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# A Rational Exponential Model for Unconstrained Non-Linear Optimization

**Huda Khalil Al-Mashhadany**
Mechanical Engineering Department
College of Engineering,
Mosul University, Mosul-Iraq.

# نموذج جديد لحل مسائل الأمثلية اللاخطية وغير المشروطة

**هدى خليل المشهداني**
قسم الهندسة الميكانيكية - كلية الهندسة
جامعة الموصل - الموصل - العراق

في هذا البحث تم اقتراح نموذج جديد أكثر عمومية من النماذج التربيعية لحل مسائل الأمثلية اللاخطية وغير المشروطة والذي يعمل على تحوير وتطوير طرق التدرج المترافق الكلاسيكية وله نفس خواص خوارزميات التدرج المترافق المطبقة على دالة تربيعية. النموذج الجديد المستخدم في خوارزميات التدرج المترافق الكلاسيكية هو النموذج الأسي الجديد والذي تم اشتقاقه وحله عددياً باستخدام بعض الدوال المختارة لاختبار كفاءة الخوارزمية الجديدة. وبصورة عامة تشير النتائج إلى أن الخوارزمية الجديدة هي تحسين للخوارزميات السابقة.

**Key Words:** *Exponential model – Non-Linear Optimization.*

## ABSTRACT

In this Paper, a new algorithm based on the non-quadratic model is suggested for solving unconstrained optimization problems which modifies the classical conjugate gradiant methods. This technique has the same properties as the classical CG-method that can be applied to a quadratic function. This algorithm is derived and evaluated numerically for some standard test functions. The results indicate that in general the new proposed algorithm is an improvement on the previous methods so it remains to be investigated.

## 1. Introduction

Conjugate Gradient methods are iterative methods, which generaqted a sequence of approximations to minimise a function f(x). The conjugate gradient method was first developed by Hestenes and Stiefel, 1952 for the solution of linear system. Fletcher and Reeves, 1964 adapted the method for solving unconstrained non-linear problems. The aim of the CG-method is trying to associate conjugacy properties with the steepest descent method to achieve both efficiency and reliability. The CG-method is based on the conjugacy property which is defined as follows:

**Definition:** A set of vectors $d_i^*$ in $E^n$ is said to be conjugate with respect to the symmetric positive definition matrix A if and only if $d_i^T$ and $Ad_j = 0$ for all $i \neq j$.

We can define the classical CG-algorithm as follows:

For given $x_0 \in R^n$ an initial estimate of the minimizer $x^*$

**Step (1):** Set $d_0 = -g_0$

**Step (2):** For i = 1, 2, …

Compute $x_i = x_{i-1} + \lambda_{i-1} d_{i-1}$

where $\lambda_{i-1}$ is the optimal step size obtained by the line search procedure.

**Step (3):** Calculate the new direction

$$d_i = -g_i + \beta_i d_i$$

where $\beta_i$ is the conjugacy coefficient and it is the following formula:

$$\beta_i = \frac{g_i (g_i - g_{i-1})}{g_{i-1}^T (g_i - g_{i-1})}$$ (Hestenes and Stiefel (H/S), 1952).

This form is considered a general form for the classical conjugate gradient.

$$\beta_i = \frac{\| g_i \|^2}{\| g_{i-1} \|^2}$$ (Fletcher and Reeves (F/R), 1964)

$$\beta_i = \frac{g_i^T (g_i - g_{i-1})}{(g_{i-1}^T - g_{i-1})}$$ (Polak and Ribier (P/R), 1969)

$$\beta_i = \frac{\| g_i \|^2}{d_{i-1}^T g_{i-1}}$$ Dixon (Dx), 1975)

When, quadratic functions and exact line search are used, all the above formula $\beta_i$'s are equivalent. However, these formulas vary according to general functions.

Several algorithms have suggested as alternative ways of modifying the classical CG-models (see Al-Assady, 1993 and Al-Assady and Al-Bayati, 1994).

The CG-methods have in general the following basic properties:

(1) The conjugacy condition.

(2) The orthogonality condition.

