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COMPUTATION OF NORMAL DEPTH IN OPEN CHANNELS

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ABSTRACT

Fixed-point iteration is an efficient technique for manual and machine calculation of normal depth. The method is applied to the solution of Manning's equation for two basic uniform flow problems in trapezoidal and circular channels. Computing normal depth when discharge and bed width, or diameter, are given, or computing bed width, or channel diameter, required to sustain a given discharge and normal depth. The iteration function has a standard form and uses only two variables, the area and hydraulic radius, for various channel geometries. It has been tested over a wide range of the variables. Convergence to the correct normal depth occurred in an average of three to four iterations regardless of the starting value used.

INTRODUCTION

Uniform flow is defined as the condition in which flow variables do not change with distance, and normal depth is the depth of uniform flow in open channels. Its occurrence in channels of uniform cross section may be relatively infrequent, however, it is a condition of such basic importance that it must be considered in all channel design problems [4].

The Manning equation is widely used for open channel uniform flow calculations. It cannot be solved explicitly for normal depth, except for triangular sections, and the determination of normal depth relied on graphical methods, tables, or trial and error solutions [1,4]. The use of digital computers has brought a shift towards the numerical solution of Manning's equation in which the Newton-Raphson method is invariably employed (e.g. Chow et al. [2] and McLatchy [6]). While effective, the Newton-Raphson method has some drawbacks. The iteration function and its derivative, which are different for each type of channel cross section, are required. The calculations may become quite lengthy and are generally not feasible on non-programmable calculators.

Fixed-point iteration, also called successive substitution, is an effective technique for solving nonlinear and transcendental equations. An initial approximation to the real root of an equation is used to calculate an improved approximation and the process is continued. The generated values will converge to the root, within tolerances set by the user, provided that the proper form of iteration function, satisfying the necessary and/or sufficient conditions for convergence, is found.

In this paper, an iteration function for solving Manning's equation is derived. The function is of standard form involving the area and hydraulic radius only and is applicable to trapezoidal and circular open channels. The function is shown to possess the necessary and/or sufficient conditions for convergence. It has been tested over a wide range of flow variables to demonstrate its accuracy and quick convergence.

Manning's equation is also solved using the Newton-Raphson method to compare convergence properties, accuracy, and the effect of the starting value on the speed of convergence.

FIXED - POINT ITERATION

Fixed-point iteration is treated in standard books on numerical analysis (eg. Conte and de Boor [3], Stark [7], and Hildebrand [5]). To obtain the real root of an equation $f(x)=0$, it is written in the equivalent form $x=g(x)$ so that the solution of the second form is also a solution of the first. A recurrence relationship, $x_{i+1}=g(x_i)$, is then used to generate successive values of x which will converge to the root ξ of the equation. The general requirements under which the recurrence relationship is useful for the solution of the problem are :

1. For a given starting value x_0, it is possible to calculate successive values $x_1, x_2, \ldots$
2. The sequence x_1, x_2, ... converges to some point ξ.
3. The limit ξ is a fixed point of g(x), that is $\xi=g(\xi)$.

Conditions for Convergence

The conditions necessary to satisfy the above requirements, i.e. to ensure convergence to the root of f(x) are stated by Conte and de Boor [3] as follows:

1. That there is an interval $I=[a,b]$ such that for all $x \in I$, $g(x)$ is defined and $g(x) \in I$. This may be restated that $a \le g(x) \le b$ for all x such that $a \le x \le b$ [5].

2. The iteration function is differentiable on $I=[a,b]$. Further, there exists a nonnegative constant $K<1$ such that for all $x \in I, |g'(x)| \le K$ over the entire region. This condition implies that $g'(x)$ is continuous on $I=[a,b]$.

An iteration function satisfying the above two conditions has exactly one fixed point ξ in I, and starting with any point x_0 in I, the sequence $x_1, x_2, \ldots$ generated by fixed-point iteration on $x_{i+1}=g(x_i)$ converges to ξ [3].

