \mathcal{R}^m_{++}. \tag{1.6}$$

Step 3. Update λ^k via

$$\tilde{\lambda}^k = \lambda^k - H\left(A\tilde{x}^k + B\tilde{y}^k - b\right). \tag{1.7}$$

The main advantage of the method of [16] is that the predictor is obtained by solving subproblems in a parallel wise. Recently, Yuan and Li [17] have developed a logarithmic quadratic proximal (LQP)-based decomposition method by applying the LQP terms to regularize the ADM subproblems. The new iterative $(x^{k+1}, y^{k+1}, \lambda^{k+1})$ in [17] is obtained via the following procedure: From a given $w^k = (x^k, y^k, \lambda^k) \in \mathcal{R}^n_{++} \times \mathcal{R}^m_{++} \times \mathcal{R}^l$, and $\mu \in (0,1)$, $(x^{k+1}, y^{k+1}, \lambda^{k+1})$ is obtained via solving the following system:

$$\begin{aligned} &f(x) - A^T\left[\lambda^k - H\left(Ax + By^k - b\right)\right] + R\left[\left(x - x^k\right) + \mu\left(x^k - X_k^2 x^{-1}\right)\right] = 0, \\ &g(y) - B^T\left[\lambda^k - H\left(Ax^{k+1} + By - b\right)\right] + S\left[\left(y - y^k\right) + \mu\left(y^k - Y_k^2 y^{-1}\right)\right] = 0, \\ &\lambda^{k+1} = \lambda^k - H\left(Ax^k + By^k - b\right), \end{aligned}$$

where $X_k = \operatorname{diag}(x_1^k, x_2^k, \ldots, x_n^k)$. Note that the LQP method was presented originally in [18]. Later, Bnouhachem *et al.* [19–21] have proposed a new inexact LQP alternating direction method by solving a series of related systems of nonlinear equations. Very recently, Li [22] presented an LQP-based prediction-correction method: the new iterate is obtained by a convex combination of the previous point and the one generated by a projection-type method along a descent direction.

In the present paper, inspired by the above cited works and by the recent works going in this direction, we proposed a new LQP-based prediction-correction method. The predictor is obtained by using the idea of He [16] to solve series of related systems of nonlinear equations, and the new iterate is obtained by a convex combination of the previous point and the one generated by a projection-type method along another descent direction. Under certain conditions, we proved the global convergence of the proposed algorithm. Preliminary numerical experiments are included to verify the theoretical assertions of the proposed method.

2 The proposed method

In this section, we recall some basic definitions and properties, which will be frequently used in our later analysis. Some useful results proved already in the literature are also summarized. The first lemma provides some basic properties of projection onto Ω.

Lemma 2.1 *Let G be a symmetric positive definite matrix and Ω be a nonempty closed convex subset of R^l, we denote $P_{\Omega,G}[\cdot]$ as the projection under the G-norm, i.e.,*

$$P_{\Omega,G}[v] = \operatorname{argmin}\{\|v-u\|_G \mid u \in \Omega\}.$$

Then, we have the following inequalities:

$$\bigl(z - P_{\Omega,G}[z]\bigr)^T G\bigl(P_{\Omega,G}[z] - v\bigr) \ge 0, \quad \forall z \in R^l, v \in \Omega; \tag{2.1}$$

$$\bigl\|P_{\Omega,G}[u] - P_{\Omega,G}[v]\bigr\|_G \le \|u-v\|_G, \quad \forall u, v \in R^l; \tag{2.2}$$

$$\bigl\|u - P_{\Omega,G}[z]\bigr\|_G^2 \le \|z-u\|_G^2 - \bigl\|z - P_{\Omega,G}[z]\bigr\|_G^2, \quad \forall z \in R^l, u \in \Omega. \tag{2.3}$$

In course, we always make the following standard assumptions:

Assumption A f is monotone with respect to $\mathcal{R}^n_{++}$ and g is monotone with respect to $\mathcal{R}^m_{++}$.

Assumption B The solution set of SVI, denoted by $\mathcal{W}^*$, is nonempty.

Now, we suggest and consider the new LQP alternating direction method (LQP-ADM) for solving SVI as follows.

Prediction step: For a given $w^k = (x^k, y^k, \lambda^k) \in \mathcal{R}^n_{++} \times \mathcal{R}^m_{++} \times \mathcal{R}^l$, and $\mu \in (0,1)$, the predictor $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k) \in \mathcal{R}^n_{++} \times \mathcal{R}^m_{++} \times \mathcal{R}^l$ is obtained via solving the following system:

$$f(x) - A^T\bigl[\lambda^k - H\bigl(Ax + By^k - b\bigr)\bigr] + R\bigl[\bigl(x - x^k\bigr) + \mu\bigl(x^k - X_k^2 x^{-1}\bigr)\bigr] = 0, \tag{2.4a}$$

$$g(y) - B^T\bigl[\lambda^k - H\bigl(Ax^k + By - b\bigr)\bigr] + S\bigl[\bigl(y - y^k\bigr) + \mu\bigl(y^k - Y_k^2 y^{-1}\bigr)\bigr] = 0, \tag{2.4b}$$

$$\tilde{\lambda}^k = \lambda^k - H\bigl(A\tilde{x}^k + B\tilde{y}^k - b\bigr). \tag{2.4c}$$