(3) The descent direction.

(4) The quadratic termination condition with exact line search (ELS).

## 2. Minimization of Quasi-Quadratic Function

In order to obtain better global rate of convergence for minimization algorithms when applied to more general (Quasi-Quadratic) functions than quadratic, we propose in this paper a new algorithm that is invariant to non-linear scaling quadratic functions. If f(x) is a quadratic function, then a function f(q(x)) is defined as a non-linear scaling of q(x) if the following condition holds.

$$f = F(q(x)), \ {}^{df} = f' > 0 \text{ and } q(x) > 0 \qquad \ldots (1)$$

where $x^*$ is the minimizer of q(x) with respect to x, Spedicato, 1976. The following properties are immediately derived from the above condition:

i) every contour line of q(x), then it is a contour line of f.

ii) if $x^*$ is a minimizer of q(x), then it is a minimizer of f.

iii) that $x^*$ is a global minimum of q(x) does not necessarily mean that it is a global minimum of f.

Various authors have published related work in this area (see Al-Bayati, 1992; Al-Assady, et. al., 1993; Al-Assady and Al-Bayati, 1994; Hu et. al., 1994).

A conjugate gradient method which minimizers the function $f(x) = (q(x))^p$, $p > 0$ and $x \in R^n$ ..... (2) in at most n step has been described by Fried (1971) and the special case.

$$F(q(x)) = \in_1 + 1/2 \in_2 q^2(x) \qquad \ldots (3)$$

where $\in_1$ and $\in_2$ are scalars, has been investigated by Boland et. a., 1979.

Tassopoulos and Storey, 1984a and 1984b, have proposed two specific models that are denoted by (T/S), they are following

$$F(q(x)) = \frac{\in_1 q(x) + 1}{\in_2 q(x)}, \ \in_2 > 0 \ and \ q > 0 \qquad \ldots (4)$$

where $\in_1$ and $\in_2$ are scalars and $q(x) = 1/2\, x^T G x + b^T x + c$ is a quadratic function and

$$F(q(x)) = \frac{\in q(x)}{1 + q(x)}, \ \in > 0 \ and \ q > 0 \qquad \ldots (5)$$

In this paper a new exponential model is investigated and tested on a set of standard test functions, on the assumption that condition (1) holds. An extended conjugate gradient algorithm is developed which is based on this new model which scales q(x) by the exponential function for the rational q(x) functions:

$$F(q(x)) = \exp\left(\frac{\in_1 q(x)}{\in_2 q(x) - 1}\right), \in_2 < 0 \qquad \dots (6)$$

We first observed that q(x) and f(q(x)) given by (6) have identical contours, though with different function rules, and they have the same unique minimum point denoted by $x^*$.

## 3. New Proposed Algorithm

For any f satisfying the condion (1) it is shown in Boland et. al., directions and the same sequence of approximations $x_i$ to the minimizer $x^*$, as does the original methods of Fletcher-Reeves, 1964, (F/R), when applied to f(x) = q(x).

In order to modify the property (iii) in the following way: "that x* is a global minimizer of q(x) implies that is a global minimizer of f", we have suggested new exponential model defined in the equation (6) based on Renpu Ge's Theorem, 1989 which is illustrated below:

$$F(x) \frac{f_1(x_1)}{g_2(x_2)}$$

$$\text{where} \quad x^T = (x_1^T, x_2^T) \qquad \dots (7)$$

$$\text{and} \quad f_1(x) > 0 \;\&\; g_2(x_2) > 0 \qquad \dots (8)$$

It follows from equation (7) and (8) that;

exp (F(x)) = exp ($f_1$ ($x_1$)/$g_2$ ($x_2$)) is a separable function.

Thus according to the following theorem which states:

**Theorem (1):**

$x^{*T} = (x_1^{*T}, x_2^{*T}, \dots, x_n^{*T})$ is a global minimizer of a rational separable function F(x) if and only if every $x_i^*$, i = 1,2 …, n is a global minimizer of fi ($x_i$).

