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THE ADJOINT PROBLEM OF A MODIFIED STURM-LIOUVILLE BOUNDARY VALUE PROBLEM

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المسألة المرافقة لمسألة حدية معدلة لشتورم ليوفيل

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في هذا البحث أعطيت الحلول لمعادلة القيم الذاتية والطيف المتقطع للمسألة المعطاة . ولقد تم الحصول على الرزولفنت والذي باستخدامه تحت شروط معينة حصلنا على المسألة المرافقة للمسألة محل الدراسة .

Key words: Adjoint problem, discrete spectrum, resolvent.

ABSTRACT

The authors give the solutions of the eigenvalue equation that is associated with a modified Stum-Liouille problem. The discrete specturm and hence the resolvent will be obtained by means of different types of solutions for the problem with different initial conditions. The adjoint problem in deduced by using the kernel resolvent.

INTRODUCTION

Consider, in the space L_2 $(0, \infty, \rho(x))$, the following modified Sturm-Liouville boundary value problem

$$-u'' + q(x)\,u = \lambda\rho(x)\,u \qquad x \in [0, \infty) \qquad (1)$$

$$u'(0) - \lambda\sum_{n=1}^{m} \alpha_n u(a_n) = 0, \qquad (2)$$

where q(x) is a complex valued function on $[0,\infty)$ which satisfies the condition

$$\int_0^\infty (1+x)|q(x)|dx < \infty \qquad (3)$$

Throughout this work, the weight function $\rho(x)$ takes the form

$$\rho(x) = \begin{cases} \beta^2 & , 0 \le x \le b \\ 1 & , b < x < \infty, \end{cases}$$

where $\beta \neq 1$; $\beta > 0$

Furthermore, λ is a complex parameter and α_n; a_n are real constants.

The inclusion of the spectral parameter, λ in the boundary condition makes this sort of problems useful in many applications, specially in the field of mathematical physics.

The spectrum of the boundary value problem (1)-(2) was investigated in the case $\rho(x) > \beta > 0$ and the boundary condition u(0)=0 holds [3]. A regular boundary value problem in which the spectral parameter appears explicitly in the boundary condition was examined in [2]. The more general case of two-points boundary value problems was thoroughly studied in [7]. The boundary value problem (1) - (2) was investigated in [1] but in this work a new technic will be introduced by using the resolvent of the stated problem to construct the adjoint problem.

The paper is organised such that in section 1, solutions for equation (1) when condition (3) holds will be presented. The discrete spectrum of the boundary value problem (1)-(2) will be given in section 2 and hence the resolvent of this boundary value problem will be obtained. In section 3, the adjoint problem which corresponds to the boundary value problem (1) - (2) will be constructed by using its resolvent.

SOME SOLUTIONS OF EQUATION (1)

Condition (3) implies directly that equation (1) could be reduced asymptotically to the equation [5]

$$-u'' = \lambda\, \rho u \qquad\qquad as\ x \to \infty.$$

Hence, the investigation of the properties of the solution of equation (1) could be carried out. We denote by $\psi(x,k)$ and $\varphi(x,k)$ the solutions of equation (1) that on the interval [0, b] satisfy the following initial conditions

$$\psi(0,k) = 1 \qquad ;\ \psi'(0,k) = 0$$

$$\varphi(0,k) = 0 \qquad ;\ \varphi'(0,k) = 1$$

$$where\ \lambda^{\frac{1}{2}} = k = \xi + i\eta\ and\ 0 \le\ arg\, k < \pi.$$

By making use of [4] and [5], one can prove the following lemmas.

