ON A MIXED BOUNDARY VALUE
PROBLEM FOR ELLIPTIC EQUATIONS

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ABSTRACT

In this work, a general form of linear and quasilinear elliptic equations is considered. In the presence of boundary singularities, some estimations for the error, according to the net method, are obtained.

INTRODUCTION

In different fields of mechanics and physics, it is necessary to solve mixed problems for elliptic equations in its general form [4]. At first, the net method has been applied and the error in the approximate solution is estimated as O (h) (h is the distance between two vertical or horizontal points), assuming the boundedness of the derivatives up to the third order of the exact solution. However, such assumption is not valid in some practical significant cases; for example, in domains with piecewise smooth boundaries, the behaviour of the solution at the corner points depends on the angles at these points [7], [8], [9], [10], [11], [12]. In [3] and [6], singular boundaries are considered for studying the net method for Laplace and Poisson's equations.

Let G be a simply bounded connected domain with piecewise analytic boundary

$$\Gamma := \Gamma^1 \cup \Gamma^2, \text{ where } \Gamma^1 = \bigcup_{i=1}^{m1} \Gamma_i^1 \text{ and } \Gamma^2 = \bigcup_{i=1}^{m2} \Gamma_i^2. \text{ Let } Z_1, z_2, .., z_m,$$

$m = m_1 + m_2$ are the corner points of G and $\pi\alpha_j. > 0$ is the interior angle at z_j. We shall call the arcs Γ_i^1 and Γ_i^2 Dirichlet and Neumann arcs, respectively. The corner z_j is called a Dirichlet corner (a mixed corner) if it lies on both neighbouring arcs of Dirichlet type (different type).

The case of linear elliptic equations:

Consider the following mixed boundary problem:

$$Lu \equiv \triangle u + a(x,y) u_x + b(x,y) u_y + c(x,y) u = f(x,y), \quad (x,y) \in G, \qquad (1)$$

$$u(x,y) = \emptyset(x,y), \quad (x,y) \in \Gamma^1, \qquad (2)$$

$$\ell u = \frac{\partial}{\partial n} u(x,y) + g(x,y) u = \emptyset(x,y), \quad (x,y) \in \Gamma^2 \qquad , (3)$$

where $\triangle$ is the Laplacian and $\emptyset(x,y)$ is a given C^3 function on every arc; $\frac{\partial}{\partial n}$ is the external normal derivative; a, b, c, g and f are given continuous functions; $|a| \leq M$

$|b| \leq M$, $M < \infty$ is constant, $c \leq 0$, $g \geq 0$.

Let $z_1 = x_1 + iy_1$ be a corner point of the boundary Γ and denote $|z - z_1| = r$.

If z_1 is a Dirichlet corner, then for $0 \leq n \leq 3$, $0 \leq k \leq n$ it has been shown that [5]:

$$\frac{\partial n}{\partial x^k \, \partial y^{n-k}} u(x,y) = O(r^n) \qquad (4)$$

while if z_1 is a mixed corner, then for all $\lambda \leq \frac{1}{2\alpha_1}$

$$\frac{\partial^n}{\partial x^k \, \partial y^{n-k}} u(x,y) = O(r^{\lambda - n} + 1) \qquad (5)$$

The corner points of Γ will be the singular points of the solution $u(x,y)$ in $G \cup \Gamma$.

Construction of the difference problem for problem (1) - (3):

Let G_h denotes the set of interior mesh points, where equation (1) is to be approximated, and Γ_h^1, Γ_h^2 denote the two sets of the boundary mesh points for which the conditions (2), (3) are applied [1].