**Proof:** See Renup-Ge, 1989.

We can conclude that $x^*$ is a global minimizer of exp(F(x)) if and only if $x_1^*$ and $x_2^*$ are global minimizers of exp $\exp\left(\frac{f_1(x_1)}{g_2(x_2)}\right)$. Further more, the monotonicity of exp(t) implies that $x^*$ is a global minimizer of F(x) if and only if $x_1^*$ and $x_2^*$ are respectively global minimizers of $f_1(x_1)$ and $g_2(x_2)$.

### 3.1 The Out Line of the New Algorithm

Given $x_0 \in R^n$ and initial estimate of the minimizer $x^*$.

**Step (1):** Set $d_0 = -g_0$

**Step (2):** For i = 1,2, …

Compute $xi = x_{i-1} + \lambda_{i-1} d_{i-1}$

where $\lambda_{i-1}$ is the optimal step size obtained by the line search procedure.

**Step (3):** Calculate

$$\ln(f) = (1-f) + \frac{(1-f)^2}{2} + \frac{(1-f)^3}{3} + \ldots$$

$$n = \lambda_{i-1}\, g_{i-1}^T\, d_{i-1}$$
$$g = \ln(f_i) - \ln(f_{i-1})$$
$$c = w\, f_{i-1} - n$$

**Step (4):** If $|w) \le 0.1$ E-5 or $|c| < 0.1$E-5,

then step $p_i = 1$ and go to step (6)

Else go to step (5)

**Step (5):** Compute

$$\rho_i = \left(\frac{f_{i-1}}{f_1}\right)\left(\frac{n}{wf_{i-1}}\right)^2$$

where the derivation of scaling $\rho_i$ will be presented below.

**Step (6):** Calculate the new direction

$$d_i = -g_i + \beta_i\, d_{i-1}$$

where bi's defined by different formulae according to variation and it is expressed as follows:

$$\beta_i = \rho_i \frac{\|g_i\|^2}{\|g_{i-1}\|^2} \qquad \text{(modified F/R, 1964)}$$

$$\beta_i = \frac{g_i^T(\rho_i g_i - g_{i-1})}{g_{i-1}^T(g_i - g_{i-1})} \qquad \text{(modified H/S, 1952)}$$

$$\beta_i = \frac{g_i^T(\rho_i g_i - g_{i-1})}{g_{i-1}^T\, g_{i-1}} \qquad \text{(modified P/R, 1969)}$$

Conjugate gradient methods are usually implemented by restarts in order to avoid an accumulation of errors affecting the search directions. It is therefore generally agreed that restarting is very helpful in practice, so we have used the following restarting criterion in our practical investigations.

$$d_i^T g_i \ge -0.8\,\|g_i\|^2 \qquad \text{If the new direction satisfies:} \qquad \ldots (9)$$

then a restart is also initiated. This new direction is sufficiently downhill.

### 3.2 The Derivation of the New Algorithm

The implementation of extended conjugate gradient method has been performed for general functions F(q(x)) of the form of equation (6). The unknown quantities $\rho_i$ were expressed in terms of available quantities of the algorithm (i.e. function and gradient values of the objective function).

It is first assume that neither $\in_1$ nor $\in_2$ is zero in eq.(6). Solving eq.(6) for q(x), then

$$q(x) = \frac{\ln(f)}{\in_1 (\ln(f) - \in_1 / \in_2)} \quad \dots (10)$$

and using the expression for $\rho_i$

$$\rho_i = \frac{f'_{i-1}}{f'_i}$$

$$= \left(\frac{f_{i-1}}{f_i}\right)\left(\frac{\ln(f_{i-1}) - \in_1 / \in_2}{\ln(f_i) - \in_1 / \in_2}\right)^2 \quad \dots (11)$$

the quantity which has to be determined explicitly is $(\in_1/\in_2)$.