Trapezoidal Channels

Consider uniform flow in a channel with a trapezoidal cross section whose bed width is B, depth of flow is y, and side slopes are 1 vertical on z horizontal (a rectangular section is a special case for which z=0). The cross sectional area is $A=y(B+zy)$ and the wetted perimeter is $P=B+2y\sqrt{(1+z^2)}$. The Manning equation is

$$\frac{Qn}{\sqrt{S}} = AR^{2/3} = A^{5/3}/P^{2/3} \tag{1}$$

in which Q is the discharge, n is the roughness coefficient, S is the longitudinal slope of the channel bottom and R is the hydraulic radius which is equal to A/P. The product $AR^{2/3}$ is called the section factor for uniform flow [1]. Substituting for A and P, Equation (1) becomes

$$\frac{Qn}{\sqrt{S}} = \frac{[y(B+zy)]^{5/3}}{\left[B+2y\sqrt{1+z^2}\right]^{2/3}} \tag{2}$$

Factoring out $y^{8/3}$ from the right-hand-side and solving for y, one form of the iteration function is

$$y=\left[\frac{Qn}{\sqrt{S}}\frac{\left(\frac{B}{y}+2\sqrt{1+z^2}\right)^{\frac{2}{3}}}{\left(\frac{B}{y}+z\right)^{\frac{5}{3}}}\right]^{\frac{3}{8}}=\left[\frac{\frac{Qn}{\sqrt{S}}}{AR^{\frac{2}{3}}}\right]^{\frac{3}{8}}y=g(y) \qquad (3)$$

A second form of the iteration function may be obtained by factoring out $y^{5/3}$, thus

$$y=\left[\frac{Qn}{\sqrt{S}}\frac{\left(B+2y\sqrt{1+z^2}\right)^{\frac{2}{3}}}{\left(B+zy\right)^{\frac{5}{3}}}\right]^{\frac{3}{5}}=\left[\frac{\frac{Qn}{\sqrt{S}}}{AR^{\frac{2}{3}}}\right]^{\frac{3}{5}}y=g(y) \qquad (4)$$

Convergence of the iteration functions

In order to ensure the existence of exactly one fixed point of each iteration function g(y) and the convergence of the iteration to this root, it must be shown that Equations (3) and (4) satisfy the two conditions stated in the section on convergence. The first condition is that in the interval I=[a,b], $a\le g(y)\le b$ for all y such that $a\le y\le b$. Considering that the normal depth, y, may have any positive value between zero and infinity, the interval I is the open interval I=[a=0, b=∞]. To show that g(y) also belongs to this open interval one proceeds in the following manner. Both A and R are increasing functions of y [1,8]. At the fixed point $y=\xi$, $Qn/\sqrt{S}=A(\xi)R(\xi)^{2/3}$. Therefore for any value $y=a<\xi$, $A(a)R(a)^{2/3}<A(\xi)R(\xi)^{2/3}$ and the value of the multiplier $N= A(\xi)R(\xi)^{2/3}/A(a)R(a)^{2/3}$ in Equations (3) and (4) will be a finite positive number greater than 1. Hence $g(a<\xi)= N^{3/8 \text{ or } 3/5}\, a>a$. At $a=\xi$ the multiplier is unity and g(a)=a. Thus, for any $y=a\le\xi$, $g(a)\ge a$. Similarly, for any $y=b\ge\xi$, $A(\xi)R(\xi)^{2/3}\le A(b)R(b)^{2/3}$ and the corresponding multiplier in the iteration functions will be a nonzero positive number equal to or smaller than 1, hence $g(b)\le b$. It is thus established that g(y) belongs to the open interval I=[a,b].

The derivative of the iteration function g(y) in Equation(3) with respect to y is:

$$g'(y)=\left(\frac{Qn}{\sqrt{S}}\right)^{\frac{3}{8}}\frac{y^{\frac{3}{8}}\left(B+2y\sqrt{1+z^2}\right)^{\frac{1}{4}}}{(B+zy)^{\frac{5}{8}}}\left[\frac{3}{8y}+\frac{1}{4}\frac{2\sqrt{1+z^2}}{B+2y\sqrt{1+z^2}}-\frac{5}{8}\frac{z}{B+zy}\right] \quad (5)$$

$Qn/\sqrt{S}$ may take any positive finite value and is always equal to $A(\xi)R(\xi)^{2/3}$, where ξ is a particular fixed point. It is the value of $g'(y)$ near a fixed point that determines whether or not the iteration converges [3]. Since we are interested in the value of $g'(y)$ near all fixed points in the interval I=[a,b], we let $\xi = y$ then

$$\frac{Qn}{\sqrt{S}}=A(y)R(y)^{\frac{2}{3}}=\frac{[y(B+zy)]^{\frac{5}{8}}}{\left[B+2y\sqrt{1+z^2}\right]^{\frac{1}{4}}} \quad (6)$$

Substituting this expression in Equation (5) one obtains,

$$g'(y)=\frac{3}{8}+\frac{1}{2}\frac{y\sqrt{1+z^2}}{B+2y\sqrt{1+z^2}}-\frac{5}{8}\frac{zy}{B+zy} \quad (7)$$