For all $(x_i, y_j) \in G_h$, we define

$$L_h U_{ij} = \frac{1}{h^2} [U_{i-1j} + U_{i+1j} + U_{i,j-1} + U_{i+1j} \quad 4 U_{ij}]$$

$$+ a_{ij} \frac{U_{ij} - U_{i-1,j}}{h} + a^+_{ij} \frac{U_{i+1,j} - U_{ij}}{h} + b^-_{ij} \frac{u_{ij} - U_{ij-1}}{h}$$

$$+ b^+_{ij} \frac{U_{ij+1} - U_{ij}}{h} + c_{ij} u_{ij},$$

Where $U_{i,j} = \ldots$ and similarly $a_{ij} \ldots$ are defined; and

$$a^-_{ij} = \frac{a_{ij} - |a_{ij}|}{2}, \quad a^+_{ij} = \frac{a_{i,j} + |a_{i,j}|}{2}$$

$$b^-_{ij} = \frac{b_{ij} - |b_{ij}|}{2}, \quad b^+_{ij} = \frac{b_{ij} + |b_{ij}|}{2}$$

If $P_1 \in \Gamma_h^1$ and $q_1 \in \Gamma^1$, where q_1 close to P_1, then we can write $U(P_1) = \varnothing (q_1)$; while if $p \in \Gamma_h^2$, hence we proceed as follows [1]:

$$\ell_h U(p) = \frac{U(p) - \mu_1 U(R^{(2)}) - \mu_2 U(R^{(1)})}{\mu h} + g(p) U(p) = \varnothing(q), \quad (6)$$

$\mu_1 \geqslant 0, \ \mu_2 \geqslant 0$

Where $\mu_1, \mu_2 \geqslant 0$ and $\mu > 0$ are certain numbers such that $\mu_1 + \mu_2 = 1$, $\mu \leqslant \sqrt{2}$; and $R^{(1)}, R^{(2)} \in G_h \cup \Gamma_h^1, q \in \Gamma^2$.

The associated difference problem for problem (1) — (3) is as follows:

Find a function U which is defined in $\overline{G}_h$ and satisfies the set of difference equations:

$$\left.\begin{array}{l} L_h U(p) = f(p), \ p \in G_h \\ U(p) = \varnothing(q), \ p \in \Gamma_h^1, \\ \quad \ell_h U_{(p)} = \varnothing(q), p \in \Gamma_h^2. \end{array}\right\} \qquad (7)$$

The uniqueness of the solution for problem (7) follows from the following lemma. give a number; better

Lemma* (The maximum principle) [4]
If the function $V \neq$ const. is defined in $\overline{G}_h$ and satisfies either of the two sets of inequalities:

$$L_h V \geqslant 0 \text{ in } G_h, \quad \ell_h V \leqslant 0 \text{ on } \Gamma_h^2$$
or
$$L_h V \leqslant 0 \text{ in } G_h, \quad \ell_h V \geqslant 0 \text{ on } \Gamma_h^2.$$

Then V takes the greatest positive or the least negative value only on Γ_h^1, respectively.

Estimating the error $\varepsilon_h = U - u$:

For any function $\psi(x, y) \in C^3(\overline{G})$ we have
$$L_h \psi_{(x,y)} = L\psi_{(x, y)} + h M_2(\psi) + M_3(\psi),$$

where $M_i(\psi)$ $(i = 2,3)$ denotes a linear combinations of the i th derivatives of ψ, and are taken in a square of side 2 h around the point (x, y). If $p \in \Gamma_h^2$, then from (6) it follows that

$$\ell_h \psi(p) = \ell_h \psi(q) + h M_2(\psi).$$

If z_1 is a Dirichlet corner, then in its neighbourhood we have

$L_h \quad \varepsilon_h = h\,O\,(r^{-3}) + h\,0\,(r^{-2}),$ $\hspace{4cm}$ (8)

and if z_1 is a mixed corner, then

$L_h \quad \varepsilon_h = h\,O\,(r^{\lambda-3} + 1) + h\,0\,(r^{\lambda-2} + 1),$ $\hspace{2cm}$ (9)

hence,

$L_h \quad \varepsilon_h = h\,O\,(r^{\lambda-2} + 1).$ $\hspace{4cm}$ (10)

Finally, if $p \in \Gamma^1{}_h$, and $q \in \Gamma^1$ is a point which corresponds to the point $\mathfrak{p}$ in the difference problem (7), then

$$\varepsilon_h\,(p) = U\,(p) - u\,(p) = \varnothing\,(q) - u\,(p) = u\,(q) - u\,(p)$$
$$= h\,M_1\,(u) = h\,O\,(r^{-1}). \hspace{3cm} (11)$$

Let $z_1 = x_1 + iy_1$ be a mixed corner of the boundary Γ, where $\pi\,\alpha_1$ is its angle. Introducing polar coordinates $z - z_1 = re^{i\theta}$ in the neighbourhood $r \leqslant r_1$ (r_1 is a small positive constant), we assume that the radius r corresponding to $\theta = 0$ is a tangent of Γ^2 at z_1 and the radius r corresponding to $\theta = \pi\,\alpha_1$ is a tangent of Γ^2.

