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FIXED POINT THEOREMS IN TOPOLOGICAL SPACES

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نظريات النقطة الثابتة في الفضاءات التوبولوجية

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تم التوصل إلى نظريتين تتعلقان بالتطبيقات الذاتية على فضاء متجهات وذلك باستخدام بعض نظريات النقطة الثابتة المعروفة من أجل أزواج التطبيقات التي تحقق شروطاً من النوع التقلّصي.

Key words and Phrases : Topological space, metric space, self-map, cluster point, fixed point

ABSTRACT

A couple of theorems for a pair of self-maps on a topological space have been obtained using which certain fixed point theorems in the literature for pairs of maps satisfying contractive type conditions have been deduced.

In this paper our objective is to bring under a single banner various fixed point theorems for a single or a pair of self-maps on a metric or a or a topological space satisfying contractive type conditions under constrains such as the satisfaction of certain inequations by the independent variables. We have partially fulfilled our objective by generalising the results of Sastry, Naidu, Rao & Rao [4], and Sastry and Babu [2 & 3] (Theorems 1 & 2) while the latter, in turn, are generalisations of the results of Sehgal [5] and those of Khan, Swaleh & Sessa [1].

Through out this paper unless otherwise stated, IN stands for the set of all natural numbers and IR^+ for the set of all non-negative real numbers.

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Theorem 1 : Let X be a topological space, ρ be a bounded below real valued function on XxX and f and h be self-maps on X. Suppose that there is an X_0 in X such that $\{h^n x_0\}_{n=1}^{\infty}$ has a cluster point z in X. Let A = $\{h^n x_0 : n \in IN\}$ and E be a subset of X. Suppose that f, h and fh are sequentially continuous at z, ρ is sequentially continuous at (z, fz) and (hz, fhz), and that

$$\rho\,(hx, fhx) < r\,(x, fx) \quad \ldots (1)$$

for all x in E. Then either A\E = is nonempty or z ∈ E. φ.

Proof: Suppose that A\E = φ. Then $h^n x_0 \in E\ \forall\ n \in IN$. Hence $\{\rho\,(h^n x_0, fh^n x_0\}_{n=1}^{\infty}$ is a monotonically decreasing sequence of real numbers. It is bounded below since ρ is bounded below on XxX. Hence it converges to a real number, say, α. Since z is a cluster point of $\{h^n x_0\}_{n=1}^{\infty}$, there exists a convergent subsequence $\{h^n k x_0\}_{k=1}^{\infty}$ of it which converges to z. Since f, h and fh are sequentially continuous at z, the sequences $\{fh^n{}_k x0\}_{k=1}^{\infty}$, $\{h^n{}_k{}^{+1} x_0)_{k=1}^{\infty}$ and $\{fh^n{}_k{}^{+1} x0\}_{k=1}^{\infty}$, converge to ρ (z, fz) and ρ (z, fz) and ρ (hz, fhz) respectively. But both of these sequences are subsequences of the convergent sequence $\{\rho\,(h^n x_0, fh^n x_0)\}_{n=1}^{\infty}$ with limit α. Hence ρ (hz, fhz) = α = ρ (z, fz). Since inequality (1) is true for x ∈ E, z ∉ E.

Corollary 1: (Theorem 2.1 of Sastry, Naidu, Rao & Rao [4]) Let X be a topological space, F: XxX → IR^+ be symmetric and sequentially continuous with F(x, x) = 0 ∀ x ∈ X. Let f and g be self-maps on X such that

1. F and gf are sequentially continuous on X
2. F (fx, gy) < max {F(x, y), F{x, fx}, F(y, gy)} for all distinct x, y in X and
3. $\{(gf)^n x_0\}_{n=1}^{\infty}$ has a cluster point for some x_0 in X.

Then either f or g has a fixed point in X.