Similarly the derivative of the second iteration function, Equation (4), is found to be

$$g'(y)=\frac{4}{5}\frac{y\sqrt{1+z^2}}{B+2y\sqrt{1+z^2}}-\frac{zy}{B+zy} \quad (8)$$

The limiting values of the derivative in Equation (7) are 0.375 and zero for y=0 and y=∞, respectively. The corresponding values of Equation (8) are -0.6 and zero. For the rectangular cross section, the values for Equation (7) are 0.375 and 0.625 and for Equation (8) zero and 0.40 . This establishes that both iteration functions also satisfy the second condition, i.e. $|g'(y)| \leq K<1$, and thus ensures the existence of exactly one fixed point in the open interval I=[0,∞] and the convergence of both functions to this fixed point regardless of the starting value of the iteration.

The computational algorithm

The choice of which of the two iteration functions to use depends on the convergence rate of the two functions in the region in which a solution is sought. This rate is dependent on the value of the derivative in the region. If $g'(y)$ is near zero in the entire region then quick convergence is assured [7].

The variation of the absolute value of the derivatives of the two functions is shown in Fig. (1) as a function of B/zy for side slopes z of 0.2 and 4. The derivative of Eq. (3), whose exponent is 0.375, increases from zero at B/zy=0 to 0.375 in the limit as $B/zy \to \infty$. The derivative of Eq. (4) whose exponent is 0.6, on the other hand, decreases from 0.6 at B/zy =0 to zero as $B/zy \to \infty$. The two curves intersect in the region near B/zy = 1.0. Evidently, faster convergence is obtained with Eq. (3) when B/zy is less than one and with Eq. (4) when B/zy is greater than one.

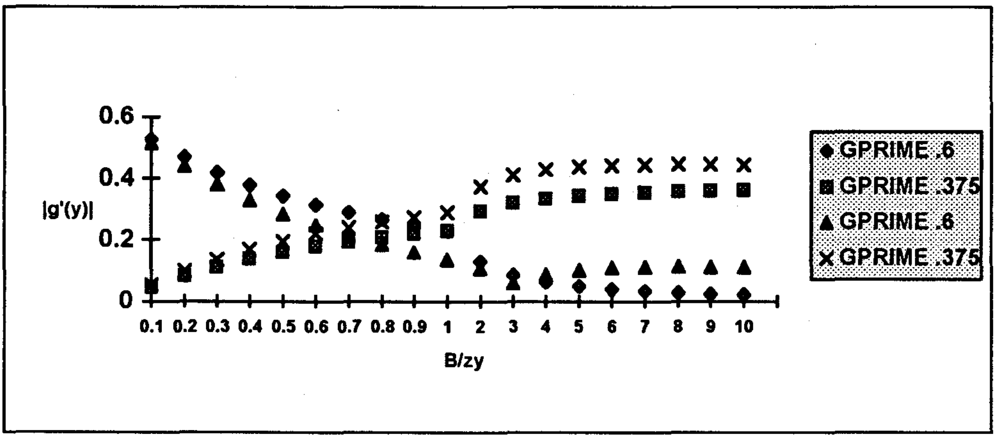

Fig. 1. Variation of the absolute value of the derivative with B/zy

A computational algorithm that takes advantage of the convergence properties of the two functions is shown in Fig. (2). The iteration is started with the seed value $y_0=B/z$, or B/zy=1, using either of the two functions. The new value y_1 is then compared with the seed value y_0. If $y_1>y_0$ then $B/zy_1<1$ and Eq. (3) is selected for the remainder of the iterations, otherwise, the algorithm switches to Eq. (4).

```
280 Enter Input Values(Q, n, S, B, z)
290 Q= Q*n/√s : ET= 0.375: Y1=B/z : P=0
300 REM START ITERATION
310 P=P+1
320 PW=B+2*Y1*SQR(1+Z^2): A1=Y1*(B+Z=Y1)
330 H1=(A1/PW)^.667: Q1=A1*H1
340 Y2=(QN/Q1)^ET)*Y1
350 IF P>1 THEN GOTO 370
360 IF Y2>Y1 THEN ET=.375 ELSE ET=.6
370 IF ABS(Y2-Y1)> 0.01 THEN Y1=Y2: GOTO 310
380 PRINT RESULTS
```