Lemma 1

The solution $\psi(x,k)$ of equation (1) on the interval [0, b] can be expressed in the form

$$\psi(x,k) = cos\,(k\beta x) + \int_0^x A(x,t)\cos(k\beta t)\, dt\,,\ 0 \le\ t \le\ x \le\ b\,,$$

where the kernel A (x,t) has summable derivatives $\acute{A}_x$, $\acute{A}_t$ and satisfies the conditions:

$$A(x,x) = \frac{1}{2}\int_0^x q(t)\ dt \qquad ;$$

$$\frac{\partial}{\partial t} A(x,t)\big|_{t=0} = 0$$

Moreover, the solution $\psi(x,k)$ is an entire function of k that behaves asymptotically, as $|k| \to \infty$, like

$$\psi(x,k) = cos(k\beta x) + O(\frac{1}{k}\exp|Im\ k\beta|x) \qquad ,$$

uniformly with respect to x on [0,b].

Lemma 2

The solution $\phi(x,k)$ can be written on the form

$$\varphi(x,k) = \frac{sin(k\beta x)}{k\beta} + \int_0^x B(x,t)\ \frac{\sin(k\beta t)}{k\beta}\ dt \quad ,\quad 0 \le t \le\ x\ \le\ b \quad ,$$

where, the kernel B (x,t) has summable derivative B_x, B_t and satisfies the conditions

$$B(x,t) = \frac{1}{2}\int_0^x q(t)\, dt \qquad ;\ B(x,0) = 0 \quad .$$

Also, this solution is an entire function and as $|k| \to \infty$ we have the following asymptotic form

$$\varphi(x,k)\ =\ \frac{sin\ (k\beta x)}{k\beta} + O\ (\frac{1}{k^2}\exp|Im\ k\beta|x) \qquad ,$$

uniformly with respect to x on [0, b].

In the following lemma, we obtain the solution F (x,k) of equation (1) which satisfies the conditions

$$\lim_{x \to \infty} e^{-ikx}\ F(x,k) = 1$$

and

$$\lim_{x \to \infty} e^{-ikx}\ F_x'(x,k) = ik.$$

The proof is to be found in [6].

Lemma 3

The solution F(x,k) of equation (1) which satisfies condition (3) and for x>b can be expressed in the form

$$F(x,k) = exp(ikx) + \int_x^\infty\ R(x,t)\ exp(ikt)\ dt \quad ,\quad b < x \le t < \infty \quad , \qquad (4)$$

where the kernel R (x,t) has continuous derivatives with respect to x and t, also it satisfies the inequalities

$$|\ R(x,t)| \ \le \frac{1}{2}\exp(\sigma_1(x))\ \sigma\left(\frac{x+t}{2}\right) \quad , \qquad (5)$$

where

$$\sigma(x) = \int_x^\infty\ |\ q(t)|\ dt \qquad and \qquad \sigma(x) = \int_x^\infty |t|\ q(t)|\ dt, \quad .$$

Moreover, if q(x) is differentiable, then R (x, k) satisfies

and
$$\frac{\partial^2 R}{\partial x^2} - \frac{\partial^2 R}{\partial t^2} = q(x)\ R(x,t) \qquad ,$$

$$\frac{dR(x,x)}{dx} = -\frac{1}{2}q(x)$$

One can see that the solution F (x,k) is an analytic function of k in the upper half plane $\eta>0$ and is continuous on the real line. For $|k| \to \infty$, the solution has the following asymptotic behaviour

$$F(x,k) = exp(ikx)\left[1 + O(\frac{1}{k})\right] \qquad ,$$

$$F_x'(x,k) = ik\ \exp\ (ikx)\left[1 + O(\frac{1}{k})\right] \quad .$$

Further, [6] equation (1) has a solution G (x,k), $x \in (b,\infty)$ such that in the domain $\eta \ge 0$, $|k| > o$, this solution

$$G(x,k) = exp(-ikx)\left[1 + O(\frac{1}{k})\right] \qquad ,$$

and

$$G_x'(x,k) = -ik\ exp\ (-ikx)\left[1 + O(\frac{1}{k})\right] \qquad ,$$

uniformly with respect to x > b.

