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# DECOMPOSITION OF THE SET OF CONDITIONALLY EXPONENTIAL CONVEX FUNCTIONS

By

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# تكوينات مجموعة الدوال الأسية المحدبة المشروطة

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في هذا البحث نستخدم نظريات الجبر $C^*$ لايجاد علاقة بين النقط المتطرفة المحلية لمجموعة الدوال الأسية المحدبة المشروطة والتوبولوجي المعرف على الفراغ الطيفي $\hat{G}$

*Key Words:* Conditionally Exponential Convex Functions, C* - algebra.

ABSTRACT

The theory of C* -algebra is used to develop a connection between the local extreme points and the topology of the spectrum space $\hat{G}$.

## INTRODUCTION

Berg, Christensen and Ressel(4) and independently Okb-El-Bab and El-Shazli(12) studied conditionally exponential convex functions on semigroups. For G compactly generated, a compact base for $E_o$ (G), the set of conditionally exponential convex functions defined on a locally compact group G, was constructed in(11). Also, the auther in(11) obtained the extreme points of that base.

In this article we use the theory of C* -algebra to develop a connection between the local extreme points and the topology of the spectrum space G. The main advantage of C* -algebraic approach, besides its generality, is that topological considerations come to the foreforent.

## DEFINITIONS AND NOTATION

Let G be a separable locally compact group equipped with left Haar measure dx and modular function Δ, where the identity element is denoted by e, and let G be the set of irreducible representations of G. If G is abelian, G is its dual. By $C_c$ (G) ($C^o_c$ (G) we denote the set of compactly supported continuous functions on G (with total left Haar integral zero(5).

Let C* (G) be the inveloping C* -algebra of $L^1$ (G) equipped with the involution # defined by $f^\#(x) = \Delta(x^{-1}) f^+_{(x)}$ where $f^+(x) = f(x^{-1})$. The dual of C* (G) is B (G) and its double dual is $W^*$ (G). For the universal representation ω we write ω (μ) to indicate that belongs to $W^*$ (G)(7).

Now let S be a separable compact convex set. A subset F of S is called a face if each line segment in S whose interior intersects F is contained in F. The complementary set F' of F is the union of all faces of S disjoint from F. If F is a closed face and F' is also a face then F is called a closed split face. In this case, S is the direct convex sum of F and F'; i.e., every $x \in S$ can be written uniquely in the form $x = \lambda y + (1 - \lambda) z$, $0 \leq \lambda \leq 1$, $y \in F$, $z \in F'$ (2).

If χ is a subset of the set ext S, of extreme points of S, its facial closure is F ∩ ext S, where F is the smallest closed split face of S containing χ. The topology defined in this way is called the facial topology. This topology is coaser than the weak * -topology (2).

In the following we write P (G) for the set of all exponentially convex functions defined on G; i.e., functions satisfying

$$\sum_{i,j=1}^{n} \psi(g_i g_j) c_i c_j \geq 0,$$

where $g_1, ..., g_n \in G$ and $c_1, ..., c_n \in R$. The set of elements from P (G) with norm equals 1 is $P_1$ (G). Clearly $P_1$ (G) is a convex set(3). Also we write $E_o$ (G) for the set of all conditionally exponential convex functions defined on G and vanishing at the group identity; i.e., functions satisfying

$$\sum_{i,j=1}^{n} [\psi(g_i) + \psi_i(g_j) - \psi(g_i g_j)] c_i c_j \geq 0,$$

where $g_1, ..., g_n \in G$ and $c_1, ..., c_n \in R$(4,12).

Elements of $E_o$ (G) can be characterized geometrically as semi-tangents to $P_1$ (G) at the identity, and if $\psi$, - $\psi \in E_o$ (G) then $\psi$ becomes a tangent vector to $P_1$ (G) at the identity.

By a Levy weight for $\psi \in E_o$ (G) we mean the linear functional (also deonoted by $\psi$) defined by

$\psi (a) = (1, \partial_\psi a), a \in$ domain $(\partial_\psi)$,

($\partial$ is a linear functional from C* (G) to C* (G) which is densely defined and $\psi |$ (ker 1)$^+ \geq 0$. If $M_\psi = N^\#_\psi N_\psi$ and $N_\psi = \{a \in$ ker $1 | \psi (a^\# a) < \infty\}$, then the weight is called local if sup $\{\rho_\alpha (a; \psi) | \alpha > 0\} = 0$ for all $a \in$ (ker 1)$^+{}_1$, where

$\rho_\alpha (a; \psi) = \inf \{\psi (s) + t\alpha | a \leq s+tz_1; t \geq 0, s \in M^+{}_\psi\}$,

$z_1$ is the central support of the weak closure of ker 1 in W* (G) and (ker 1)$_1$ is the unit ball of ker 1(6).