Fig. 2. Computational algorithm for trapezoidal channels

Circular Open Channels

The geometry of uniform open channel flow in circular sections is expressed in terms of the diameter d and the angle θ subtended by the wetted perimeter, Fig. (3). The depth is $y=d(1-\cos(\theta/2))/2$, the flow area $A=d^2(\theta-\sin\theta)/8$, the wetted perimeter $d\cdot\theta/2$, and Manning's equation becomes

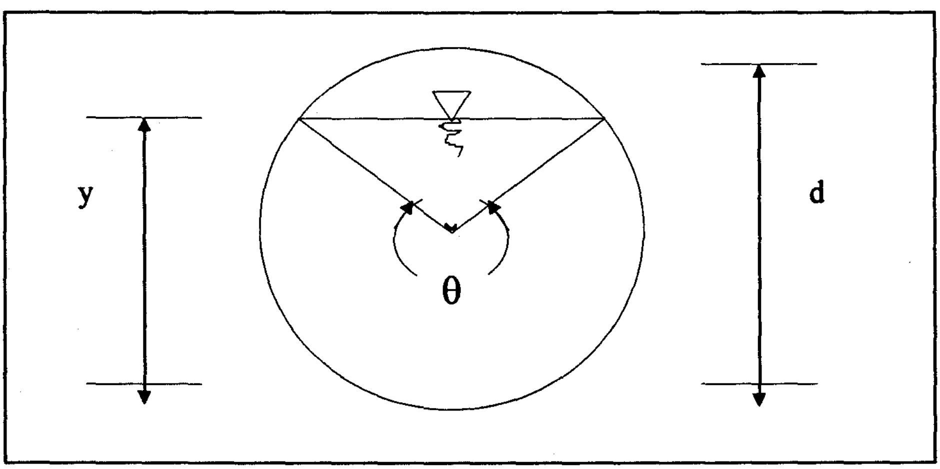

Fig. 3. Definition sketch of circular channels

$$\frac{Qn}{\sqrt{S}}=\frac{2^{\frac{2}{3}}d^{\frac{8}{3}}}{32}\frac{(\theta-\sin\theta)^{\frac{5}{3}}}{\theta^{\frac{2}{3}}} \tag{9}$$

The iteration function is obtained by factoring out $\theta^{8/3}$ from the right hand side and solving,

$$\theta=[\frac{Qn}{\sqrt{S}}\frac{32}{2^{\frac{2}{3}}d^{\frac{8}{3}}}\frac{\theta^{\frac{2}{3}}}{(\theta-\sin\theta)^{\frac{5}{3}}}]^{\frac{3}{8}}\theta=[\frac{\frac{Qn}{\sqrt{S}}}{A(\theta)R(\theta)^{\frac{2}{3}}}]^{\frac{3}{8}}\theta=g(\theta) \tag{10}$$

Convergence of the iteration function

The choice of interval in which to seek a solution for normal depth in circular channels is governed by the restriction on the values that the angle θ may take (zero to 2π) and by the specific nature of the variation of the section factor, $AR^{2/3}$, with θ. Recalling that $AR^{2/3}$ has a maximum at $\theta=1.68\pi$ (or $y=0.938d$), this fixes the upper bound, b, of the interval I at $b=1.68\pi$, since the section factor is an increasing function only up to $\theta=1.68\pi$ where the discharge is a maximum. In practical terms this depth may be considered the full depth in the conduit [1]. The case of $\theta=0$ is of no practical interest and the interval becomes the open interval $I=[a=0, b=1.68\pi]$. Using similar arguments as those for the trapezoidal case, it is easily shown that for any $\theta=a\leq\xi$, $g(a)\geq a$, and for any $\theta=b\geq\xi$, $g(b)$ is positive and $g(b)\leq b$. This, however, does not necessarily lead to satisfaction of the first condition since for some high values of the fixed point ξ, $g(a)$ will be greater than the upper bound b, which necessitates placing a restriction on the value that the lower bound a may take.

Numerical experimentation has shown that setting the lower bound $a=0.5\pi$ ensures that in the closed interval $I=[a=0.5\pi , b=1.68\pi]$, $a\leq g(\theta)\leq b$ for all θ such that $a\leq\theta\leq b$. It must be emphasized that this restriction on the lower bound of the interval I does not place any restriction on how small a value the root ξ may have. It is merely a restriction on the initial value of θ used to start the iteration, to ensure that the sequence does not lead to values of θ greater than 1.68π, which may cause $g(b)$ to be greater than b and lead to θ values greater than 2π in subsequent iterations.