Correction step: The new iterate $w^{k+1}(\alpha_k) = (x^{k+1}, y^{k+1}, \lambda^{k+1})$ is given by

$$w^{k+1}(\alpha_k) = (1-\sigma)w^k + \sigma P_{\mathcal{W}}\bigl[w^k - \alpha_k G^{-1} d\bigl(w^k, \tilde{w}^k\bigr)\bigr], \quad \sigma \in (0,1), \tag{2.5}$$

where

$$\alpha_k = \frac{\varphi_k}{\|w^k - \tilde{w}^k\|_G^2}, \tag{2.6}$$

$$\varphi_k := \|w^k - \tilde{w}^k\|_M^2 + \left(\lambda^k - \tilde{\lambda}^k\right)^T \left(A\left(x^k - \tilde{x}^k\right) + B\left(y^k - \tilde{y}^k\right)\right), \tag{2.7}$$

$$d\left(w^k, \tilde{w}^k\right) = \begin{pmatrix} f(\tilde{x}^k) - A^T\tilde{\lambda}^k + A^T H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)) \\ g(\tilde{y}^k) - B^T\tilde{\lambda}^k + B^T H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)) \\ A\tilde{x}^k + B\tilde{y}^k - b \end{pmatrix}$$

and

$$G = \begin{pmatrix} (1+\mu)R + A^T HA & 0 & 0 \\ 0 & (1+\mu)S + B^T HB & 0 \\ 0 & 0 & H^{-1} \end{pmatrix},$$

$$M = \begin{pmatrix} R + A^T HA & 0 & 0 \\ 0 & S + B^T HB & 0 \\ 0 & 0 & H^{-1} \end{pmatrix}.$$

We need the following result in the convergence analysis of the proposed method.

Lemma 2.2 [17] *Let q be a monotone mapping of u with respect to $\mathcal{R}_+^n$ and $R \in \mathcal{R}^{n\times n}$ be positive definite diagonal matrix. For given $u^k > 0$, if we let $U_k := \operatorname{diag}(u_1^k, u_2^k, \dots, u_n^k)$ and u^{-1} be an n-vector whose jth element is $1/u_j$, then the equation*

$$q(u) + R\left[\left(u - u^k\right) + \mu\left(u^k - U_k^2 u^{-1}\right)\right] = 0 \tag{2.8}$$

has a unique positive solution u. Moreover, for any $v \geq 0$, we have

$$(v - u)^T q(u) \geq \frac{1+\mu}{2}\left(\|u - v\|_R^2 - \left\|u^k - v\right\|_R^2\right) + \frac{1-\mu}{2}\left\|u^k - u\right\|_R^2. \tag{2.9}$$

In the next theorem, we show that α_k is lower bounded away from zero, and it is one of the keys to prove the global convergence results.

Theorem 2.1 *For given $w^k \in \mathcal{R}_{++}^n \times \mathcal{R}_{++}^m \times \mathcal{R}^l$, let $\tilde{w}^k$ be generated by* (2.4a)-(2.4c), *then we have the following*:

$$\varphi_k \geq \frac{2-\sqrt{2}}{2}\left\|w^k - \tilde{w}^k\right\|_G^2 \tag{2.10}$$

and

$$\alpha_k \geq \frac{2-\sqrt{2}}{2}. \tag{2.11}$$

Proof It follows from (2.7) that

$$\begin{aligned} \varphi_k &= \left\|w^k - \tilde{w}^k\right\|_M^2 + \left(\lambda^k - \tilde{\lambda}^k\right)^T \left(A\left(x^k - \tilde{x}^k\right) + B\left(y^k - \tilde{y}^k\right)\right) \\ &= \left\|x^k - \tilde{x}^k\right\|_R^2 + \left\|Ax^k - A\tilde{x}^k\right\|_H^2 + \left\|y^k - \tilde{y}^k\right\|_S^2 \end{aligned}$$

$$+ \|By^k - B\tilde{y}^k\|_H^2 + \|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2 \\ + (\lambda^k - \tilde{\lambda}^k)^T (A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)). \tag{2.12}$$

By using the Cauchy-Schwarz inequality, we have

$$(\lambda^k - \tilde{\lambda}^k)^T (A(x^k - \tilde{x}^k)) \geq -\frac{1}{2}\left(\sqrt{2}\|A(x^k - \tilde{x}^k)\|_H^2 + \frac{1}{\sqrt{2}}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2\right) \tag{2.13}$$

and

$$(\lambda^k - \tilde{\lambda}^k)^T (B(y^k - \tilde{y}^k)) \geq -\frac{1}{2}\left(\sqrt{2}\|B(y^k - \tilde{y}^k)\|_H^2 + \frac{1}{\sqrt{2}}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2\right). \tag{2.14}$$