Proof : From condition 2 and the symmetry F we have F(fgfx, gfx) < F(fx, gfx) if gfx ≠ fx and fx and F(fx, gfx) < F(x, fx) if x ≠ fx so that F(fgfx, gfx) < F(x, fx) if gfx ≠ fx and fx ≠ x. On taking ρ = F, h = gf and E = {x ∈ X: fx ≠ x. and gfx ≠ fx} in Theorem 1 it follows that either f has a fixed point B or g has a fixed point in f(B), where B is the closure of A in X.

Remark 1 : In Corollary 1 the condition: "F(x, x) = 0 ∀ x ∈ X" is redundant.

Corollary 2 : Let X be a T_1-topological space. ρ be a nonnegative real valued sequentially continuous function on XxX with ρ(x, x) = 0 ∀ x ∈ X, f be a self-map on X and $x_0 \in X$ be such that $\{f^n x_0\}_{n=1}^{\infty}$ has a cluster point z in X. Suppose that f and f^2 are sequentially continuous at z and that

$$\rho(fx, fy) < \max\,\{\rho(x, y), (\mathrm{Max}\,\{\rho(x, fx), \rho(y, fy)\} + [\rho(x, fy)\,\rho(fx, y)]^{1/2}\} \quad \ldots(2)$$

for all distinct x, y in X. then z is the only fixed point of f in X.

Proof : On taking y = fx in inequality (2) we obtain ρ(fx, f^2x) <ρ(x, fx) ∀ x ∈ X with fx ≠ x. On taking h = f and E = {x ∈ X: fx ≠ x} in Theorem 1 we conclude that either $f(f^m x_0) = f^m x_0$ for some m ∈ IN or fx = z. If $f^{m+1}x_0 = f^m x_0$, then $f^n x_0 = f^m x_0$ ∀ n ≥ m so that in view of the hypothesis that z is a cluster point of $\{f^n x_0\}$∞n=1 and X is T_1 it follows that z = fmx0. Hence z is a fixed point of f in either case. From inequality (2) it is evident that f has at most one fixed point in X. Hence z is the only fixed point of f in X.

Corollary 3 : (Theorem 2.2 of Sastry & Babu [2] Let (X, d) be a metric space, f be a continuous self-map on X and Ψ: IR^+ → IR^+ be a continuous map with ψ(t) = 0 if t = 0. Suppose that for some x0 ∈ X the sequence $\{f^n x_0\}$ has a cluster point z in X and that there are nonnegative constants a, b, c such that a + b ≤ 1, a + c ≤ and

$$\psi(d(fx, fy)) < a\,\psi\,(d(x, y)) + {}^{b}/_{2}\,[\psi(x, fx) + \psi(y, fy)] + c[\psi(x, fy)\,\psi\,(fx, y)]^{1/2}$$

for all distinct x, y in X. then z is the unique fixed point of f in X.

Proof : Let ρ = ψod. then ρ is a nonnegative real valued continuous function on X×X with ρ(x, x) = 0 ∀ × ∈ X, and for all distinct x, y in X we have

$$\rho(fx, fy) < a\rho(x,y) + {}^{b}/_{2}\,[\rho(x, fx) + \rho(y, fy)] + c[\rho(x, fy)\,\rho(fx, y)]^{1/2}$$

$$\leq a\rho(x, y) + b \max\,\{\rho(x, fx), \rho(y, fy)\} + c[\rho(x, fy)\,\rho(fx, y)^{1/2}\ (\because b \geq 0)$$

$$\leq a\rho(x, y) + (1 - a) \max\,\{\rho(x, fx), \rho(y, fy)\} + (1 - a)\,[\rho(x, fy)\,\rho(fx, y)]^{1/2}$$

(a + b ≤ 1, a + c ≤ 1 and ρ is nonnegative)

$$= a\rho\,(x, y) + (1 - 1)\,[\max\,\{\rho(x, fx), \rho(y, fy)\} + [\rho(x, fy)\,\rho(fx, y)]^{1/2}$$

$$\leq \max\,\{\rho(x, y), (\max\,\{\rho(x, fx), \rho(y, fy)\}) + [\rho(x, fy)\,\rho(fx, y)]^{1/2}$$

$$(\because 0 \leq a \leq 1).$$

Hence Corollary 3 follows from Corollary 2.