The derivative of the iteration function, Equation (10), can be shown to be

$$g'(\theta) = \frac{5}{8}\left[2 - \frac{\theta(1-\cos\theta)}{\theta - \sin\theta}\right] \tag{11}$$

The value of $g'(\theta)$ is undefined when $\theta=0$. Taking the limit as $\theta \to 0$, $g'(0)=-0.625$. Numerical evaluation of Equation (11) indicates that $g'(\theta)$ increases with increasing θ, reaching a value of 0.9999 when $\theta=1.68\pi$. The second condition for convergence, that $|g'(\theta)|<1$ in the interval $I=[0, 1.68\pi]$, is thus satisfied ensuring the existence of exactly one fixed point in this interval and the convergence of the iteration function to it. As a consequence of this condition the iteration will always converge to the lower value of depth in the region where it is possible to have two different depths for the same discharge, one above and one below the value of 0.938d.

Bed Width and Channel Diameter

Another type of uniform flow problem in trapezoidal and circular channels is that of computing bed width, or channel diameter, required to sustain a given discharge and normal depth. Without going into details, the iteration function for trapezoidal channels is

$$B = \left[\frac{Qn}{\sqrt{S}} / AR^{\frac{2}{3}}\right] B \tag{12}$$

with the derivative, $g'(B)$, equal to zero as $zy/B \to 0$ and to 1.0 as $zy/B \to \infty$. For circular channels the iteration function is cast in terms of y/d and takes the form

$$\frac{y}{d} = \left[\frac{AR^{\frac{2}{3}}}{Qn/\sqrt{S}}\right]^{\frac{3}{8}} \left(\frac{y}{d}\right) \tag{13}$$

with derivative, $g'(y/d)$, equal to 0.81 at $\theta=0.1\pi$ and equal to -0.94 at $\theta=1.9\pi$.

THE NEWTON – RAPHSON METHOD

In the Newton – Raphson iteration method the root x of an equation of the form f(x)=0 is found by expanding f(x) in a Taylor series about a neighboring point x_0 and retaining only the first term of the expansion thus,

$$f(x) = f(x_0) + (x-x_0)\, f'(x_0) = 0$$

Solving for x,

$$x = x_0 - \frac{f(x_0)}{f'(x_0)} \qquad , f'(x_0) \neq 0$$

or, more generally, in a form suitable for iteration

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad , f'(x_i) \neq 0 \qquad (14)$$

Convergence of the Iteration Function

The iteration function Eq.(14), is seen to be a special case of x=g(x) in which g(x)=x-f(x)/f′(x), thus $g'(x)=f(x)\ f''(x)/[f'(x)]^2$. Now, if $f'(x)\neq 0$ and $f''(x)$ is finite, there follows $g'(\xi)=0$ so that the convergence factor (the ratio of the error in x_{k+1} to the error in x_k) tends to zero when and if $x_k \rightarrow \xi$[5]. If the curve representing y=f(x) possesses turning points or inflections in the interval between the initial estimate x_0 and the true root ξ, or between x_0 and x_1 , the iteration may not converge to ξ. However, if f′(x) and f″(x) do not change sign in the interval (x_0, ξ), and if $f(x_0)$ and $f''(x_0)$ have the same sign, so that the iteration is initiated at a point at which the curve representing y=f(x) is concave away from the x-axis, it can be seen that successive iterates must tend to $x=\xi$ and that they all lie between x_0 and ξ [5].

Trapezoidal Channels

In applying the Newton-Raphson method, Manning's equation is written in the form

$$f(y) = AR^{2/3} - \frac{Qn}{\sqrt{S}} \qquad (15)$$

Taking the derivative of f(y),

$$f'(y) = \frac{2}{3}\frac{A}{R^{1/3}}\frac{dR}{dy} + R^{2/3}\frac{dA}{dy}$$

$$= AR^{2/3}\left[\frac{2}{3R}\frac{dR}{dy} + \frac{1}{A}\frac{dA}{dy}\right]$$

$$f'(y) = AR^{2/3}\left[\frac{(B+2zy)(5B+6y\sqrt{1+z^2}) + 4zy^2\sqrt{1+z^2}}{3y(B+zy)(B+2y\sqrt{1+z^2})}\right] \tag{16}$$