Substituting (2.13) and (2.14) into (2.12), we get

$$\begin{aligned}
\varphi_k &\geq \frac{2-\sqrt{2}}{2}\left(\|Ax^k - A\tilde{x}^k\|_H^2 + \|By^k - B\tilde{y}^k\|_H^2 + \|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2\right) \\
&\quad + \|x^k - \tilde{x}^k\|_R^2 + \|y^k - \tilde{y}^k\|_S^2 \\
&\geq \frac{2-\sqrt{2}}{2}\left(\|Ax^k - A\tilde{x}^k\|_H^2 + \|By^k - B\tilde{y}^k\|_H^2 + \|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2\right) \\
&\quad + (2-\sqrt{2})\left(\|x^k - \tilde{x}^k\|_R^2 + \|y^k - \tilde{y}^k\|_S^2\right) \\
&= \frac{2-\sqrt{2}}{2}\left(\|w^k - \tilde{w}^k\|_G^2 + (1-\mu)\|x^k - \tilde{x}^k\|_R^2 + (1-\mu)\|y^k - \tilde{y}^k\|_S^2\right) \\
&\geq \frac{2-\sqrt{2}}{2}\|w^k - \tilde{w}^k\|_G^2.
\end{aligned}$$

Therefore, it follows from (2.6) and (2.10) that

$$\alpha_k \geq \frac{2-\sqrt{2}}{2}$$

and this completes the proof. □

3 Basic results

In this section, we prove some basic properties, which will be used to establish the sufficient and necessary conditions for the convergence of the proposed method. The following results are due to applying Lemma 2.2 to the LQP system in the prediction step of the proposed method.

Lemma 3.1 *For given* $w^k = (x^k, y^k, \lambda^k) \in \mathcal{R}_{++}^n \times \mathcal{R}_{++}^m \times \mathcal{R}^l$, *let* $\tilde{w}^k$ *be generated by* (2.4a)-(2.4c). *Then for any* $w^* = (x^*, y^*, \lambda^*) \in \mathcal{W}^*$, *we have*

$$(x_*^k - \tilde{x}^k)^T \{(1+\mu)R(x^k - \tilde{x}^k) - f(\tilde{x}^k) + A^T\tilde{\lambda}^k + A^T HA(x^k - \tilde{x}^k) \\ - A^T H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k))\} \leq \mu\|x^k - \tilde{x}^k\|_R^2 \tag{3.1}$$

and

$$(y_*^k - \tilde{y}^k)^T \{(1+\mu)S(y^k - \tilde{y}^k) - g(\tilde{y}^k) + B^T\tilde{\lambda}^k + B^T HB(y^k - \tilde{y}^k) \\ - B^T H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k))\} \leq \mu\|y^k - \tilde{y}^k\|_S^2, \tag{3.2}$$

where

$$w_*^k = (x_*^k, y_*^k, \lambda_*^k) := P_{\mathcal{W}}\big[w^k - \alpha_k G^{-1} d(w^k, \tilde{w}^k)\big]. \tag{3.3}$$

Proof Applying Lemma 2.2 to (2.4a) by setting $u^k = x^k$, $u = \tilde{x}^k$, $v = x_*^k$ in (2.9) and

$$q(u) = f(\tilde{x}^k) - A^T\big[\lambda^k - H(A\tilde{x}^k + By^k - b)\big],$$

we get

$$\begin{aligned}&(x_*^k - \tilde{x}^k)^T\big\{f(\tilde{x}^k) - A^T\big[\lambda^k - H(A\tilde{x}^k + By^k - b)\big]\big\}\\ &\quad\geq \frac{1+\mu}{2}\big(\|\tilde{x}^k - x_*^k\|_R^2 - \|x^k - x_*^k\|_R^2\big) + \frac{1-\mu}{2}\|x^k - \tilde{x}^k\|_R^2.\end{aligned} \tag{3.4}$$

Recall

$$(x_*^k - \tilde{x}^k)^T R(x^k - \tilde{x}^k) = \frac{1}{2}\big(\|\tilde{x}^k - x_*^k\|_R^2 - \|x^k - x_*^k\|_R^2\big) + \frac{1}{2}\|x^k - \tilde{x}^k\|_R^2. \tag{3.5}$$

Adding (3.4) and (3.5), we obtain

$$\begin{aligned}&(x_*^k - \tilde{x}^k)^T\big\{(1+\mu)R(x^k - \tilde{x}^k) - f(\tilde{x}^k) + A^T\tilde{\lambda}^k + A^T HA(x^k - \tilde{x}^k)\\ &\quad - A^T H\big(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\big)\big\} \leq \mu\|x^k - \tilde{x}^k\|_R^2.\end{aligned}$$