Next, we employ lemma (1), (2) and (3) along with the technic of continuation [3] on $(0,\infty)$ in order to obtain the solution F(x,k), hence

$$F(x,k) = \begin{cases} F(b,k)\ \psi(x,k) + F'(b,k)\ \phi(x,k) & ,\quad 0 \le x \le b \\ exp(ikx) + \int_x^\infty R(x,t)\ exp(ikt)\ \ dt & ,\quad b < x < \infty \end{cases} \qquad (6)$$

Furthermore, as $|k| \to \infty$, we have

$$F(x,k) = \begin{cases} \exp(ikb)\left[\cos k\beta(x-b) + \frac{i}{\beta}\sin\ k\beta(x-b)\right](1 + O(\frac{1}{k})) & ,\quad 0 \le x \le b \\ exp(ikx)(1 + O(\frac{1}{k})) & ,\quad b < x < \infty \end{cases} \qquad (7)$$

Similarly, we find

$$G(x,k) = \begin{cases} \exp(-ikb)\left[\cos k\beta(x-b) + \frac{i}{\beta}\sin\ k\beta(x-b)\right](1 + O(\frac{1}{k})) & ,\quad 0 \le x \le b \\ exp(-ikx)(1 + O(\frac{1}{k})) & ,\quad b < x < \infty \end{cases}$$

ON THE DISCRETE SPECTRUM AND THE RESOLVENT OF THE BOUNDARY VALUE PROBLEM (1) - (2).

In this section, we give the discrete spectrum and construct the resolvent of the problem (1) - (2) by using F(x.k). Lemma (4) and lemma (5) are used in the study, however the proofs are shown in [6].

Lemma 4

The boundary value problem (1) - (2) has not positive eigenvalues.

Lemma 5

The necessary and sufficient conditions that $\lambda = k^2 \neq o$ be an eigenvalue to the boundary value problem (1) - (2) is

$$Z(k) = F'(0,k) - k^2 \sum_{n=1}^{m} \alpha_n F(a_n,k) = 0 \quad ,\quad \eta > 0 \quad . \qquad (8)$$

Since Z (k) is a holomorphic function in the upper half plane, thus the following lemma can be easily proved [1].

Lemma 6

The boundary value problem (1) - (2) has no more than a countable complex set of eigenvalues for which its limit points lie only on the real line.

Theorem 1

If q(x) satisfies the condition $\exp(\varepsilon x)q(x) \in L_1(0,\infty)$ for $\varepsilon>0$, then the zeros of Z (k) consist of finite complex numbers of eigenvalues and possibly finite real numbers of singular spectrum.

Proof

From lemma 6, it is sufficient to show that the set of complex zeros of z(k), n>o has no limit point on. Since for equation (1) has a solution on the form (4). Thus, in view of (4) and from the assumption of the theorem, we find.

$$|R(x,t)| < c\exp\left[-\frac{\varepsilon}{2}(x+t)\right] , \qquad (9)$$

where c is a positive constant

$$Also \quad \int_x^\infty \left[|R_x'(x,t)| + |R_t'(x,t)| \right] \exp\left(\frac{\varepsilon}{2}t\right) dt < \infty \quad . \qquad (10)$$

In view of (9) and (10), one can show that the integral in (4) converges for $n > -\varepsilon/2$.

Applying the analytic continuation theorem implies that the function F (x,k) is a solution of (1) not only for $\eta \geq 0$ but also for $\eta > -\varepsilon/2$. Thus Z(k) is a holomorphic function for $\eta > -\varepsilon/2$. Therefore the set of complex zeros of Z(k) has no limit points on the real line. Hence, the set of complex zeros Z(k) is finite in a closed domain. Furthermore, Z(k) may have only real zeros of singular spectrum on the real axis. This completes the proof.