Substituting in Eq.(14) and simplifying, the iteration function becomes

$$y_{i+1} = y_i - \frac{1 - \dfrac{Qn/\sqrt{S}}{\left(B+zy^2\right)^{5/3} / \left(B+2y\sqrt{1+z^2}\right)^{2/3}}}{\dfrac{(B+2zy)(5B+6y\sqrt{1+z^2}) + 4zy^2\sqrt{1+z^2}}{3y(B+zy)(B+zy\sqrt{1+z^2})}} \tag{17}$$

$$= y_i - \frac{1 - \dfrac{Qn/\sqrt{S}}{A_i^{5/3}/P_i^{2/3}}}{\dfrac{5(B+2zy_i)}{3A_i} - \dfrac{4\sqrt{1+z^2}}{3P_i}} s \tag{17.a}$$

Circular Open Channels

The Manning equation for circular open channels may be written in a normalized form

$$f(\theta) = \frac{(\theta - \sin\theta)^{5/3}}{\theta^{2/3}} - \frac{32}{2^{2/3}d^{8/3}}\frac{Qn}{\sqrt{S}} \tag{18}$$

with the derivative

$$f'(\theta)=\left[\frac{\theta-\sin\theta}{\theta}\right]^{5/3}\left[\frac{5}{3}\frac{\theta-\theta\cos\theta}{\theta-\sin\theta}-\frac{2}{3}\right] \tag{19}$$

The value of $f'(\theta)$ is undefined at $\theta=0$. Taking the limit as $\theta\to 0$, $f'(0)=0$ which violates the condition that $f'(\theta)\neq 0$. This , however, is of no practical consequence since it represents a condition of no flow and can be excluded from consideration.

Substituting Eqs. (18) and (19) into Eq. (14) and simplifying, the resulting iteration function is:

$$\theta_{i+1}=\theta_i-\frac{\theta_i-\dfrac{\dfrac{32}{2^{2/3}d^{8/3}}\dfrac{Qn}{\sqrt{S}}}{(1-\dfrac{\sin\theta_i}{\theta_i})^{5/3}}}{\dfrac{5}{3}\dfrac{1-\cos\theta_i}{1-\dfrac{\sin\theta_i}{\theta_i}}-\dfrac{2}{3}} \tag{20}$$

TESTING OF THE ITERATION FUNCTIONS

The iteration functions were systematically tested over a wide range of channel cross section and normal depth values. The main objectives were to determine the accuracy of the computed values, the speed of convergence and the effect of the starting values on speed of convergence.

Target values of the section factor for uniform flow, $Qn/\sqrt{S}=AR^{2/3}$, were generated for a given channel geometry and normal depth. The computational algorithm was then used to calculate the normal depth corresponding to this target value.

Trapezoidal Channels

The basic target generating algorithm consisted of three loops. The outer loop generated values of B/zy from 0.2 to 1.0 in increments of 0.2, from 2 to 10 in increments of 2, and from 20 to 100 in increments of 20. The middle loop generated values of the side slope, z,from 0.5 to 5 in increments of 0.5. The inner loop generated ten values of normal depth, which shall be termed the true depth, between 0.1 and 1.0. The same procedure was repeated using depths between 1.0 and 10. The total number of target values was therefore 3000. The iteration started with the seed value y=B/z, and was continued until the absolute difference between two successive iterations was equal to or smaller than 0.01.

Fixed point iteration

Both iteration functions, Equation (3) and Equation (4) whose exponents are 0.375 and 0.6, respectively, were tested, since the algorithm automatically selects the iteration function depending on the value of B/zy. The results of these tests are summarized in Table (1) which shows the average number of iterations and the root-mean-square (RMS) difference between the true depth and the computed depth. The algorithm converged to the correct root in an average number of iterations varying between 4.5 and 3.0 depending on the value of B/zy. In general the iterations averaged 3.8 for $1 \le y \le 10$ and 3.1 for $0.1 \le y \le 1.0$..

Table 1. Results of Testing the Fixed – Point Iteration Function for Trapezoidal Channels

B/zy	$0.1 \le y \le 1.0$		$1 \le y \le 10$	
	Avg. No. of Iterations	RMS Error	Avg. No. of Iterations	RMS Error
0.2	3.6	3.8×10^{-4}	4.5	3.9×10^{-4}
0.4	3.6	7.0×10^{-4}	4.7	7.4×10^{-4}
0.6	3.3	1.3×10^{-4}	4.7	1.1×10^{-3}
0.8	2.8	1.6×10^{-4}	4.4	1.4×10^{-3}
2	3.4	4.8×10^{-4}	4.4	4.7×10^{-4}
4	3.1	2.8×10^{-4}	3.9	2.3×10^{-4}
6	3.0	1.5×10^{-4}	3.8	1.1×10^{-4}
8	3.0	8.5×10^{-5}	3.8	7.1×10^{-5}
10	3.0	5.5×10^{-5}	3.7	7.0×10^{-5}
20	3.0	2.7×10^{-5}	3.4	5.9×10^{-5}
40	3.0	1.8×10^{-5}	3.0	2.6×10^{-5}
60	3.0	1.1×10^{-5}	3.0	1.2×10^{-5}
80	3.0	1.1×10^{-5}	3.0	7.1×10^{-6}
100	3.0	8.2×10^{-6}	3.0	4.6×10^{-6}

