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ON ESTIMATES OF THE RATE OF CONVERGENCE IN THE GLOBAL LIMIT THEOREMS FOR HOMOGENEOUS MARKOV CHAINS

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حول تقدير معدل التقارب في نظريات النهايات العالميه في سلاسل ماركوف المتجانسه

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تم في هذا البحث الحصول على تقدير لحد الباقي في نظريات النهايه لمجموع موزون لمجموعه من المتغيرات العشوائيه تكون سلسله ماركوف متجانسه ولها عدد اختياري من الحالات الممكنه. النتائج التي نحصل عليها تجعل من الممكن أن نقدر معدل التقارب في هذه النظريات في مقياس الفراغ Lp ، $1 \leq p < \infty$ بدون ان يتطلب الأمر أن يكون العزم المطلق من الدرجة الثالثة للإحتمالات الانتقالية محدوداً.

KEY WORDS . *homogeneous Markov chain , global limit theorem, rate of convergence ,Lp space*

ABSTRACT.

In this paper some estimates are obtained for the remainder term in the limit theorems for the weighted sum of random variables forming a homogeneous Markov chain with arbitrary set of possible states . The achieved results make it possible to estimate the rate of convergence in these theorems in the metric of the space L_p , $1 \leq p < \infty$ without requiring that the absolute third order moment for the transition probabilities be bounded .

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1. INTRODUCTION.

Let $x_1, x_2, \ldots$ be a homogeneous Markov process with arbitrary set of possible states X , defined by the transition probability function $p(\xi, A)$, $\xi \in X$, $A \in \Im x$ (σ - algebra of subsets of X) and the initial distribution:

$$p(x_1 \in A) = \pi(A), A \in \Im x.$$

Suppose that $p(.,.)$ satisfies the condition of uniform ergodicity

$$\sup | p(\xi, A) - p(\eta, A) | = p < 1. \quad (1.1)$$
$$(\xi, \eta \in X, A \in \Im x$$

It is well known that (see [2]) when (1.1) holds a stationary distribution $p(A), A \in \Im x$ exists . Moreover (see [9]) , if $\int_X f^2(\eta)\, p(d\eta) < \infty$, then the limit.

$$\sigma^2 = \lim_{n\to\infty} E\left\{\left[\frac{1}{\sqrt{n}}(f(x_1)+\ldots+f(x_n))\right]^2\right\} = $$
$$= E_p\left[f^2(x_1)\right] + 2\sum_{r=1}^{\infty} E_p\left[f(x_1)f(x_{r+1})\right],$$

exists , where $f(\cdot)$ is a real measurable function defined on X and Ep indicates that x_1 has the distribution $p(\cdot)$. The initial distribution $\pi(\cdot)$ and the transition probability function completely define the sequence of r.vs.

$$f(x_1),\ f(x_2), \ldots, f(x_n), \ldots \quad (1.2)$$

Below , one can assume without loss of generality that

$$\int f(\eta)\, p(d\eta) = 0, \sigma^2 = 1.$$

Let $\{a_{n,k}\}_1^{\infty}$ be a sequence of real numbers that satisfy the following conditions :

$$\sum_{k=1}^{\infty} a_{n,k}^2 = 1, \quad \gamma_n = \sup_k |a_{n,k}| \to 0 \text{ as } n \to \infty \quad (1.3)$$

Put ,

$$S_n = \sum_{k=1}^{\infty} a_{n,k} f(x_k), \quad \Delta_n = \sup_{x\in X} |w_n(x) - \Phi(x)|,$$

$$\Delta_{np} = \left(\int_{-\infty}^{\infty} |w_n(x) - \Phi(x)|^p\, dx\right)^{1/p}, \quad 1 \le p < \infty,$$

where ,$w_n(k)$ is the distribution function of S_n and $\Phi(x)$ is the standard normal distribution function .