Let $\alpha\,(r,\theta)$ be an angle such that:

$$t_g\,\alpha\,(r,\theta) = \frac{b\,(r\cos\theta,\ r\sin\theta)}{a(r\cos\theta,\ r\sin\theta)}$$

and satisfies in the neighbourhood $r \leqslant r_1$ the following inequality

$0 \leqslant \alpha\,(r,\theta) - 0 \leqslant \pi/2.$ $\hspace{5cm}$ (12)

Now, consider the function

$g_1\,(x,y) = r^{\lambda}(A\cos\lambda\,\theta - \sin\lambda\,\theta - 1),$ $\hspace{3cm}$ (13)

where $0 < \lambda < 1/2\,\alpha_1, \leqslant 1$ and A is a sufficiently large positive number.

The function $g_1\,(x,y)$ is positive and satisfies the inequality

$g_1\,(x,y) \geqslant dr^{\lambda}, d > 0.$ $\hspace{5cm}$ (14)

In virtue of (12) — (14), for the points $r \geqslant k_1 h$ ($k_1 > 0$), the following inequalities hold:

$L_h\,g_1(x,y) \leqslant -d_1\,\lambda^2 r^{\lambda-2} - d_2\,\lambda\,r^{\lambda-1}, (x,y) \in G_h,$ $\hspace{1.5cm}$ (15)

$\ell_h\,g_1(x,y) \geqslant \dfrac{\lambda}{2}\,r^{\lambda-1}, (x,y) \in \Gamma_h^2,$ $\hspace{2.5cm}$ (16)

where d_1, d_2 are positive constants independent upon h.

According to (9) — (11), (14) — (16) and for sufficiently large positive number K_1, the following inequalities hold:

$L_h\,(K_1\,h^{\lambda}\,g_1 \pm \varepsilon_h) < 0, (x,y) \in ?$ $\hspace{3cm}$ (17)

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$$\ell_h (K_1 h^\lambda g_1 \pm \epsilon_h) > 0, (x, y) \in ? \qquad (18)$$

The Maximum Principal then yields

$$(K_1 h^\lambda g_1 \pm \epsilon_h) > 0. \qquad (19)$$

Let z_1 be a Dirichlet corner and $0 < \lambda < 1/2 \alpha_1$, $\lambda \leq 1$, A and ξ are sufficiently large and small positive numbers, respectively. Further, let $\alpha(r,\theta)$ be an angle such that

$$t_g \alpha (r,0) = \frac{b (r \cos \theta, r \sin \theta)}{a (r \cos \theta, r \sin \theta)}$$

and in the neighbourhood $r \leq r_1$ satisfies the inequality:

$$0 < \theta + 2 \lambda (\theta + \xi) - \alpha (r, \theta) < \pi. \qquad (20)$$

Now, let us consider the positive function
$$g_1 (x, y) = r^{-2\lambda} [A \sin 2 \lambda (0 + \xi) - 1]. \qquad (21)$$

It can be shown that $g_1(x, y)$ satisfies the inequality
$$g_1(x, y) \geq dr^{-2\lambda}, d > 0. \qquad (22)$$

In view of (20) — (22), there exists a number $k_1 > 0$ such that for the points $r \quad k_1$ h, the following inequality holds

$$L_h g_1 (x, y) \leq -d_3 \lambda^2 r^{-2\lambda^{-2}} - d_4 \lambda r^{2\lambda -1}, (x, y) \quad G_h, \qquad (23)$$

Where d_3 and d_4 are positive constants and independent upon h.