Given a continuous unitary representation U of G on a Hilbert space H, a continuous map c:G → H such that c (e) = 0 and c (xy) = c (x) +U (x) c (y) is called a l-cocycle for U. The additive group of such cocycles is denoted by Z' (U). The subgroup B' (U) of 1-cobounaries is that set of cocycles of the form c (x) = U (x) $\xi - \xi$, $\xi \in$ H. The quotient H' (U) = z' (U)/B' (U) is called the first cohomology group of U(8).

Finally, for $\psi, \phi \in E_o$ (G) we say that $\psi$ dominates $\psi$ if $\psi - \phi \in E_o$ (G). If $\phi$ and $\psi$ dominate each other then they are equivalent, and they are weakly equivalent if one is equivalent to a positive multiple of the other(6).

## A DECOMPOSITION THEOREM FOR $E_o$ (G)

In this section we study the connection between the topology on G and the structure of $E_o$ (G). Let $C^*_e$ (G) be the smallest C* -algebra containg C* (G) and has an identity and let S be its state space which is compact in the relative weak *-topology.

If C* (G) has an identity then S coincides with $P_1$ (G). Otherwise $P_1$ (G) is a split face in S, and S is the direct convex sum of $P_1$ (G) and the state $f_o$, defined by $f_o | C^*$ (G) = 0.

Let $\{O_n | n \geq 1\}$ be a collection of relatively weak * -open subsets of ext S such that $o_n \subset \bar{o}_n \cap$ ext S $\subset O_{n-1}$ and $\cap_n O_n = \{1\}$. If Up is the irreducible representation of G obtained from $p \in \bar{o}_n \cap$ ext S then we define the two sided ideal In = $\cap$ {ker $U_p$} $\forall$n. The closure of $O_n$ in the facial topology on ext S is $F_n \cap$ ext S $F_n = I^\perp{}_n$ is the closed split face of S annihilated by $I_n$. Finally, let $q_n \in$ W* (G) and $z_1$ be the central supports of $I_n$ and ker 1, respectively. Since $O_{n-1} \supset O_n \supset \{1\}$, we have $q_{n-1} \leq q_n \leq z_1$.

In the following we prove that the Levy weights of elements of $E_o$ (G) are bounded on $I_n$.

***Lemma 3.1***

If V is a weak-* open neighborhood of the constant function 1 is S, then there exist $\delta > 0$ and $g \in C_c$ (G) such that i) $g \geq 0$, ii) $0 \leq \omega (g) \leq 1$, iii) $(1, \omega (g)) = \int g (x) dx = 1$, and iv) for each $f \in S \backslash V$ we have $(f, \omega (g) < 1 - \delta$.

***Proof***

Let $L = \{g \in C_c (G) | g \geq 0, \int g (x) dx = 1\}$ and D (g) = $\{f \in S | g \in L$ and $(f, \omega (g\#* g) = 1\}$ where * denotes the usual convolution. For $f \in D (g)$ we can write $f = \lambda p + (1-\lambda) f_o$ for some $p \in P_1$ (G) and $0 \leq \lambda \leq 1$. Simple calculations show that $\lambda = 1$ and f = p; i.e., $f \in P_1$ (G). Now, for $f \in \cap \{D (g) | g \in L\}$ we have

$\int\int (1-f (xy)) g (x) g (y) dxdy = 0$, hence

$(1-f (xy)) g (x) g (y) dxdy = 0$ on G $\times$ G. Choosing the support of g to contain any compact set in G we get f = 1.