Limit theorems for sums of r.vs. forming a homogeneous Markov chain with an arbitrary set of states and the remainder terms in these theorems were investigated in a number of works (see , for example , [1] , [5] , [6] , [9] , [10] , and [12] .) . Estimates , in the uniform metric , for the remainder term , in the central limit theorem of the weighted sum of the sequence (1.2) were given in [7] .

In the present paper some estimates are given , in the metric of the space L_p , $1 \le p < \infty$, for the remainder term in the central limit theorem for the weighted sum of the sequence (1.2) . The results make it possible to estimate the rate of convergence in the considered metric without assuming the finiteness of the absolute third order moment of the transition probabilities . Results pertaining to the case of the existence of absolute moments of the transition probabilities of orders not less than the third are given in the monograph [12] .

For $n \ge 1$, set

$$X_n = \left\{\xi : |f(\xi)| \le \gamma_n^{-1}\right\}, \quad \overline{X}_n = X \setminus X_n,$$

$$\beta_n = \sup_{\xi\in X}\left\{\int_{X_n} |f(\eta)|^3 p(\xi, d\eta) + \gamma_n^{-1}\int_{\overline{X}_n} f^2(\eta) p(\xi, d\eta)\right\},$$

$$\tau = \int_X |f(\xi)|\, \pi(d\xi) \int_X |f(\eta)|\, |s(\xi, d\eta)| + $$
$$+ \frac{1}{2}\int_X \pi(d\xi) \int_X f^2(\eta) |s(\xi, d\eta)|,$$

$$\delta_n = \int_X |f(\xi)| p(d\xi) \int_{\overline{X}_n} |f(\eta)|\, |s(\xi, d\eta)|$$
$$+ \int_{\overline{X}_n} |f(\xi)| p(d\xi) \int_X |f(\eta)|\, |s(\xi, d\eta)|,$$

$$m_{k\pi} = \int_X |f(\xi)|^k \pi(d\xi), \text{ and}$$

$$m_k = \sup_{\xi\in X} \int_X |f(\eta)|^k p(\xi, d\eta), \quad k = 1,2,$$

where , $|s(., A)|$ is the complete variation of the measure

$$s(., A) = \sum_{k=1}^{\infty}\left(p^{(k)}(., A) - p(A)\right), \quad A \in \Im_X$$

and $p^{(k)}(., A)$, $A \in \Im_X$ is the transition probability function after $k \ge 1$ steps .

Note that the measure $s(., A)$, $A \in \Im_X$ exists in accordance with condition (1.1).

2. MAIN RESULTS.

THEOREM 1. If $m_{2\pi} < \infty$ and

$$\int_X \pi(d\xi) \int_X f(\eta) p^{(k)}(\xi, d\eta) = 0 \qquad , \ k = 1, 2, \ldots$$

then,

$$\Delta_{np} \leq C(\rho) \max(\beta_n \gamma_n + \delta_n, m_1 m_1 \pi \gamma_n, m_2 \pi \gamma_n, \tau \gamma_n), \qquad (2.1)$$

where $C(\rho) \leq \dfrac{C^{1/p}}{(1-\rho)^{3+1/p}}$, $1 \leq p < \infty$

Here and below C, C_1 denote absolute positive constants, in general distinct.

THEOREM 2. If $\pi(.) = p(.)$, then

$$\Delta_{np} \leq C(\rho)(\beta_n \gamma_n + \delta_n) \qquad (2.2)$$

Repeating the arguments given in [3] - [4], one can see that the estimate (2.2) has a universal character.

THEOREM 3. If then the central limit theorem holds for the sequence of r.vs. (1.2), i.e.

$\Delta_{np} \to 0$ as $n \to \infty$, for any initial distribution $\pi(.)$.

Now, let $g(x)$, $x > 0$ be an increasing function such that:

$$\lim_{x \to \infty} g(x) = \infty \ , \ \text{and} \ \frac{x}{g(x)} \text{ is non decreasing}.$$

Put, $m_g = \sup_{\zeta \in X} \int_X f^2(\eta) g(|f(\eta)|) p(\zeta, d\eta)$.