Taking into consideration the relations (8), (11), (22) and (23), then for sufficiently large positive number K_1 we have

$$L_h (K_1 h^{2\lambda} g_1 \pm \epsilon_h) < 0, \qquad (24)$$

$$(K_1 h^{2\lambda} g_1 \pm \epsilon_h) > 0. \qquad (25)$$

If we continue the process of constructing majorant functions, we can suppose that the function $\gamma_O(x, y)$ is a continuous solution in $\overline{G}$ for the problem

$$L \gamma (x, y) = -1, (x, y) \in G,$$

$$\gamma (x, y) = 1, (x, y) \in \Gamma^1,$$

$$\ell \gamma (x, y) = 1, (x, y) \in \Gamma^2.$$

It is noticable that $\gamma_0 (x, y) > O$ in $\overline{G}$ (by lemma and the behaviour of its derivatives near the corners is the same as that of the derivatives of u (x, y).

Now, consider the function

$$S_h(x, y) = \sum_{i=1}^{m_1 - m_2} K_i \, h^{v_i} \, g_i(x, y) + K_0 \, h^s \, \gamma_0(x, y),$$

where $v_i = \lambda$ or 2λ depending on the type of the corner z_i, the number K_O is taken sufficiently large such that for every i and $| z - z_i | \geqslant k_i h$, the function

$S_h(x, y) - K_i h^{v_i} g_i(x, y) > 0$ in $\overline{G}_h$ and has discrete negative operators L_h and ℓ_h in G_h and on Γ^2_h, respectively.

Let $G_h{}^*$ denotes the set of points $p \in G_h$ which lie outside the neighbourhood $| z - z_i | < k_i h$ of the corners, Hence according to the construction of $S_h(x, y)$ we have

$$L_h \left[s_h(p) \pm \varepsilon_h(p) \right] \leqslant 0.$$

Therefore, $\{ S_h(p) \pm \varepsilon_h(p) \}$ takes its minimum value on the boundary of $G_h{}^*$.

This value can not be attained at a point $p \in \Gamma_h^2$, since on Γ_h^2:
$\ell_h \left[S_h(p) \pm \varepsilon_h(p) \right] \geqslant 0.$

If the minimum value is attained at a point $p \in \Gamma_h^1$, then
$S_h(p) \pm \varepsilon_h(p) \geqslant 0.$
and consequently

$$| \varepsilon_h(q) | \leqslant S_h(q), \quad \forall q \in \overline{G}_h{}^*.$$

The above results can be formulated in the following theorem.

Theorem Let $U(p)$ is the solution of the difference problem (7) and $u(x, y)$ is the solution of the differential problem (1) — (3). Further, let $\overrightarrow{G} \subseteq G$ such that $\overrightarrow{G}$ does not contain any corner points of G. Then $\forall p \in \overline{G} \cap \overline{G}_h$ we have

$$U(p) - u(p) = O(h^s),$$

where s is a number satisfying the following conditions:
$s \leqslant 1;$
$s < 1/2 \, \alpha_j$, if z_j is a mixed corner;
$s < 1/\alpha_j$, if z_j is a Dirichlet corner.

Corollary 1. If all the mixed corners are acute and the Dirichlet corners are convex, then
$U(p) - u(p) = O(h).$
The case of quasilinear elliptic equations:
Let $f(x, y, z, v, \omega)$ be a given continuous function defined $\forall (x, y) \in \overline{G}$

and $\forall$ z, v, ω . Assume that the partial derivatives f_z, f_v and f ω exist and satisfy the following inequalities

$$f_z \geq \bar{\eta} > 0, \; |\; f_v \;|, \; |\; f\; \omega \;| \leq \overline{M} < \infty, \; \overline{M} = \text{const.}$$

Let Lu = $\triangle$ u — f (x, y, u, u_x, u_y); $\varnothing$ (x, y) and g(x, y) $\geq$ 0 are continuous functions. Now, the problem is to find a continuous function u (x, y) in $\overline{G}$, twice continuously differentiable in G and satisfies both the equation