Similarly, applying Lemma 2.2 to (2.4b), substituting $u^k = y^k$, $u = \tilde{y}^k$, $v = y_*^k$, and replacing R, n with S, m, respectively, in (2.9) and

$$q(u) = g(\tilde{y}^k) - B^T\big[\lambda^k - H(Ax^k + B\tilde{y}^k - b)\big],$$

we get

$$\begin{aligned}&(y_*^k - \tilde{y}^k)^T\big\{g(\tilde{y}^k) - B^T\big[\lambda^k - H(Ax^k + B\tilde{y}^k - b)\big]\big\}\\ &\quad\geq \frac{1+\mu}{2}\big(\|\tilde{y}^k - y_*^k\|_S^2 - \|y^k - y_*^k\|_S^2\big) + \frac{1-\mu}{2}\|y^k - \tilde{y}^k\|_S^2.\end{aligned} \tag{3.6}$$

Recall

$$(y_*^k - \tilde{y}^k)^T S(y^k - \tilde{y}^k) = \frac{1}{2}\big(\|\tilde{y}^k - y_*^k\|_S^2 - \|y^k - y_*^k\|_S^2\big) + \frac{1}{2}\|y^k - \tilde{y}^k\|_S^2. \tag{3.7}$$

Adding (3.6) and (3.7), we have

$$\begin{aligned}&(y_*^k - \tilde{y}^k)^T\big\{(1+\mu)S(y^k - \tilde{y}^k) - g(\tilde{y}^k) + B^T\tilde{\lambda}^k + B^T HB(y^k - \tilde{y}^k)\\ &\quad - B^T H\big(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\big)\big\} \leq \mu\|y^k - \tilde{y}^k\|_S^2.\end{aligned}$$

The assertion of this lemma is proved. □

Theorem 3.1 *Let $w^* \in \mathcal{W}^*$, $w^{k+1}(\alpha_k)$ be defined by* (2.5) *and*

$$\Theta(\alpha_k) := \|w^k - w^*\|_G^2 - \|w^{k+1}(\alpha_k) - w^*\|_G^2, \tag{3.8}$$

then we have

$$\Theta(\alpha_k) \geq \sigma\bigl(\|w^k - w_*^k - \alpha_k(w^k - \tilde{w}^k)\|_G^2 + 2\alpha_k\varphi_k - \alpha_k^2\|w^k - \tilde{w}^k\|_G^2\bigr). \tag{3.9}$$

Proof It follows from (3.1) and (3.2) that

$$\begin{pmatrix} x_*^k - \tilde{x}^k \\ y_*^k - \tilde{y}^k \\ \lambda_*^k - \tilde{\lambda}^k \end{pmatrix}^T \begin{pmatrix} (1+\mu)R(x^k - \tilde{x}^k) - f(\tilde{x}^k) + A^T\tilde{\lambda}^k + A^T HA(x^k - \tilde{x}^k) - A^T H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)) \\ (1+\mu)S(y^k - \tilde{y}^k) - g(\tilde{y}^k) + B^T\tilde{\lambda}^k + B^T HB(y^k - \tilde{y}^k) - B^T H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)) \\ H^{-1}(\lambda^k - \tilde{\lambda}^k) - (A\tilde{x}^k + B\tilde{y}^k - b) \end{pmatrix}$$
$$\leq \mu\|x^k - \tilde{x}^k\|_R^2 + \mu\|y^k - \tilde{y}^k\|_S^2,$$

which implies

$$2\alpha_k(w_*^k - \tilde{w}^k)^T\bigl(G(w^k - \tilde{w}^k) - d(w^k, \tilde{w}^k)\bigr) - 2\alpha_k\mu\|x^k - \tilde{x}^k\|_R^2 - 2\alpha_k\mu\|y^k - \tilde{y}^k\|_S^2 \leq 0. \tag{3.10}$$

Since $w^* \in \mathcal{W}^*$ and $w_*^k = P_{\mathcal{W}}[w^k - \alpha_k G^{-1}d(w^k, \tilde{w}^k)]$, it follows from (2.3) that

$$\|w_*^k - w^*\|_G^2 \leq \|w^k - \alpha_k G^{-1}d(w^k, \tilde{w}^k) - w^*\|_G^2 - \|w^k - \alpha_k G^{-1}d(w^k, \tilde{w}^k) - w_*^k\|_G^2. \tag{3.11}$$

From (2.5), we get

$$\begin{aligned} \|w^{k+1}(\alpha_k) - w^*\|_G^2 &= \|(1-\sigma)(w^k - w^*) + \sigma(w_*^k - w^*)\|_G^2 \\ &= (1-\sigma)^2\|w^k - w^*\|_G^2 + \sigma^2\|w_*^k - w^*\|_G^2 \\ &\quad + 2\sigma(1-\sigma)(w^k - w^*)^T G(w_*^k - w^*). \end{aligned}$$