THEOREM 4. If $\pi(.) = p(.)$, and $m_g < \infty$

$$\text{then,} \ \Delta np \leq C(\rho) \frac{1}{g(\gamma_n^{-1})} (2m_g + \delta_g), \qquad (2.3)$$

where,

$$\delta_g = \int_X |f(\xi)| p(d\xi) \int_X |f(\eta)| g(|f(\eta)|) |s(\xi, d\eta)| +$$

$$+ \int_X |f(\xi)| g(|f(\xi)|) p(d\xi) \int_X |f(\eta)| |s(\xi, d\eta)|.$$

COROLLARY. If $g(x) = x^\delta$, $x > 0$, $0 < \delta \leq 1$, then

$$\Delta_{np} \leq C(\varrho) \gamma_n^\delta m_{2+\delta}, \qquad (2.4)$$

where, $m_{2+\delta} = \sup_{\zeta \in X} \int_X |f(\eta)|^{2+\delta} p(\zeta, d\eta)$.

3. AUXILIARY RESULTS.

In proving the formulated theorems we apply the method of characteristic functions (c.fs.) in combination with the general spectral theory of linear operators using the appropriate material from [11].

Let P(t) be a linear operator on the Banach space of all measurable bounded complex valued functions $g(\xi)$, $\xi \in X$, defined by (cf. [9]).

$$P(t)g(.) = \int_X e^{it f(\eta)} g(\eta) p(., d\eta).$$

For any complete additive complex valued set function $\mu(A)$, $A \in \mathfrak{I}_X$, and function $g(.)$ mentioned above set

$$\langle g, \mu \rangle = \int_X g(\eta) \mu(d\eta).$$

It is not hard to deduce that the c.f. $\Psi_{n\pi}(t)$ of the r.v. is given by

$$\Psi_{n\pi}(t) = E_\pi(\exp(itS_n)) =$$

$$= \left\langle \prod_{k=1}^{\infty} P(a_{n,k+1}\, t) \psi \ , \ \Pi(a_{n,1}\, t) \right\rangle, \qquad (3.1)$$

where the measure $\Pi(t)$ is defined by

$$\Pi(t) = \Pi(t, A) = \int_A e^{it f(\eta)} g(\eta) \ (d\eta) \ , \ \text{and} \ \Psi \equiv 1.$$

Let R(z, t) be the resolvent operator for P(t), = R(z) = R(z, 0) (the operator P = P(0) is generated by the kernel p(.,.)), and I_1, I_2 are circles with centers at 1 and 0 and radii, $\varrho_1 = (1-\varrho)/3$, $\varrho_2 = (1+2\varrho)/3$, respectively. Set.

$$P_1(t) = \frac{1}{2\pi i} \int_{I_1} R(z,t)\, dz \quad \text{and}$$

$$P_2(t) = \frac{1}{2\pi i} \int_{I_2} R(z,t)\, dz. \qquad (3.2)$$

By lemma (1.1) in [9], (3.1) can be written as

$$\Psi_{n\pi}(t) = \prod_{k=1}^{\infty} \lambda(a_{n,k+1}\, t) + \qquad (3.3)$$

$$+ \prod_{k=1}^{\infty} \lambda(a_{n,k+1}\, t)(H_1(a_{n,k}\, t) - 1) + H_2(a_{n,k}\, t),$$

where,

$$\lambda(a_{n,k+1}\, t) = \langle P(a_{n,k+1}\, t)\ P_1(a_{n,k+1}\, t) \psi \ , \ p \rangle$$

$$/ \langle P_1(a_{n,k+1}\, t) \psi \ , \ p \rangle,$$

$$H_1(a_{n,k}\, t) = \langle P_1(a_{n,k+1}\, t) \psi \ , \ \Pi(a_{n,1}\, t) \rangle \ , \ \text{and}$$

$$H_2(a_{n,k}\, t) =$$

$$= \left\langle \prod_{k=1}^{\infty} P(a_{n,k+1}\, t) P_2(a_{n,k+1}\, t) \psi \ , \ \Pi(a_{n,1}\, t) \right\rangle.$$