During every iteration $(\in_1/\in_2)$ musst be evaluated as a function of known available quantities. from the relation

$$g_i = f'_i \, G(x_i - x^*) \quad \dots (12)$$

$$g_{i-1} = f'_{i-1} \, G(x_{i-1} - x^*) \quad \dots (13)$$

where G is the Hessian matrix and $x^*$ is the minimum point, we get:

$$\rho_i = \frac{f'_{i-1}}{f'_i}$$

$$= \left(\frac{g^T_{i-1}}{G^T_i}\right)\left(\frac{x_1 - x^*}{x_{i-1} - x^*}\right) \quad \dots (14)$$

Furthermore,

$$g^T_{i-1}(x_i - x^*) = g^T_{i-1}(x_{i-1}\lambda_{i-1} d_{i-1} - x^*)$$

$$= g^T_{i-1}((x_{i-1} x^*) + \lambda_{i-1} g^T_{i-1} d_{i-1}$$

and

$$g^T_{i-1}(x_i - x^*) = g^T_i((x_i \lambda_{i-1} d_{i-1} - x^*)$$

$$= g^T_i (x_i - x^*)$$

$$\text{since } g^T_i d_{i-1} = 0$$

Therefore, we can write expression $\rho_i$ as follows:

from (4) and (5), we get:

$$\rho_i = \frac{g^T_{i-1}(x_{i-1} - x^*) + \lambda_{i-1} g^T_{i-1} d_{i-1}}{g^T_i (x_i - x^*)} \quad \dots (15)$$

Therefore,

$$\rho_i = \frac{f'_{i-1}(x_{i-1} - x^*)TG(x_{i-1} - x^*) + \lambda_{i-1} g_{i-1}^T d_{i-1}}{f'_i (x_i - x^*)^T G(x_i - x^*)}$$

Therefore,

$$\rho_i = \frac{2f'_{i-1}\, q_{i-1} + \lambda_{i-1}\, g_{i-1}^T\, d_{i-1}}{2f'_i\, q_i}$$

$$= \rho_i \left(\frac{q_{i-1}}{q_i}\right) + \frac{\lambda_{i-1}\, g_{i-1}^T\, d_{i-1}}{2f'_i\, q_i} \qquad \ldots (16)$$

The quantities ($q_{i-1}/q_i$) and ($F'_i\, q_i$) can be rewritten:

$$\frac{q_{i-1}}{q_i} = \left(\frac{\ln(f_{i-1})}{\ln(f_i)}\right)\left(\sqrt{\frac{f_{i-1}}{f_i}}\right)\left(\frac{1}{\sqrt{\rho_i}}\right) \qquad \ldots (17)$$

$$f'_i\, q_i = \frac{f_i \ln(f_i)(\ln(f_i) - \in_i / \in_2)}{\in_i / \in_2} \qquad \ldots (18)$$

Substituting (17) and (18) in (16), gives:

$$\rho_i = \rho_i \left(\frac{1}{\sqrt{\rho_i}}\right)\left(\sqrt{\frac{f_{i-1}}{f_i}}\right)\left(\frac{\ln(f_{i-1})}{\ln(f_i)}\right) + \frac{\lambda_{i-1}\, g_{i-1}^T\, d_{i-1} / 2)\,(\in_1 / \in_2)}{-f_i \ln(f_i)\,(\ln f_i - \in_i / \in_2)} \qquad \ldots (19)$$

Using the transformation:

$$\lambda_{i-1}\, g_{i-1}^T\, d_{i-1} = 2n \qquad \ldots (20)$$

Substituting (20) in (19), gives:

$$\rho_i = \left(\sqrt{\rho_i}\right)\left(\sqrt{\frac{f_{i-1}}{f_i}}\right)\left(\frac{\ln(f_{i-1})}{\ln(f_i)}\right) + \frac{n\,(\in_i / \in_2)}{-f_i \ln(f_i)\,(\ln\ f_i - \in_i / \in_2))} \qquad \ldots (21)$$

from (11) and (21), it follows that:

$$\left(\frac{f_{i-1}}{f_i}\right)\left(\frac{\ln(f_i) - \in_i / \in_2}{\ln(f_i) - \in_i / \in_2}\right)^2 =$$