Using the following identity:

$$2(a+b)^T Gb = \|a+b\|_G^2 - \|a\|_G^2 + \|b\|_G^2$$

for $a = w^k - w_*^k$, $b = w_*^k - w^*$, and (3.11), we obtain

$$\begin{aligned} \|w^{k+1}(\alpha_k) - w^*\|_G^2 &= (1-\sigma)^2\|w^k - w^*\|_G^2 + \sigma^2\|w_*^k - w^*\|_G^2 \\ &\quad + \sigma(1-\sigma)\bigl\{\|w^k - w^*\|_G^2 - \|w^k - w_*^k\|_G^2 + \|w_*^k - w^*\|_G^2\bigr\} \\ &= (1-\sigma)\|w^k - w^*\|_G^2 + \sigma\|w_*^k - w^*\|_G^2 - \sigma(1-\sigma)\|w^k - w_*^k\|_G^2 \\ &\leq (1-\sigma)\|w^k - w^*\|_G^2 + \sigma\|w^k - \alpha_k G^{-1}d(w^k, \tilde{w}^k) - w^*\|_G^2 \\ &\quad - \sigma\|w^k - \alpha_k G^{-1}d(w^k, \tilde{w}^k) - w_*^k\|_G^2 - \sigma(1-\sigma)\|w^k - w_*^k\|_G^2. \end{aligned} \tag{3.12}$$

Using the definition of $\Theta(\alpha_k)$ and (3.12), we get

$$\begin{aligned} \Theta(\alpha_k) &\geq \sigma\|w^k - w_*^k\|_G^2 + 2\sigma\alpha_k(w_*^k - w^k)^T d(w^k, \tilde{w}^k) \\ &\quad + 2\sigma\alpha_k(w^k - w^*)^T d(w^k, \tilde{w}^k). \end{aligned} \tag{3.13}$$

Using the monotonicity of f and g, we obtain

$$\begin{pmatrix}\tilde{x}^k - x^* \\ \tilde{y}^k - y^* \\ \tilde{\lambda}^k - \lambda^*\end{pmatrix}^T \begin{pmatrix} f(\tilde{x}^k) - A^T\tilde{\lambda}^k \\ g(\tilde{y}^k) - B^T\tilde{\lambda}^k \\ A\tilde{x}^k + B\tilde{y}^k - b\end{pmatrix} \geq \begin{pmatrix}\tilde{x}^k - x^* \\ \tilde{y}^k - y^* \\ \tilde{\lambda}^k - \lambda^*\end{pmatrix}^T \begin{pmatrix} f(x^*) - A^T\lambda^* \\ g(y^*) - B^T\lambda^* \\ Ax^* + By^* - b\end{pmatrix} \geq 0$$

and consequently

$$\begin{aligned}(\tilde{w}^k - w^*)^T d(w^k, \tilde{w}^k) &\geq (\tilde{w}^k - w^*)^T \begin{pmatrix} A^T H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)) \\ B^T H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)) \\ 0\end{pmatrix} \\ &= (A\tilde{x}^k + B\tilde{y}^k - b)^T H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)) \\ &= (\lambda^k - \tilde{\lambda}^k)^T H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k))\end{aligned}$$

and it follows that

$$\begin{aligned}(w^k - w^*)^T d(w^k, \tilde{w}^k) \geq{}& (w^k - \tilde{w}^k)^T d(w^k, \tilde{w}^k) \\ &+ (\lambda^k - \tilde{\lambda}^k)^T H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)).\end{aligned} \tag{3.14}$$

Applying (3.14) to the last term on the right side of (3.13), we obtain

$$\begin{aligned}\Theta(\alpha_k) \geq{}& \sigma \|w^k - w_*^k\|_G^2 + 2\sigma\alpha_k (w_*^k - w^k)^T d(w^k, \tilde{w}^k) \\ &+ 2\sigma\alpha_k \{(w^k - \tilde{w}^k)^T d(w^k, \tilde{w}^k) \\ &+ (\lambda^k - \tilde{\lambda}^k)^T H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k))\} \\ ={}& \sigma\{\|w^k - w_*^k\|_G^2 + 2\alpha_k (w_*^k - \tilde{w}^k)^T d(w^k, \tilde{w}^k) \\ &+ 2\alpha_k (\lambda^k - \tilde{\lambda}^k)^T H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k))\}.\end{aligned} \tag{3.15}$$

Adding (3.10) (multiplied by σ) to (3.15), we get

$$\begin{aligned}\Theta(\alpha_k) \geq{}& \sigma\{\|w^k - w_*^k\|_G^2 + 2\alpha_k (w_*^k - \tilde{w}^k)^T G(w^k - \tilde{w}^k) \\ &- 2\alpha_k\mu \|x^k - \tilde{x}^k\|_R^2 - 2\alpha_k\mu \|y^k - \tilde{y}^k\|_S^2 \\ &+ 2\alpha_k (\lambda^k - \tilde{\lambda}^k)^T H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k))\} \\ ={}& \sigma\{\|w^k - w_*^k - \alpha_k(w^k - \tilde{w}^k)\|_G^2 - \alpha_k^2 \|w^k - \tilde{w}^k\|_G^2 \\ &+ 2\alpha_k \|w^k - \tilde{w}^k\|_G^2 - 2\alpha_k\mu \|x^k - \tilde{x}^k\|_R^2 - 2\alpha_k\mu \|y^k - \tilde{y}^k\|_S^2 \\ &+ 2\alpha_k (\lambda^k - \tilde{\lambda}^k)^T H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k))\}\end{aligned}$$

and by using the notation of φ_k in (2.7), the theorem is proved. □

From the computational point of view, a relaxation factor $\gamma \in (0, 2)$ is preferable in the correction. We are now in a position to prove the contractive property of the iterative sequence.