Using (3.2) one can get (see [9] p. 394)

$$\lambda(a_{n,k}\,t) = \left(1 + \sum_{r=1}^{\infty} B_r(a_{n,k}\,t)\right) \Big/ \left(1 + \sum_{r=1}^{\infty} C_r(a_{n,k}\,t)\right), \tag{3.4}$$

where, $B_r(a_{n,k}\,t) = \frac{1}{2\pi i}\left\langle \int_{I_1} z R(z) A^r(z, a_{n,k}\,t) dz\, \psi, p \right\rangle$,

$$C_r(a_{n,k}\,t) = \frac{1}{2\pi i}\left\langle \int_{I_1} R(z) A^r(z, a_{n,k}\,t) dz\, \psi, p \right\rangle, \text{ and}$$

$$A(z, a_{n,k}\,t) = \left(P(a_{n,k}\,t) - P(0)\right) R(z) .$$

LEMMA 1. For $|t| < \left(10\sqrt{2}\, M^3(\rho)\, \beta_n\, \gamma_n\right)^{-1}$,

$$\left|\lambda(a_{n,k}\,t) - 1\right| \le \frac{9}{10} M^2(\rho)\, m_2\, \gamma_n\, |t|^2 , \tag{3.5}$$

$$\left|\lambda(a_{n,k}\,t) - 1 - B_1(a_{n,k}\,t) - \left(B_2(a_{n,k}\,t) - C_2(a_{n,k}\,t) \right)\right| \le \frac{18}{25} M^3(\rho)\, m_1^3\, \gamma_n\, |t|^3 , \tag{3.6}$$

and

$$\left| B_1(a_{n,k}\,t) + B_2(a_{n,k}\,t) - C_2(a_{n,k}\,t) + \frac{1}{2} a_{n,k}^2 t^2 \right| \le \le \frac{4}{3} M(\rho)\, \beta_n\, \gamma_n\, |t|^3 + 2\left(\beta_n\, \gamma_n + \delta_n\right) |t|^2 . \tag{3.7}$$

PROOF :

On noticing that $C_1(a_{n,k}\,t) = 0$, we have.

$$\lambda(a_{n,k}\,t) - 1 = B_1(a_{n,k}\,t) + \sum_{r=2}^{\infty}\left(B_r(a_{n,k}\,t) - C_r(a_{n,k}\,t) \right) + + \sum_{r=1}^{\infty}\left(B_r(a_{n,k}\,t) - C_r(a_{n,k}\,t) \right) \sum_{j=1}^{\infty} (-1)^j C^j(a_{n,k}\,t) \tag{3.8}$$

$$\lambda(a_{n,k}\,t) - 1 - B_1(a_{n,k}\,t) - \left(B_2(a_{n,k}\,t) - C_2(a_{n,k}\,t) \right) = = \sum_{r=3}^{\infty}\left(B_r(a_{n,k}\,t) - C_r(a_{n,k}\,t) \right) + + \sum_{r=1}^{\infty}\left(B_r(a_{n,k}\,t) - C_r(a_{n,k}\,t) \right) \sum_{j=1}^{\infty} (-1)^j C^j(a_{n,k}\,t), \tag{3.9}$$

where, $C(a_{n,k}\,t) = \sum_{r=2}^{\infty} C_r(a_{n,k}\,t)$.