L u = O, (x, y) ϵ G, (26)

and the boundary conditions

u (x, y) = $\varnothing$ (x, y), (x, y) ϵ Γ^1, (27)

$\ell u = \dfrac{\partial}{\partial n}$ u (x, y) + g (x, y) u = $\varnothing$ (x, y), (x, y) ϵ Γ^2, (28)

Suppose that the point P_o (x_o, y_o) ϵ G. Denote the points (x_o + h, y_o) (x_o, y_o + h), (x_o — h, y_o), (x_o, y_o — h) by P_{o1}, P_{o2}, P_{o3}, P_{o4}, respectively. The points P_{o1}, P_{o2}, P_{o3}, P_{o4} we shall call neighbouring points of P_o.
Denote the set of mesh points P_1, P_2, P_3, ..., P_{N1} ϵ G for which P_{ik} ϵ $\overline{G}$

(i = $\overline{1, N_1}$; k = $\overline{1, 4}$) by G_h. The set of mesh points P_{N1} + 1, P_{N1} + 2, ...,

$P_{N2} \epsilon$ G , where the distances from Γ^2 are less than $\sqrt{2}$ h, is denoted by Γ_h^2. The set of points $P_{N2}+1$, $P_{N2}+2$, ..., P_N which are neighbouring points of the points of G_h $\cup \Gamma_h^2$ is denoted by Γ_h^1.

Construction of the difference problem for problem (26) — (28):
If $\forall$ P_j (j = $\overline{N_1 + 1, N_2}$), we can find the points $R_j^{(1)}$ and R_j $^{(2)}$ such that the condition (28) can be approximated as follows

$$\ell_h \; U \; (P_j) \equiv \frac{U \; (p_j) \; — \; n_j^{(1)} \; U \; (R_j^{(2)}) \; — \; n_j^{(2)} \; U \; (R_j^{(1)})}{\mu_j h} + g \; (P_j) \; U \; (P_j)$$
$$= \varnothing \; (q),$$

where q ϵ Γ^2, O $< \mu_j \leq \sqrt{2}$ and $\eta_j^{(1)}$, $\eta_j^{(2)}$ $\geq$ 0 where $\eta_j^{(1)}$ + $\eta_j^{(2)}$ = 1,

then we approximate L and ℓ by the following operators:

$L_h \; u_j = \triangle h \; u_j — f[x_j, y_j, u_j, D_{h,xo}(u_j), D_{h,yo}(u_j)], j = \overline{1, N_1},$ (29)
$\tilde{\ell}_h \; u_j = (1 + g_j \; \mu_j \; h) \; u_j — \eta_j^{(1)} u \; (R^{(2)}_j) — \eta_j^{(2)} u(R_j^{(1)}), j = \overline{N_1 + 1, N_2},$ (30)
where
$\triangle_h \; u_j = 1_{h2} \; (\sum_{k=1}^{4} u_{jk} — 4 \; u_j), D_{h,xo}(u_j) = \dfrac{1}{2h}(u_{j1} — u_{j3}),$
$D_{h,yo}(uj) = \dfrac{1}{2h} (u_{j2} — u_{j4}), \tilde{\ell}_h \; u_j = \mu_j \; h \; \ell_h \; u_j.$

Since one of the two points $R_j^{(1)}$, $R_j^{(2)}$ is an interior point, then without loss of generality we can take $R_j^{(1)}$ as an interior point, and therefore $\eta_j^{(2)} \neq 0$, $\forall$ j = $\overline{N_1 + 1, N_2}$.

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The associated difference problem for the problem (26) — (28) is as follows: Find a function U_j is defined in $\overline{G}_h$ and satisfies the problem:

$$L_h\, U_j = O, \; j = \overline{1,\, N_1},$$

$$U_j = \varnothing_j, j = \overline{N_2 + 1, N}, \tag{31}$$

$$\tilde{\ell}_h\, U_j = \overline{\varnothing}_j, \; j = \overline{N_1 + 1,\, N_2},$$

where $\overline{\varnothing}_j = \mu_j\, h\, \varnothing\,(q)$ and $q \in \Gamma^2$ is the point of intersection of Γ^2 with the normal passing through $P_j \in \Gamma^2_h$.