Theorem 2

If $Z(k) \neq 0$, then the resolvent T_λ is an integral operator which may be expressed in the form

$$T_\lambda(\rho v) = \int_0^\infty T(x,t,k)\,\rho(t)\,v(t)\,dt \quad , \quad v(t) \in L_2(0,\infty) \quad ,$$

where

$$T(x,t,k) = T_0(x,t,k) + \frac{k^2 F(x,k)}{Z(k)} \sum_{n=1}^{m} \alpha_n T_0(a_n,t,k) \quad , \qquad (11)$$

with

$$T_0(x,t,k) = \begin{cases} \dfrac{1}{2ik} F(x,k)\left[\dfrac{G'(0,k)\,F(t,k)}{F'(0,k)} - G(t,k)\right], & t \leq x - \\ \dfrac{1}{2ik} F(t,k)\left[\dfrac{G'(0,k)\,F(x,k)}{F'(0,k)} - G(x,k)\right], & t \geq x \end{cases} \qquad (12)$$

One should notice here that all numbers $\lambda=k^2$, n > 0 and $Z(k)\neq 0$ belong to the resolvent set of the problem (1) - (2).

Proof

It follows directly from lemma (5) that all numbers $\lambda=k^2$, $Z(k)\neq 0$, n > 0 belong to the resolvent set of the problems (1) - (2). Since $\lambda = k^2$ is not an eigenvalue of the problem (1) - (2), thus the resolvent T_λ should exist. This implies the existence of a non-trivial solution to the equation

$$-u'' + q(x)u - \lambda\rho(x)u = \rho v \qquad (13)$$

which belongs to $L_2(0, \infty, \rho(x))$ and satisfies condition (2). Let $u_0(x,k) \in L_2(0, \infty, \rho(x))$ be that solution of equation (13) such that ú (0) = 0. This solution has the form

$$u_0(x,k) = \int_0^\infty T_0(x,t,k)\,\rho(t)\,v(t)\,dt \quad ,$$

where T_0 is given by (12).

Hence equation (13) has a general solution $u(x,k) \in L_2(0, \infty, \rho(x))$ which can be written on the form [6]

$$u(x,k) = u_0(x,k) + C\,F(x,k)$$

where C is an arbitrary constant which could be determined. Since u (x, k) satisfies condition (2), hence

$$C = \frac{\lambda_n \sum_{n=1}^{m} \alpha_n \int_0^\infty T_0(a_n,t,k)\,\rho(t)\,v(t)\,dt}{F'(0,k) - \lambda_n \sum_{n=1}^{m} \alpha_n F(a_n,k)}$$

Upon substituting into (13), we get

$$u(x,k) = \int_0^\infty T(x,t,k)\,\rho(t)\,v(t)\,dt, \quad , \qquad (14)$$

where

$$T(x,t,k) = \begin{cases} \dfrac{1}{2ik} F(x,k)\left[\dfrac{G'(0,k)\,F(t,k)}{F'(0,k)} - G(t,k)\right] \\ \quad + \dfrac{k^2 F(x,k)}{Z(k)}\left[\sum_{n=1}^{m} \dfrac{\alpha_n F(a_n,k) G'(0,k) F(t,k)}{2ikF'(0,k)} - G(t,k)\right] & ; t \leq x \\ \dfrac{1}{2ik} F(t,k)\left[\dfrac{G'((0,k)\,F(x,k)}{F'(0,k)} - G(x,k)\right] \\ \quad + \dfrac{k^2 F(x,k)}{Z(k)}\left[\sum_{n=1}^{m} \dfrac{\alpha_n F(t,k) G'(0,k)\,F(a_n,k)}{2ikF'(0,k)} - G(a_n,k)\right] & ; t \geq x \end{cases} \quad . \qquad (15)$$

CONSTRUCTION OF THE ADJOINT PROBLEM TO THE PROBLEM (1) - (2).

This section is devoted to the construction of the adjoint problem to the problem (1) - (2). This will be achieved by making use of the resolvent T (x,t,k).