Now, the validity of (3.5) and (3.6) follows from (3.8) and (3.9) on noticing that

$$\| R(z) \| \le M(\rho) ,$$

$$\| A(z, a_{n,k}\,t) \| \le M(\rho)\, m_1\, \gamma_n\, |t| = \theta M(\rho), \quad \text{say},$$

and for r = 1,2 , ...

$$\left|B_r(a_{n,k}\,t) - C_r(a_{n,k}\,t)\right| \le \frac{1}{3}\left(\theta M(\rho)\right)^r ,$$

$$\left|C_r(a_{n,k}\,t)\right| \le \left(\theta M(\rho)\right)^r , \tag{3.10}$$

$$\text{and} \quad \left|C(a_{n,k}\,t)\right| \le \frac{\left(\theta M(\rho)\right)^2}{1 - \theta M(\rho)} < \frac{1}{306} . \tag{3.11}$$

Now, it is easy to see that

$$\left| B_1(a_{n,k}\,t) + B_2(a_{n,k}\,t) - C_2(a_{n,k}\,t) + + \frac{1}{2} a_{n,k}^2\, t^2 \right| \le M_1 + M_2, \tag{3.12}$$

where, $M_1 = \left| B_1(a_{n,k}\,t) - \int_X \frac{\left(it a_{n,k}\, f(\eta)\right)^2}{2} p(d\eta) \right|$, ,

$$M_2 = \Big| B_2(a_{n,k}\,t) - C_2(a_{n,k}\,t) - - \int_X \left(it a_{n,k} f(\xi)\right) p(d\xi) \int_X \left(it a_{n,k}\, f(\eta)\right) s(\xi, d\eta) \Big| .$$

It is not hard to get that

$$M_1 \le \left(\frac{|t|^3}{6} + |t|^2 \right) \beta_n\, \gamma_n \quad . \tag{3.13}$$

and , $M_2 \le 2\, M(\rho)\, \beta_n\, \gamma_n\, |t|^3 + 2\, \delta_n\, t^2$. (3.14)

Relation (3.7) follows from (3.12) - (3.14) .

The next lemma can be shown analogously to lemma 1 .

LEMMA 2. For $|t| < \left(10\sqrt{2}\, M^3(\rho)\, \beta_n\, \gamma_n\right)^{-1}$,

$$\left| \frac{d}{dt} \lambda(a_{n,k}\,t) \right| \le \frac{14}{15} M^2(\rho)\, m_2\, \gamma_n\, |t| ,$$

$$\left| \frac{d}{dt}\left(\lambda(a_{n,k}\,t) - 1 - B_1(a_{n,k}\,t) - \left(B_2(a_{n,k}\,t) - - C_2(a_{n,k}\,t) \right) \right) \right| \le 3\sqrt{2}\, M^3(\rho)\, \beta_n\, \gamma_n\, |t|^2$$

and

$$\left| \frac{d}{dt}\left(B_1(a_{n,k}\,t) + B_2(a_{n,k}\,t) - C_2(a_{n,k}\,t) + \frac{1}{2} a_{n,k}^2\, t^2 \right) \right| \le \le 4M(\rho)\, \beta_n\, \gamma_n\, |t|^2 + 4\left(\beta_n\, \gamma_n + \delta_n\right) |t|$$

Using lemmas 1 and 2 and by means of certain additional arguments one can prove

LEMMA 3. For $|t| < \left(10\sqrt{2}\, M^3(\rho)\, \beta_n\, \gamma_n \right)^{-1}$,

$$\left|\prod_{k=1}^{\infty} \lambda(a_{n,k}\, t) - e^{-t^2/2}\right| \le \Big[h(\rho)\,\beta_n\,\gamma_n\,|t|^3 + 2\left(\beta_n\,\gamma_n + \delta_n\right)|t|^2\Big]\, e^{-t^2/5} \tag{3.15}$$

and

$$\left|\frac{d}{dt}\left(\prod_{k=1}^{\infty} \lambda(a_{n,k}\, t) - e^{-t^2/2}\right)\right| \le$$

$$\le \Big[\frac{50}{3}\, h(\rho)\,\beta_n\,\gamma_n\, \max(\,|t|^2\,,|t|^4) + 4\left(\beta_n\,\gamma_n + \delta_n\right)\max(\,|t|\,,|t|^3\,)\Big]\, e^{-t^2/5}, \tag{3.16}$$

where, $h(\rho) = \frac{3}{\sqrt{2}}\, M^3(\rho) + \frac{1}{10}\, M^2(\rho) + \frac{4}{3}\, M\,(\rho)$.