Both iteration functions possess robustness in that they are insensitive to the starting value of iteration. Equation (3) (exponent 0.375) converged to the correct root in an average of 6.5 and 7.0 iterations when the starting value is 5 and 0.2 times the true depth, respectively. The corresponding values for Equation (4) were 6.6 and 6.0 iterations. Starting the iteration with a value 500 times the depth increased the average number of iterations by 0.5 in both iteration functions.

Although the tolerance limit was set at 0.01m, it was observed that the computed depth was generally within ±0.001m of the true depth or better. Tests with very small and very large B/zy, z and y values indicated no limitations of either iteration function as to convergence or accuracy regardless of the starting value, only the speed of convergence was affected.

Problems of best hydraulic section are easily solved by fixed-point iteration. The relationship between bed width and depth, $B=2y(\sqrt{(1+z^2)}-z)$, is simply inserted in the algorithm so that the iteration function is entered with the value of depth and its corresponding bed width. For z values equal to or greater than 0.9, the value of B/zy is smaller than 1.0 for best hydraulic sections and Equation (3) converges quicker in this range. For the general problem however, the same computational algorithm may be used.

The iteration function for computing bed width was similarly tested and found effective. Starting with a B value 5 times the correct bed width, the iteration function converged in an average of 12.8 and 5.2 iterations when B/zy was smaller than 1.0 or greater than 2.0, respectively. The average number of iterations remained practically the same even when the starting value was 50 times the correct bed width.

Newton-Raphson iteration

Four values were used to start the iteration. The first two, considered an educated guess of the root, were the upper and lower limits of normal depths in the target generating algorithm. A value 10 times larger than the upper limit and another 10 times smaller than the lower limit were used as extreme guesses. The results of testing are shown in Table (2) as the variation of the number of iterations with B/zy for each starting value. Each number in the table is the average number of iterations in 100 computations of normal depths for the particular B/zy value.

In general, the number of iterations decreases with increasing B/zy and levels off at about B/zy=4. More important is the apparent sensitivity of the speed of convergence to the starting value. Starting from the lower extreme value required an average (in 3000 computations) of over 1000 iterations for convergence, while

Table 2. Results of Testing the Newton-Raphson Iteration for Trapezoidal Channels

B/zy	$0.1 \leq y \leq 1.0$ Starting Value				$1.0 \leq y \leq 10$ Starting Value			
	0.01	0.1	1.0	10	0.1	1.0	10	100
0.2	3308	58.1	7.6	72.7	3309	58.8	8.2	73.5
0.4	2048	41.4	7.3	68.4	2048	42.0	7.9	69.2
0.6	1629	34.7	7.0	65.3	1630	35.4	7.7	66.1
0.8	1424	31.3	6.9	63.0	1424	32.0	7.6	63.9
1.0	1303	29.1	6.7	61.4	1303	29.8	7.4	62.1
2	1075	24.9	6.5	56.6	1076	25.6	7.2	57.4
4	979	23.2	6.4	53.7	979	23.9	6.9	54.4
6	953	22.9	6.3	52.7	954	23.6	6.9	53.6
8	942	22.8	6.3	52.3	943	23.4	6.9	53.1
10	936	22.6	6.3	52.1	937	23.4	6.9	52.9
20	927	22.6	6.3	51.9	928	23.2	6.9	52.6
40	925	22.6	6.3	51.9	925	23.2	6.9	52.4
60	924	22.5	6.3	51.8	924	23.2	6.9	52.4
80	924	22.5	6.3	51.8	924	23.3	6.9	52.4
100	924	22.5	6.3	51.8	924	23.3	6.9	52.4
Overall mean	1281	28.2	6.6	57.2	1281	29	7.2	57.9
Maximum RMS error, m	3.8×10^{-4}	1.8×10^{-4}	4.8×10^{-4}	3.5×10^{-4}	2.3×10^{-5}	9×10^{-6}	2.2×10^{-5}	2.3×10^{-5}

starting from the upper extreme value required an average of about 60 iterations. It is also notable that approaching the root from above is more efficient than approaching it from below. Thus starting at the upper limit of normal depths required an average of 6.9 iterations as compared to 28.6 iterations for starting at the lower limit.