If V is a weak-* open neighborhood of 1 in S, then S\V is compact and

$$\phi = (S\backslash V) \cap (\cap \{D (g) | g \in L\})$$
$$= (S\backslash V \cap (\bigcap_{g \in L} \bigcap_{\in > 0} \{f \in S | (f, \omega (g^\# * g)) \geq 1-\in\}) .$$

By the finite intersection property, there exist $\in_1, ..., \in_n > 0$ and $g_1, ..., g_m \in L$ such that

$$\phi = \bigcap_{k=1}^{n} \bigcap_{\ell=1}^{m} \{f \in S | (f, \omega (g^\#_\ell * g_\ell)) \geq 1 - \in_k \} \cap (S\backslash V).$$

Now, conditions i) -iv) are easily satisfied for $\delta = \min (\in_k /m)$ and $g = \frac{1}{m} \sum_{\ell=0}^{m} g^\#_\ell * g_\ell$ and the lemma follows.

***Lemma 3.2***

For each $n \geq 1$ there exist $g_n \in C_c$ (G) with $g_n \geq 0$, $\int g_{n(x)} dx = 1$ and $B_n \in$ W* (G) such that $q_n = B_n \omega (\delta_e - g_n)$ where $\delta_e$ is the point mass at e.

***Proof***

Suppose that $V_n$ is a neighborhood of the identity is S containted in the face $F_n$ annihilated by $I_n$. For this neighborhood we construct $g_n \in C_c$ (G) and $\delta_n > 0$ as in Lemma 3.1. Then

$$|| q_n \bullet \omega (g_n) || = \sup \{(p, q_n \omega (g_n)) | p \in S\}$$
$$= \sup \{(p, \omega (g_n)) | p \in S\backslash F_n\}$$

$$\le \sup \{(p, \omega(g_n)) \mid p \in S \backslash V_n\} \le 1 - \delta_n.$$

Thus the geometric series $\sum_{k=0}^{\infty} q_n (\omega (g_n))^k$ converges to an element

$B_n \in W^* (G)$ of norm at most $\sum_{k=0}^{\infty} (1-\delta_n)^k = \delta^{-1}_n$. clearly,

$B_n \omega (\delta_e - g_n) = q_n$. //

***Lemma 3.3.***

If $\psi$ is the Levy weight of an element of $E_o$ (G) then there exists $p_n \in P$ (G) such that $\psi | I_n = p_n | I_n$ for all values of n.

***Proof***

Let $a \in I_n \subset \ker 1$. Then

$$0 \le \psi (a^{\#} a) = \psi (q_n a^{\#} a q_n)$$
$$= \psi (\omega (\delta_e - g_n) B_n a^{\#} a B_n \omega (\delta_e - g_n))$$
$$= \psi (\omega (\mu_n) B_n a^{\#} a B_n \omega (\mu_n)),$$

where $\mu_n$ is the measure defined by $d\mu_n = \delta_e - g_n \; dx$. It is clear that $\mu \in M^o{}_c$ (G), the set of compactly supported Borel measure on G of total mass zero. Since

$$\sum_{k=0}^{\infty} a(\omega (g_n)^k = \sum_{k=0}^{\infty} a q_n (\omega (g_n))^k \to a B_n$$

we have a $B_n \in I_n \subset C^*$ (G) Applying Lemma 2.1 in (11) we obtain

$$\psi (a^{\#} a) = (- \psi^{\mu}{}_{n}, B_n \; a^{\#} a B_n)$$
$$\le - \psi^{\mu}{}_{n(e)} \; || B_n ||^2 \; || a ||^2 \le - \delta^{-2}{}_n \psi^{\mu}{}_{n(e)} \; || a ||^2$$

This shows that $\psi | I_n$ is a bounded weight. On the other hand, we define $p_n$ by the product $p_n (.) = -\psi^{\mu n} (B_n . B_n)$. Then $p_n \in P$ (G) and $\psi | I_n = p_n | I_n$. //

***Corollary 3.4***

$\psi$ dominates $p_n (e) - p_n$ in $E_o$ (G).

***Theorem 3.5***

Suppose that $p_n \psi \in E_o$ (G) has a local Levy weight and F is the closed split face of S given by $F = \cap_n F_n$. Then for each $\mu \in M^o{}_c$ (G) such that $-\psi^{\mu} (e) \ne 0$ we have $-\psi^{\mu}$-$\psi^{\mu}$ (e) $\in$ F.