Following the approach given in [2], we can prove that the solution U_j of the difference problem (31), can be obtained as the limit of Seidel successive approximations:

$$\overset{(m+1)}{U_j} = S_j\, (\overset{(m)}{U_{j1}} \quad \overset{(m)}{U_{j2}}, \; \overset{(m)}{U_{j3}}, \; \overset{(m)}{U_{j4}} \quad, \; J = \overline{1,\, N_1},$$

$$\overset{(m+1)}{U}{}^j = \frac{1}{1+g_j\mu_j\,h}[\eta_j^{(1)}\, \overset{(m-1)}{U}\,(R_j^{(2)}) + n_j^{(2)}\, \overset{(m+1)}{U}\,(R_j^{(1)}) + \overline{\varnothing}_j], \; j = \overline{N_1 + 1,\, N_2^*},$$

$$\overset{(m+1)}{U_j} = \overset{(m)}{U_j} = \varnothing_j, \; j = \overline{N_2+1,\, N}, \; m = o, 1, \dots,$$

where $S_j\,(\omega|_1, \omega_2, \omega_3, \omega_4,), j = 1, N_1$ are continuous functions according to its arguments. Further, the rate of convergence can be estimated by:

$$\overset{max}{\overline{G}_h} \quad |\overset{(m-1)}{U_j} - U_j| \le p^m\,|U_j^{(o)} - U_j|,$$

where $O < p < 1$ is the ratio factor and $U_j^{(o)}$ are the intial approximations.

Estimating the error $\delta_h = U - u$:

Making use of equation (29), we have

$$L_h\, U = \Delta_h U - f(x,\, y,\, U,\, D_{h,xo}\,(U),\, D_{h,yo}\,(U)\,),$$

$$L_h u = \Delta_h U - f\,(x,\, y,\, u,\, D_{h,xo}(U_j),\, D_{h,yo}\,(U)\,)$$
$$= \Delta\, u - f\,(x,\, y,\, u,\, u_x,\, u_y) + h\, M_3\,(u) + h\, M_2\,(u),$$

consequently,

$$\mathcal{l}_h\delta_h \equiv L_h\, U - L_h\, u = \Delta_h\,\delta_h - \tilde{f}_z\,\delta_h - \tilde{f}_v\, D_{h,\,xo}(\,\delta_h)-$$

$$- \tilde{f}\omega\, D_{h,yo}\,(\,\delta_h\,) = h\, M_3\,(u) + h\, M_2\,(u). \tag{32}$$

let z_1 be a corner point. As in § 2, let $\alpha a(r,\theta)$ be an angle such that

$$t_g\,\alpha\,(r,\theta) = \frac{b\,(r\,\cos\,\theta,\, r\,\sin\,\theta)}{a\,(r\,\cos\,\theta,\, r\,\sin\,\theta)}$$

b (r cos θ , r sin θ) = — f ω (r cos θ , r sin θ, z (r cos θ, r sin θ),
v (r cos θ, r sin θ), ω (r cos θ, r sin θ)).
a (r cos θ, r sin θ) = — f_v (r cos θ, r sin θ, z (r cos θ, r sin θ),
v (r cos θ, r sin θ), ω(r cos θ, r sin θ)).

Assume that

c (r cos θ, r sin θ) = — f_z (r cos θ, r sin θ, z (r cos θ, r sin θ),
v (r cos θ, r sin θ), ω (r cos θ, r sin θ)).
If z_1 is a Dirichlet corner and inequality (20) is satisfied, then using (4), (32) we have

$$\mathcal{L}_h \quad \delta_h = h\, O\, (r^{-3}) + h\, O\, (r^{-1}).$$

Finally, if z_1 is a mixed corner and inequality (12) is satisfied, then taking into account (5), (32) we obtain.

$$\mathcal{L}_h \quad \delta_h = h\, O\, (r^{\lambda -3} + 1) + h\, O\, (r^{\lambda -1} + 1).$$

Corollary 2. The computations for the operator $\mathcal{L}_h$ is a straight forward extension to that used in the proof of Theorem.

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