Theorem 3.2 *Let $w^* \in \mathcal{W}^*$ be a solution of SVI and let $w^{k+1}(\gamma\alpha_k)$ be generated by* (2.5). *Then $\{w^k\}$ and $\{\tilde{w}^k\}$ are bounded sequences, and*

$$\|w^{k+1}(\gamma\alpha_k) - w^*\|_G^2 \le \|w^k - w^*\|_G^2 - c\|w^k - \tilde{w}^k\|_G^2, \tag{3.16}$$

where

$$c := \frac{\sigma\gamma(2-\sqrt{2})^2(2-\gamma)}{4} > 0.$$

Proof It follows from (3.9), (2.10), and (2.11) that

$$\begin{aligned} \|w^{k+1}(\gamma\alpha_k) - w^*\|_G^2 &\le \|w^k - w^*\|_G^2 - \sigma\left(2\gamma\alpha_k\varphi_k - \gamma^2\alpha_k^2\|w^k - \tilde{w}^k\|_G^2\right) \\ &= \|w^k - w^*\|_G^2 - \gamma(2-\gamma)\alpha_k\sigma\varphi_k \\ &\le \|w^k - w^*\|_G^2 - \frac{\sigma\gamma(2-\sqrt{2})^2(2-\gamma)}{4}\left(\|w^k - \tilde{w}^k\|_G^2\right). \end{aligned}$$

Since $\gamma \in (0,2)$, we have

$$\|w^{k+1} - w^*\| \le \|w^k - w^*\| \le \cdots \le \|w^0 - w^*\|,$$

and thus $\{w^k\}$ is a bounded sequence.

It follows from (3.16) that

$$\sum_{k=0}^{\infty} c\|w^k - \tilde{w}^k\|_G^2 < +\infty,$$

which means that

$$\lim_{k\to\infty}\|w^k - \tilde{w}^k\|_G = 0. \tag{3.17}$$

Since $\{w^k\}$ is a bounded sequence, we conclude that $\{\tilde{w}^k\}$ is also bounded. □

4 Convergence of the proposed method

In this section, we prove the global convergence of the proposed method. The following results can be proved by using the technique of Lemma 5.1 in [19].

Lemma 4.1 *For given $w^k = (x^k, y^k, \lambda^k) \in \mathcal{R}^n_{++} \times \mathcal{R}^m_{++} \times \mathcal{R}^l$, let $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ be generated by* (2.4a)-(2.4c). *Then for any $w = (x, y, \lambda) \in \mathcal{W}$, we have*

$$\begin{aligned} &(x - \tilde{x}^k)^T\left(f(\tilde{x}^k) - A^T\tilde{\lambda}^k - A^THA(x^k - \tilde{x}^k) + A^TH\left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right)\right) \\ &\quad \ge (x^k - \tilde{x}^k)^TR\left\{(1+\mu)x - (\mu x^k + \tilde{x}^k)\right\} \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &(y - \tilde{y}^k)^T\left(g(\tilde{y}^k) - B^T\tilde{\lambda}^k - B^THB(y^k - \tilde{y}^k) + B^TH\left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right)\right) \\ &\quad \ge (y^k - \tilde{y}^k)^TS\left\{(1+\mu)y - (\mu y^k + \tilde{y}^k)\right\}. \end{aligned} \tag{4.2}$$

Proof Similarly as in (3.4), we have

$$\begin{aligned}&(x-\tilde{x}^k)^T\{f(\tilde{x}^k)-A^T[\lambda^k-H(A\tilde{x}^k+By^k-b)]\}\\&\quad\geq\frac{1+\mu}{2}\left(\|\tilde{x}^k-x\|_R^2-\|x^k-x\|_R^2\right)+\frac{1-\mu}{2}\|x^k-\tilde{x}^k\|_R^2,\end{aligned}$$

which implies

$$\begin{aligned}&(x-\tilde{x}^k)^T\left(f(\tilde{x}^k)-A^T\tilde{\lambda}^k-A^THA(x^k-\tilde{x}^k)+A^TH\left(A(x^k-\tilde{x}^k)+B(y^k-\tilde{y}^k)\right)\right)\\&\quad\geq\frac{1+\mu}{2}\left(\|\tilde{x}^k-x\|_R^2-\|x^k-x\|_R^2\right)+\frac{1-\mu}{2}\|x^k-\tilde{x}^k\|_R^2.\end{aligned}$$

By a simple manipulation, we have

$$\begin{aligned}&\frac{1+\mu}{2}\left(\|\tilde{x}^k-x\|_R^2-\|x^k-x\|_R^2\right)+\frac{1-\mu}{2}\|x^k-\tilde{x}^k\|_R^2\\&\quad=(1+\mu)x^TRx^k-(1+\mu)x^TR\tilde{x}^k-(1-\mu)(\tilde{x}^k)^TRx^k-\mu\|x^k\|_R^2+\|\tilde{x}^k\|_R^2\\&\quad=(1+\mu)x^TR(x^k-\tilde{x}^k)-(x^k-\tilde{x}^k)^TR(\mu x^k+\tilde{x}^k)\\&\quad=(x^k-\tilde{x}^k)^TR\{(1+\mu)x-(\mu x^k+\tilde{x}^k)\},\end{aligned}$$

and the assertion (4.1) is proved. Similarly we can prove the assertion (4.2). □

Now, we are ready to prove the convergence of the proposed method.