The accuracy of the computed normal depths was very good. Although the tolerance limit was set at 0.01m, it was observed that the maximum RMS difference between the computed and true depths did not exceed 0.0005m in all computations.

Circular Channels

The target generating algorithm generated normalized section factors for uniform flow, $Qn/d^{8/3}\sqrt{S}$, for 20 values of θ between 0.1π and 2π, termed the true θ. The iteration was started with a given value of θ and was continued until the absolute difference of θ values in two successive iterations was 0.010 rad or less.

Fixed point iteration

The average number of iterations, over the whole range of true θ values, was 6.6 and 8.0 iterations when the starting value was $\pi/2$ and π, respectively. The average number of iterations was approximately the same when the value of true θ was equal to or smaller than 1.68π. Values of true θ between 1.68 and 2π, are in the range where it is possible to have two different depths (θ's) for the same discharge [1]. In this range the iteration function converged to the lower value of θ (1.43π-1.68π) as indicated earlier. Starting with $\theta=1.5\pi$ resulted in faster convergence with an average of 6.4 iterations. The accuracy of the iteration function was such that the ratio of the computed normal depth to the true normal depth was between 0.99 and 1.01. Reducing the error tolerance to 0.005 radians increased the average number of iterations by about 1.5 with the accuracy improving only in the range of very low y/d ratios which are not of practical importance.

The iteration function for computing channel diameter required to sustain a given discharge and normal depth has been tested over the range of $\theta=0.1\pi$ to 1.8π. For θ values between $0.5\,\pi$ and 1.8π corresponding to $0.146\leq y/d\leq 0.975$, the average number of iterations is 12.6 and 11.3 when the starting θ value is $\pi/2$ and π, respectively. The ratio of computed to true depth varied between 1.0 and 1.01. For θ values smaller than 0.4π, corresponding to $y/d\leq 0.097$, the average number of iterations is 15 and 17.5 for starting values of $\pi/2$ and π, respectively.

Newton Raphson iteration

The iteration function is subject to the same restrictions applicable to fixed-point iteration regarding the interval [0 to 1.68π] in which to seek a solution. This required restricting the starting θ to values between 0.8π and 1.3π, since starting iteration with values outside of this range produced θ values greater than 1.68π and prevented convergence by leading to θ values greater than 2π. It is emphasized again that this restriction on the lower and upper bounds of the interval I does not place any restriction on the value the root ξmay have. It is merely a restriction on the initial value of θ used to start the iteration, to ensure that the sequence does not lead to values of θ greater than 1.68π. With this method the iteration will also converge to the lower value of depth in the region where it is possible to have two depths for the same discharge.

The average number of iterations, over the whole range of true θ values, varied between 4.1 and 4.4 iterations when the starting value was 1.3π and π, respectively. The accuracy of the iteration function was such that the ratio of the computed normal depth to the true normal depth was between 0.999 and 1.00. Reducing the error tolerance to 0.005 radians did not significantly change the average number of iterations nor improve the accuracy of the computed y/d ratios.

COMPUTATIONAL ECONOMY

The computational economy of fixed-point iteration is demonstrated by comparison with the Newton-Raphson method. The cost of computation is directly related to the total number of computer operations involved and comparison may be made on this basis. Examination of the present and two other routines for solving Manning's equation [2,6] shows that when counting exponentiation as two operations and excluding the constants, the minimum number of operations required for each iteration is 25 and 22 for trapezoidal and circular sections, respectively. The corresponding numbers for fixed-point iteration are 13 and 7.

Thus, using the best guess for the starting value, the Newton-Raphson method requires an average of 172.5 operations compared to 45.1 operations for fixed-point iteration. The solution for circular sections requires, on average, 53.3 operations with the Newton-Raphson method compared to 44.8 with fixed-point iteration.

CONCLUSIONS

Fixed-point iteration is applied to the solution of Manning's equation of uniform open channel flow. The iteration function has a standard form and uses only two variables, the area and hydraulic radius, for various channel geometries. The method is tested over a wide range of discharge and cross sectional parameters and is found to have excellent convergence properties. The iteration function is simple, insensitive to the starting value, and can also be used on hand calculators. Comparison with the Newton-Raphson method shows that, for trapezoidal sections, fixed-point iteration requires at most one-fourth of the computational effort while for circular sections it requires about 80% of the computational effort of the Newton-Raphson iteration.

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