***Proof***

If the Levy weight for $\psi$ is local then, by definition, $\psi$ does not dominate any semitangent of the form p (e) -p, p $\in$ P (G). This implies that $\psi | I_n = 0$ for all $n \ge 1$ and hence $\psi^{\mu} | I_n = 0$ for each $\mu \in M^o{}_c$ (G). This means that $-\psi^{\mu}$ is annihilated by the closed two-sided ideal $\overline{\cap_n I_n} = F^{\perp}$. //

***Theorem 3.6***

Each $\psi \in E_o$ (G) can be written uniquely in the form $\psi = \psi_1 + \psi_2, \psi_1, \psi_2 \in E_o$ (G), where

i) for each $\mu \in M^o{}_c$ (G) such that $-\psi^{\mu}{}_1 (e) \ne 0$, $-\psi^{\mu}{}_1 / -\psi^{\mu}{}_1 (e) \in F$

ii) $\psi_2 = \lim_n (p_n (e) - p_n)$, where $p_n \in P$ (G) and $p_n | I_n = \psi^{\mu} | I_n$.

***Proof***

We note that if $\psi_2 = \lim_n (p_n (e) - p_n)$, then from Corollary 3.4, $\psi_2$, $\psi_1$, $= \psi - \psi_2$ belong to $E_o$ (G). Moreover, $\psi_1$ vanishes on each $I_n$ and the proof of Theorem 3.5 applies. //

Now, let U be a factor representation of G. By Proposition 5.2.7 of(7) and Corollary 2 of(1) ker U is a primitive ideal of C* (G).

***Definition 3.7(9)***

A factor representation U of G is said to be separated from the trivial representation if there exist disjoint open sets $V_1$ and $V_2$ in Prim (G), the primitive ideal space of C* (G), such that ker 1 $\in V_1$ and ker U $\in V_2$.

A group G has a property (P) if each factor representation of G on a separable Hilbert space which is separated from the trivial representation has a trivial first cohomology group $H^1$ (U).

In fact, every locally compact group has this property. As an application of Theorem 3.6 we have:

***Theorem 3.8***

Let G be a separable locally compact group.

i) If $\hat{F} = \cap\{ \bar{o} | o$ is an open neighborhood of the trivial representation in G} and if U $\in$ G-F, then $H^1 (U) = 0$

ii) G has property (P).

***Proof***

i) We need the following Lemma for proving this part.

***Lemma 3.9***

If U is an irreducible representation of G and if c $\in$ V' (U) then $\psi (x) = || c (x) ||^2/2$ generates an extreme ray in $E_o$ (G); i.e., each of its dominated elements is either a tangent vector or weakly equivalent to $\psi$.

***Proof***

Let $\phi \in E_o$ (G) be dominated by $\psi$ and let $\tilde{G} = GX_mR$ be the multiplier extension of G by R w.r.t. the trivial action of G on R, defined by the multiplier

$$m(g, h) = - (c (h), c (y)) \text{ and } \psi' (g, s) = \psi (g) + s \in E_o (G).$$

Construct the corresponding representation $(U_{\psi'}, H_{\psi'})$ of $\widetilde{G}$ ; $U_{\psi'}$ is the trivial extension of U to G, therefore irreducible. It is to be noted that $\psi'$ is extreme in $E_o$ (G). Now extend $\phi$ to G by $\phi'$ (g, s) = $\phi$ (g). Clearly, $\psi'$ dominates $\phi'$ in $E_o$ (G) and since $\psi'$ is extreme there exists $\lambda > 0$ such that $\lambda\psi = \phi'$; i.e., $\lambda\psi = \phi$. //

***Lemma 3.10***

B' (U) is precisely the set of bounded 1-cocycles of U. The proof follows directly from 3.7 of[10].

***Lemma 3.11***

Let c be a cocycle for the representation U of G and let $\psi(x) = \| c(x) \|^2/2 \in E_o(G)$. Then for each $\mu \in M^o{}_c$ (G) such that $-\psi^\mu(e) = 1$ we have $-\psi^\mu$ is a diagonal coefficient of U.