Theorem 4.1 *The sequence $\{w^k\}$ generated by the proposed method converges to some w^∞ which is a solution of SVI.*

Proof It follows from (3.17) that

$$\lim_{k\to\infty}\|x^k-\tilde{x}^k\|_R=0,\qquad\lim_{k\to\infty}\|y^k-\tilde{y}^k\|_S=0\tag{4.3}$$

and

$$\lim_{k\to\infty}\|\lambda^k-\tilde{\lambda}^k\|_{H^{-1}}=\lim_{k\to\infty}\|A\tilde{x}^k+B\tilde{y}^k-b\|_H=0.\tag{4.4}$$

Moreover, (4.1) and (4.2) imply that

$$\begin{aligned}&(x-\tilde{x}^k)^T\left(f(\tilde{x}^k)-A^T\tilde{\lambda}^k\right)\\&\quad\geq(x^k-\tilde{x}^k)^TR\{(1+\mu)x-(\mu x^k+\tilde{x}^k)\}\\&\qquad+(x-\tilde{x}^k)^T\left(A^THA(x^k-\tilde{x}^k)-A^TH\left(A(x^k-\tilde{x}^k)+B(y^k-\tilde{y}^k)\right)\right)\end{aligned}$$

and

$$\begin{aligned}&(y-\tilde{y}^k)^T\left(g(\tilde{y}^k)-B^T\tilde{\lambda}^k\right)\\&\quad\geq(y^k-\tilde{y}^k)^TS\{(1+\mu)y-(\mu y^k+\tilde{y}^k)\}\\&\qquad+(y-\tilde{y}^k)^T\left(B^THB(y^k-\tilde{y}^k)-B^TH\left(A(x^k-\tilde{x}^k)+B(y^k-\tilde{y}^k)\right)\right).\end{aligned}$$

We deduce from (4.3) that

$$\begin{cases} \lim_{k\to\infty}(x-\tilde{x}^k)^T\{f(\tilde{x}^k)-A^T\tilde{\lambda}^k\}\geq 0, & \forall x\in\mathcal{R}^n_{++}, \\ \lim_{k\to\infty}(y-\tilde{y}^k)^T\{g(\tilde{y}^k)-B^T\tilde{\lambda}^k\}\geq 0, & \forall y\in\mathcal{R}^m_{++}. \end{cases} \tag{4.5}$$

Since $\{w^k\}$ is bounded, it has at least one cluster point. Let w^∞ be a cluster point of $\{w^k\}$ and the subsequence $\{w^{k_j}\}$ converges to w^∞. It follows from (4.4) and (4.5) that

$$\begin{cases} \lim_{j\to\infty}(x-x^{k_j})^T\{f(x^{k_j})-A^T\lambda^{k_j}\}\geq 0, & \forall x\in\mathcal{R}^n_{++}, \\ \lim_{j\to\infty}(y-y^{k_j})^T\{g(y^{k_j})-B^T\lambda^{k_j}\}\geq 0, & \forall y\in\mathcal{R}^m_{++}, \\ \lim_{j\to\infty}(Ax^{k_j}+By^{k_j}-b)=0 \end{cases}$$

and consequently

$$\begin{cases} (x-x^\infty)^T\{f(x^\infty)-A^T\lambda^\infty\}\geq 0, & \forall x\in\mathcal{R}^n_{++}, \\ (y-y^\infty)^T\{g(y^\infty)-B^T\lambda^\infty\}\geq 0, & \forall y\in\mathcal{R}^m_{++}, \\ Ax^\infty+By^\infty-b=0, \end{cases}$$

which means that w^∞ is a solution of SVI.