Denote by L^*_λ the operator, adjoint of the operator L_λ which is generated by the problem (1) - (2). Let us denote by $D(L^*_\lambda)$ as the domain of all functions $y(x) \in (L_2(0, \infty, \rho(x))$ which satisfy the conditions

(i) the function y' (x) has a continuous derivative in all entire intervals $[0,a_1),(a_1,a_2) \;,\ldots\; (a_n,\infty)$;

(ii) $\bar{y}'(a_n+0) - \bar{y}'(a_n-0) = \lambda\alpha_n \bar{y}(0)$;

(iii) $y'(0) = 0$;

(iv) $y(t)$ is twice differentiable when $x \neq a_n, n = 1,m$ and $-y'' + \bar{q}(x)y \in L_2(0,\infty,\rho(x))$.

Theorem 3

Let the above conditions be satisfied. Then the adjoint problem of the problem (1) - (2) has the form

$$-y'' + \overline{q(x)}\,y + k^2 \sum_{n=1}^{m} \alpha_n \delta(x-a_n)\,y(0) = \lambda\rho y \quad ,$$

$$y'(0) = 0 \quad .$$

Proof

If $u \in D(L_\lambda)$ *and* $y \in D(L^*_\lambda)$, *then*

$$(L_\lambda u, y) = \int_0^\infty (-u'' + qu)\, y\, dt$$
$$= (-u'', y) + (qu, y)$$
$$= \left[-u'\bar{y} + u\bar{y}l\right]_0^\infty - \int_0^\infty u\, \bar{y}''\, dt + \int_0^\infty qu\, \bar{y}\, dt$$
$$= u'(0)\, \bar{y}(0) - u(0)\, \bar{y}'(0) + \int_0^\infty (-y'' + qy)\, u\, dt$$
$$= \int_0^\infty (-\bar{y}'' + \overline{qy})\, u\, dt - u(0)\, \bar{y}'(0) + \lambda \sum_{n=1}^{m} \alpha_n\, u(a_n)\, \bar{y}(o)$$
$$= \int_0^\infty (-\bar{y}'' + \overline{qy})\, u\, dt + \lambda\, \bar{y}(0) \sum_{n=1}^{m} \alpha_n \int_0^\infty \delta(t - a_n)\, u(t)\, dt - u(0)\bar{y}'(0) = 0.$$

Now to prove that $\bar{y}'(0)=0$, differentiate (15), as $t \le x$ to obtain

$$T_t'(0, x, \bar{k}) = 0$$

Similarly as $t \ge x$, we obtain

$$T_t'(o, x, \bar{k}) = 0$$

Since

$$y(t) = \int_0^\infty \overline{T(t, x, \bar{k})}\, \rho(x)\, v(x)\, dx \quad , \quad v(x) \in L_2(0, \infty, \rho(x))$$

thus

$$\bar{y}'(0) = \int_0^\infty T_t(0, x, \bar{k})\, \rho(x)\, v(x)\, dx = 0$$

Hence

$$(L_\lambda u, y) = \int_0^\infty (-\bar{y}'' + q(t)\, \bar{y}(t) + k^2 \sum_{n=1}^{m} \alpha_n\, \delta(t - a_n)\, \bar{y}(0))\, u\, dt$$

$$= (u, -y'' + \bar{q}(t) y + k^2 \sum_{n=1}^{m} \alpha_n\, \delta(t - a_n)\, y(0))$$

$$= (u, L^*_\lambda\, y).$$

Thus, we conclude that the adjoint problem of the problem (1)-(2) is in the form

$$-y'' + \bar{q}(x)\, y + k^2 \sum_{n=1}^{m} \alpha_n\, \delta(x - a_n)\, y(0) = \lambda \rho y,$$
$$y'(0) = 0\ .$$

Note: In the forthcoming article we plan to study a general form of (1) - (2) in which the boundary condition takes the form

$$y'(0) - \lambda \sum_{n=1}^{m} \alpha_n\, y(a_n) + \lambda \int_0^\infty k(x) y(x) dx = 0$$

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