***Proof***

The proof follows immediately because,

$-\psi^\mu(g) = -\int\int \psi(xgy)\, d\mu(x)\, d\mu(y)$

$= -\int\int (U(g)\, c(x), c(y))\, d\mu(x)\, d\mu(y)$

$= (U(g)\, c_\mu, c_\mu)$. //

***Proof of i)***

Let U $\in$ G-F, c $\in$ Z' (U) and $\psi(x) = \| c(x) \|^2/2$. By Lemma 3.9, $\psi$ generates an extreme ray in $E_o$ (G). In fact $\psi$ is either local or bounded. If $\psi$ is boudned then by Lemma 3.10 the result follows. If it is local, we choose $\mu \in M^o{}_c$ (G) such that $-\psi^\mu(e) = 1$. By Theorem 3.5 we have $-\psi^\mu \in$ F. In the same time, by Lemma 3.11, (see also Theorem 4.2 in [11], there is a diagonal coefficient p of U such that $-\psi^\mu = (p+\bar{p})/2$. Clearly, $\bar{p}$ is also a diagonal coefficient of $\bar{U} \in$ G-F. Now p and $\bar{p}$ belong to the set of extreme points of $P_1$ (G), say ext $P_1$ (G). So p, $\bar{p} \in$ F', the complementary face of F. This makes a contradiction with $-\psi^\mu \in$ F and hence H' (U) = 0.

Before starting on part ii) we have to prove the following:

***Lemma 3.12***

Suppose that U is a representation of G on a separable Hilbert space and it has a direct integral decomposition U (.) $= \int_S U(s,.)\, d\mu(s)$ over some probability space (S, $\mu$ (s). For H' (U) = 0 it is necessarily that there exist open sets $V_1$ and $V_2$ in G, $V_1 \cap V_2 = \phi$ such that $V_1$ contains the trivial representation and $V_2$ contains almost every U (s.,).

***Proof***

Let c $\in$ Z' (U). By Theorem 13.2[13] c has a decomposition in the form $(c(.) = \int_s c(s,.)\, d\mu(s)$ where s $\in$ S and c (s,.) $\in$ Z' (U (s,.)).

By part i), for almost every s $\in$ S, there exists $\lambda_s > 0$ and a unit vector $\xi_s$ in the Hilbert space of U (s,.) such that c (s, x) $= \lambda_s (U(s, x)\, \xi_s - \xi_s$. Let p (s, x) $\xi_s, \xi_s)$. Then

$\psi(x) = \int_S \lambda^2{}_s (1-p(s, x))\, d\mu(s)$

$= \int_S \lambda^2{}_s\, d\mu(s) - \int_S \lambda^2{}_s p(s, x)\, d\mu(s)$.

Now, $H_1$ (U) = 0 if $\psi$ (x) is bounded and this is true if $\lambda_s \in L^2$ (S, u). In fact, there exists an open set O in $P_1$ (G) containing the identity and excluding almost every p (s,.). Let g be the non-negative function in Cc (G) of Lemma 3.1 which corresponds to O and let $\delta > 0$ be such that for p $\in P_1$ (G) - O, (p, g) < 1 - $\delta$. By Fubini's Theorem we have

$\infty > \int \psi(x)\, g(x)\, dx = \int_s \lambda^2{}_s (1-p(s, x))\, g(x)\, dx d\mu(s) \geq$

$\delta \int_s \lambda^2{}_s\, d\mu(s)$, so that $\lambda_s \in L_2 (S, \mu)$. //

***Proof of ii)***

Let the assumptions of Lemma 3.12 be given and suppose that there exist disjoint open sets $V_1$ and $V_2$ in Prim (G) such that ker 1 $\in V_1$ and ker U $\in V_2$. For a e C* (G), $\| U(a) \|$ = ess.$_S$ sup $\| U(s, a) \|$, so that if a $\in$ ker U, then a $\in$ ker U (s,.) for almost every s $\in$ S. Let I = ker U and I (s) = ker U (s,.). Excluding a $\mu$-null set, then I = $\cap$ {I (s) | s $\in$ S}. Since factor representations are homogeneous, then for each measurable subset E of positive $\mu$-measure there exists $E_o$ E such that I = $\cap$ {I (s) | s $\in E_o$}. Evidently, the map s $\rightarrow$ I (s) is a measurable function from S into Prim (G), hence the set E = {s | I (s) $\notin V_1$} is measurable. Now, suppose that $\mu$, E > 0. Then I = $\cap$ {I (s) | s $\in E_o$} $\subset$ {I (s) | s $\in E_o$} $\subset V^c{}_1$. Since $V_1$ is open we arrive to a contradiction, and the proof can be completed by applying Lemma 3.12. //

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