LEMMA 4. For $|t| < \left(\,2M^2(\rho)\, m_1\, \gamma_n\,\right)^{-1}$,

$$\left|H_2\left(a_{n,k}\, t\right) - 1\right| \le \left(\,\frac{1}{2}\, m_{2\pi} + 5M^3(\rho)\,\beta_n + \tau\right)\gamma_n\, |t|^2\,, \tag{3.17}$$

$$\left|H_2\left(a_{n,k}\, t\right)\right| \le \left(\frac{18}{17}\, M^3(\rho)\, m_2 + \tau\right)\gamma_n\, |t|^2\,, \tag{3.18}$$

$$\left|\frac{d}{dt}\, H_1\left(a_{n,k}\, t\right)\right| \le$$

$$\le \left(\,m_{2\pi} + \frac{21}{2}\, M^3(\rho)\, \beta_n + \frac{1}{17}\, M(\rho)\, m_1\, m_{1\pi} + 2\,\tau\right)\gamma_n\, |t| \tag{3.19}$$

and

$$\left|\frac{d}{dt}\, H_2\left(a_{n,k}\, t\right)\right| \le$$

$$\le \left(\,\frac{81}{34}\, M^3(\rho)\, m_2 + \frac{1}{17}\, M^2(\rho)\, m_1\, m_{1\pi} + \tau\right)\gamma_n\, |t|\,. \tag{3.20}$$

PROOF.

The proof of (3.17) and (3.18) is similar to that of (4.4) and (4.6), respectively, in [7], and the proof of (3.19) and (3.20) is obtained analogously by using the following representations:

$$H_1\left(a_{n,k}\, t\right) - 1 = J_1 + J_2 + J_3$$

$$H_2\left(a_{n,k}\, t\right) = \sum_{r=1}^{\infty} D_r\left(a_{n,k}\, t\right),$$

where,

$$J_1 = \int_X \left(e^{it a_{n,1} f(\xi)} - 1 - it a_{n,1} f(\xi)\right) \pi(d\xi)\,,$$

$$J_2 = \int_X e^{it a_{n,1} f(\xi)}\, \pi(d\xi) \int_X \left(e^{it a_{n,k+1} f(\eta)} - 1\right) s(\xi, d\eta)\,,$$

$$J_3 = \sum_{r=2}^{\infty} \frac{1}{2\pi i} \int_X e^{it a_{n,1} f(\xi)} \left(\int_{I_1} R(z)\, A^r\left(z\,,\, a_{n,k+1}\, t\right) dz\, \psi\right)(\xi)\, \pi(d\xi)\,,$$

$$D_r\left(a_{n,k}\, t\right) = $$

$$= \frac{1}{2\pi i}\left\langle \int_{I_2} zR(z) B^r\left(z\,,\, a_{n,k}\, t\right) dz\, \psi\,,\, \Pi\left(a_{n,1}\, t\right)\right\rangle,$$

and, $B\left(z\,,\, a_{n,k}\, t\right) = \prod_{k=1}^{\infty} A\left(z\,,\, a_{n,k+1}\, t\right)$.

4. PROOFS OF THE THEOREMS.

We first prove theorem 1.