Corollary 4: (Theorem 2.1 of Sastry & Babu [3] Let f be a continous self-map on a metric space (X, d) such that for some $x_0 \rightarrow X \in$ the sequence $\{f^n x_0)$ has a cluster point z in X, and ψ: $IR^+ \rightarrow IR^+$ be a continuous map with $\psi(t) = 0$ iff $t = 0$. Suppose that

$$\psi(d(fx, fy)) < \max \{\psi(d(x, y)), \psi(d(x, fx)), \psi(d(y, fy))\} \quad \ldots(3)$$

for all distinct x, y in X. then z is the unique fixed point of f in X.

Proof : Corollary 4 follows from Corollary 2 on taking $\rho = \psi o d$.

Note 1 : As observed by Sastry & Babu [3] Theorem 5 of Sehgal [5] follows from Corollary 4 on taking $\psi(t) = t \ \forall \ t \in IR^+$.

Theorem 2 : Let X be a topological space, ρ be a bounded below real valued function on XxX and f and h be self-maps on X. Suppose that there is an x_0 in X such that $\{h^n x_0)^\infty_{n=1}$ has a cluster point z in X. Let $A = \{h^n x_0 : n \in IN\}$ and E_1, E_2 be subsets of X. Suppose that f and h are sequentially continuous at z, ρ is sequentially continuous at (z, fz) and (fz, hz),

$$\rho(fx, hx) < \rho(x, fx) \quad \ldots (4)$$

for all x in E_1 and

$$\rho(hx, fhx) < \rho(fx, hx) \quad \ldots (5)$$

for all x in E_2. Then either $A \backslash (E_1 \cap E_2)$ is nonempty or $z \notin E_1$.

Proof : Suppose that $A \backslash (E_1 \cap E_2) = \phi$. Then $h^n x_0 \in (E_1 \cap E_2) \ \forall \ n \in IN$. Hence from inequalities (4) and (5) we have $\rho(h^{n+1}x_0, fh^{n+1}x_0) < \rho(fh^n x_0, h^{n+1}x_0) < \rho(h^n x_0, fh^n x_0) \ \forall \ n \in IN$. Hence $\{\rho(h^n x_0, fh^n x_0)\}^\infty_{n=1}$ is a monotonically decreasing sequence of real numbers. It is bounded below since ρ is bounded below on XxX. Hence it converges to a real number, say, α. Hence $\{\rho(fh^n x_0, h^{n+1}x_0)\}^\infty_{n=1}$ also converges to α. Since z is a cluster cluster point of $\{h^n x_0\}^\infty_{n=1}$, there exists a convergent subsequence $\{h^{n_k}x0\}^\infty_{k=1}$ of it which converges to z. Since f and h are sequentially continuous at z, the sequences $\{fh^{n_k}x_0\}^\infty_{k=1}$ and $\{h^{n_k+1}x_0\}^\infty_{k=1}$ converge respectively to fz and hz. Since ρ is sequentially continuous at (z, fz) and (fz, hz) it follows that the sequences $\{\rho(h^{n_k}x_0, fh^{n_k}x0)\}^\infty_{k=1}$ and $\{\rho(fh^{n_k}x0, h^{n_k+1}x_0)\}^\infty_{k=1}$ converge to $\rho(z, fz)$ and $\rho(fz, hz)$ respectively. But the first one is a subsequence of $\{\rho(h^n x_0, fh^n x_0)\}^\infty_{n=1}$ which converges to α and the second one is a subsequence of $\{\rho(fh^n x_0, h^{n+1}x_0)\}^\infty_{n=1}$ which also converges to α. Hence $\rho(fz, hz) = \alpha = \rho(z, fz)$. Since inequality (4) is true for all $x \in E_1$, $z \notin E_1$.