Now, we prove that the sequence $\{w^k\}$ converges to w^∞. Since

$$\lim_{k\to\infty}\|w^k-\tilde{w}^k\|_G=0 \quad\text{and}\quad \{\tilde{w}^{k_j}\}\to w^\infty$$

for any $\epsilon>0$, there exists an $l>0$ such that

$$\|\tilde{w}^{k_l}-w^\infty\|_G<\frac{\epsilon}{2} \quad\text{and}\quad \|w^{k_l}-\tilde{w}^{k_l}\|_G<\frac{\epsilon}{2}. \tag{4.6}$$

Therefore, for any $k\geq k_l$, it follows from (3.16) and (4.6) that

$$\|w^k-w^\infty\|_G\leq\|w^{k_l}-w^\infty\|_G\leq\|w^{k_l}-\tilde{w}^{k_l}\|_G+\|\tilde{w}^{k_l}-w^\infty\|_G<\epsilon.$$

This implies that the sequence $\{w^k\}$ converges to w^∞ which is a solution of SVI. □

5 Preliminary computational results

In this section, we report some numerical results of the proposed method, we consider the following optimization problem with matrix variables:

$$\min\left\{\frac{1}{2}\|X-C\|_F^2 \,\middle|\, X\in S^n_+\right\}, \tag{5.1}$$

where $\|\cdot\|_F$ is the matrix Fröbenius norm, *i.e.*, $\|C\|_F=(\sum_{i=1}^n\sum_{j=1}^n|C_{ij}|^2)^{1/2}$,

$$S^n_+=\{H\in\mathcal{R}^{n\times n}\mid H^T=H, H\succeq 0\}.$$

Note that the matrix Fröbenius norm is induced by the inner product

$$\langle A,B\rangle=\operatorname{Trace}(A^TB).$$

Table 1 Numerical results for problem (5.1) with $r = 0.5$, $s = 5$

Dimension of the problem	The proposed method		The method in [22]	
	k	CPU (sec.)	k	CPU (sec.)
100	48	1.34	70	2.32
300	53	9.73	78	13.04
500	56	34.12	82	46.82
700	57	90.03	85	122.17

Table 2 Numerical results for problem (5.1) with $r = 1$, $s = 10$

Dimension of the problem	The proposed method		The method in [22]	
	k	CPU (sec.)	k	CPU (sec.)
100	109	2.14	125	2.54
300	123	21.5	140	22.25
500	128	78	147	87.81
700	132	195.17	152	223.41

Note that problem (5.1) is equivalent to the following:

$$\begin{aligned} &\min \frac{1}{2}\|X-C\|_F^2 + \frac{1}{2}\|Y-C\|_F^2 \\ &\quad \text{s.t. } X - Y = 0, \\ &\qquad X, Y \in S_+^n, \end{aligned} \tag{5.2}$$

by attaching a Lagrange multiplier $Z \in \mathcal{R}^{n\times n}$ to the linear constraint $X - Y = 0$, the Lagrange function of (5.2) is

$$L(X,Y,Z) = \frac{1}{2}\|X-C\|_F^2 + \frac{1}{2}\|Y-C\|_F^2 - \langle Z, X-Y\rangle,$$

which is defined on $S_+^n \times S_+^n \times \mathcal{R}^{n\times n}$. If $(X^*, Y^*, Z^*) \in S_+^n \times S_+^n \times \mathcal{R}^{n\times n}$ is a KKT point of (5.2), then (5.2) can be converted to the following variational inequality: find $u^* = (X^*, Y^*, Z^*) \in \Omega = S_+^n \times S_+^n \times \mathcal{R}^{n\times n}$ such that

$$\begin{cases} \langle X - X^*, (X^* - C) - Z^*\rangle \ge 0, \\ \langle Y - Y^*, (Y^* - C) + Z^*\rangle \ge 0, \quad \forall u = (X,Y,Z) \in \Omega, \\ X^* - Y^* = 0. \end{cases} \tag{5.3}$$

Problem (5.3) is a special case of (1.3)-(1.4) with matrix variables, where $A = I_{n\times n}$, $B = -I_{n\times n}$, $b = 0$, $f(X) = X - C$, $g(Y) = Y - C$ and $\mathcal{W} = S_+^n \times S_+^n \times \mathcal{R}^{n\times n}$.

For simplification, we take $R = rI_{n\times n}$, $S = sI_{n\times n}$ and $H = I_{n\times n}$ where $r > 0$ and $s > 0$ are scalars. In all tests we take $\mu = 0.5$, $C = \text{rand}(n)$ and $(X^0, Y^0, Z^0) = (I_{n\times n}, I_{n\times n}, 0_{n\times n})$ as the initial point in the test. The iteration is stopped as soon as

$$\max\{\|X^k - \tilde{X}^k\|, \|Y^k - \tilde{Y}^k\|, \|Z^k - \tilde{Z}^k\|\} \le 10^{-6}.$$

All codes were written in Matlab, we compare the proposed method with that in [22]. The iteration numbers, denoted by k, and the computational time for problem (5.1) with different dimensions are given in Tables 1 and 2.

Tables 1 and 2 show that the proposed method is more flexible and efficient. Moreover, it demonstrates computationally that the new method is more effective than the method presented in [22] in the sense that the new method needs fewer iterations and less computational time.

6 Conclusions

In this paper, we propose a new logarithmic-quadratic proximal alternating direction method (LQP-ADM) for solving structured variational inequalities. Each iteration of the new LQP-ADM includes a prediction step, where a prediction point is obtained by using the idea of He [16], and a correction step, where the new iterate is generated by a convex combination of the previous iterate and the one generated by a projection-type method along a new descent direction. Global convergence of the proposed method is proved under mild assumptions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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