We have $\quad \Delta_{np}^{p} \le \Delta_n^{p-1}\, \Delta_{n_1}$. (4.1)

According to theorem 1 in [7], we have

$$\Delta_n \le \frac{C_1}{(1-\rho)^3}\left(\,\beta_n\,\gamma_n + \delta_n\,\right) + C_2\, m_{1\pi}\, \gamma_n\,. \tag{4.2}$$

Now, from (3.3), using lemmas (3) and (4) and some additional arguments, one can see that.

$$\left|\psi_{n\pi}(t) - e^{-t^2/2}\right| \le \{3\, h(\rho)\,\beta_n\,\gamma_n\, \max\left(|t|^2\,,|t|^3\right) + \left[\,2\left(\beta_n\,\gamma_n + \delta_n\right) + \left(\frac{1}{2}\, m_{2\pi} + \tau\right)\gamma_n\,\right]|t|^2\}\, e^{-t^2/5} + \left[\,\frac{18}{17}\, M^3(\rho)\, m_2 + \tau\right]\gamma_n\, |t|^2 \tag{4.3}$$

and

$$\left|\frac{d}{dt}\left(\psi_{n\pi}(t) - e^{-t^2/2}\right)\right|$$

$$\le 26\, h(\rho)\,\beta_n\,\gamma_n\, \max\left(|t|\,,|t|^4\right) e^{-t^2/5} + \left[\,4\left(\beta_n\,\gamma_n + \delta_n\right) + \frac{9}{5}\left(\,m_{2\pi} + 2\,\tau\right)\gamma_n\,\right] \max\left(|t|\,,|t|^3\,\right) e^{-t^2/5} + \frac{1}{17}\, M(\rho)\, m_1\, m_{1\pi}\, \gamma_n\, |t|\; e^{-t^2/5} + \left[\,\frac{81}{34}\, M^3(\rho)\, m_2 + \frac{1}{17}\, M^2(\rho)\, m_1\, m_{1\pi} + \tau\right]\gamma_n\, |t|\,. \tag{4.4}$$

From (4.3) and (4.4), using the well known smoothing lemma, due to Esseen, of the metric of the space (see for example [8], p. 28, theorem 1.5.4) one can get.

$$\Delta_{n1} \le \frac{C}{(1-\rho)^4}\, \max\left(\,\beta_n\gamma_n + \delta_n\,,\, m_1\, m_{1\pi}\, \gamma_n\,,\, m_{2\pi}\, \gamma_n\,\right). \tag{4.5}$$

Finally from (4.1), (4.2) and (4.5) we get (2.1).

The proof of theorem 2 goes along the lines of the proof of theorem 1 , where the condition is taken into account .

It suffices to prove theorem 3 under the condition (the general case reduces to this partial one (see [2]) .) . By theorem 1 it suffices to show that

$$\beta_n \gamma_n + \delta_n \to 0 \quad as \quad n \to \infty \tag{4.6}$$

Since $m2 < \infty$ then $\sup_{\xi} \int_{\bar{X}_n} f^2(\eta)p(\xi,d\eta) \to 0$

Hence , $\beta_n \gamma_n \to 0 \quad as \quad n \to \infty$ (4.6)

On the other hand , one can show that

$$\delta_n \le \frac{4}{1-\rho} m_2^{1/2} \left(\sup_{\xi \in X} \int_{\bar{X}_n} f^2(\eta)p(\xi,d\eta) \right)^{1/2} . \tag{4.8}$$

Relation (4.6) follows from (4.7) and (4.8) .

Now , let us prove theorem 4 .
In accordance with the properties of the functions , $\frac{x}{g(x)}$ and $g(x)$ one can , easily, get that

$$\sup_{\xi \in X} \int_{X_n} |f(\eta)|^3 p(\xi,d\eta) \le \frac{\gamma_n^{-1}}{g(\gamma_n^{-1})} m_g ,$$

and , $$\sup_{\xi \in X} \int_{\bar{X}_n} f^2(\eta)p(\xi,d\eta) \le \frac{\gamma_n^{-1}}{g(\gamma_n^{-1})} m_g .$$

Hence, $$\beta_n \gamma_n \le \frac{2}{g(\gamma_n^{-1})} m_g . \tag{4.9}$$

Similarly , it can be shown that

$$\delta_n \le \frac{1}{g(\gamma_n^{-1})} \delta_g . \tag{4.10}$$

Finally , (2.3) follows from (2.2) , (4.9) and (4.10) .

The corollary is proved analogously .

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