Remark 2 : Corollary 1 with the additional conclusion (as stated in [4] that any cluster point of $\{(gf)^n x_0\}^\infty_{n=1}$ is a fixed point of f if $x_n \neq x_{n+1} \ \forall \ n \in IN$, where $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ (n = 0, 1, 2, ...), can be deduced from Theorem 2 by taking $\rho = F$, h = gf, $E_1 = \{x \in X: fx \neq x\}$ and $E_2 = \{x \in X: gfx \neq fx\}$.

Corollary 5 : (Theorem 3.1 of Sastry & Babu [3] Let f and g be self-maps on a metric space (X, d) such that for some $x_0 \in X$ the sequence $\{(gf)^n x_0\}$ has a cluster point z in X and f and gf are continuous at z. Let ψ: $IR^+ \rightarrow IR^+$ be a continuous map with $\psi(t) = 0$ iff $t = 0$. Suppose that

$$\psi(d(fx, gy)\} < \max \{\psi d(x, y)), \psi(d(x, fx)), \psi(d(y, gy))\}$$

for all distinct x, y in X. then either f or g has a fixed point in X. If both f and g have fixed points, then their fixed point sets are singletons and are equal.

Proof : Let $\rho = \psi o d$. Then ρ is a symmetric nonnegative real valued continuous function on XxX with $r(x, x) = 0$ for all x in X and

$$\rho(fx, gy) < \max \{\rho(x, y), \rho(x, fx), \rho(y, gy)\} \quad \ldots(6)$$

for all distinct x, y in X. On taking y = fx in inequality (6) we obtain

$$\rho(fx, gfx) < \rho(x, fx) \text{ if } fx \neq x$$

and on replacing x with gfx and y with fx in inequality (6) and on using the symmetry of ρ we obtain

$$\rho(gfx, fgfx) < \rho(fx, gfx) \text{ if } gfx \neq fx.$$

On taking h = gf, $E_1 = = \{x \in X: fx \neq x\}$ and $E_2 = \{x \in X: fgx \neq fx\}$ in Theorem 2 we can conclude that either f or g has fixed point in X. The second conclusion of the corollary is evident from inequality (6).

Note 2 : Corollaries 4 & 6 are the generalisations obtained by Sastry & Babu [3] for a theorem of Khan, Swaleh & Sessa [1].

Remark 3 : In Corollaries 3, 4 & 5 the condition: "$\psi(t) = 0$ iff $t = 0$" may be replaced by the weaker condition: "$\psi(0) = 1$".

Sastry & Babu [3] posed the following

Problem : Find an exapmple of a metric space X, a continuous function ψ: $IR^+ \rightarrow IR^+$ which vanishes only at zero and a self-map f on X with a fixed point z satisfying inequality (3) for all distinct x, y in X and such that for no x in $X \backslash \{z\}$ the sequence $\{f^n x\}$ has a convergent subsequence.

The following exapmple provides an answer to the above problem with $\psi(t) = t \ \forall$ t in IR^+.

Example : Let X = {0, 1, 2...}. Define d on XxX as $\frac{1}{2^n}$

$d(x, y) = 0$ if $x = y$, $d(0, n) = d(n,0) = 1 +$ $\forall\, n \in \mathbb{N}$ and $d(m, n) = 1 + \frac{1}{2^m} + \frac{1}{2^n}$ if $m, n \in \mathbb{N}$ are distinct. Define $f\colon X \to$ as $f0 = 0$ and $fn = n + 1$ $\forall\, n \in \mathbb{N}$. Then d is a metric on X, $d(fx, fy) < d(x, y)$

for all distinct x, y in X., 0 is the only fixed point of f in X and for no x in $X \setminus \{0\}$ the sequence $\{f^n x\}$ has a convergent subsequence.

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