$$\left(\sqrt{\frac{f_{i-1}}{f_i}}\right)\left(\frac{\ln(f_{i-1}) - \in_1 / \in_2}{\ln(f_i) - \in_i / \in_2}\right)\left(\sqrt{\frac{f_{i-1}}{f_i}}\right)\left(\frac{\ln(f_{i-1})}{\ln(f_i)}\right) + \frac{n\,(\in_1 / \in_2)}{[-f_i \ln\ (f_i)\,(\ln(f_i) - \in_1 / \in_2)]}$$

$$[\ln(f_{i-1}) - \in_1/\in_2]^2 = \left(\frac{\ln(f_{i-1})}{\ln(f_i)}\right)(\ln(f_{i-1}) - \in_1/\in_2).$$

$$(\ln(f_i) - \in_1/\in_2) - \frac{n(\in_1/\in_2)(\ln(f_i) - \in_1/\in_2)}{f_{i-1}\ln(f_i)}$$

$$\frac{\in_1}{\in_2}\left[1 - \frac{\ln(f_{i-1})}{\ln(f_i)} - \frac{n}{f_{i-1}\ln(f_i)}\right] = \left[\ln(f_{i-1}) - \left(\frac{\ln^2(f_{i-1})}{\ln(f_i)}\right) - \frac{n}{f_{i-1}}\right]$$

$$\frac{\in_1}{\in_2}\left[\frac{f_{i-1}\ln(f_i) - f_{i-1}\ln f_{i-1}) - n}{f_{i-1}\ln(f_i)}\right] \qquad \left[\frac{f_{i-1}\ln(f_i)\ln(f_{i-1}) - f_{i-1}\ln^2(f_{i-1}) - n\ln(f_{i-1})}{f_{i-1}\ln(f_i)}\right]$$

$$\frac{\in_1}{\in_2} = \left[\frac{f_{i-1}\ln(f_{i-1})(\ln(f_i) - \ln(f_{i-1})) - nf_{i-1}}{f_{i-1}(\ln(f_i) - \ln(f_{i-1})) - n}\right]$$

Using the following transformation:

$w = \ln(f_i) - \ln(f_{i-1})$

$$\frac{\in_1}{\in_2} = \left[\frac{wf_{i-1}\ln(f_{i-1}) - nf_{i-1}}{wf_{i-1} - n}\right] \qquad \ldots (22)$$

and substituting eq (22) in eq (11), we get:

$$\rho_i = \left(\frac{f_{i-1}}{f_i}\right)\left(\frac{n}{wf_{i-1}}\right)^2$$

## 4. Numerical Results and Conclusion

In order to test the effectiveness of the new algorithm which has been used to extend the standard CG-method, the comparative tests involve several well-known test functions (see Appendix) has been chosen and solved numerically by utilize the new and established methods.

Tables (1,) (2) and (3) utilize the comparison between our proposed new algorithm which is corresponding to the new non-quadratic model represented in equation (6), dennoted by (NEW), the classical CG-method, denoted by (CG), the rational model of tassopoulos and Storey (T/S) for low; intermediate and high dimensions.

The identical lineral search was used, namely, the cubic fitting procedure described by (Bundy, 1984) and also used in each case so that $\|g_{i-1}\| < 1 \times 10^{-5}$. Specifically quantity the number of function calls (NOF), the number of interations (NOI).

**Table (1)**

**Comparison of different metods for non-quadratic models**

**$2 \leq n \leq 10$**

| Test function | n-dimension | CG-NOI(NOF) | T/S NOI(NOF) | H/N NOI(NOF) | New NOI (NOF) |
|---|---|---|---|---|---|
| Rosen | 2 | 31 (73) | 31 (73) | 31 (73) | 31 (73) |
| Wood | 4 | 28 (61) | 36 (78) | 36 (75) | 31 (65) |
| Powell | 4 | 50 (114) | 38 (96) | 51 (109) | 30 (67) |
| Miele | 4 | 57 (178) | 46 (139) | 57 (178) | 30 (67) |
| Dixon | 10 | 22 (46) | 18 (44) | 21 (44) | 18 (38) |
| Total | | 188 (472) | 169 (430) | 196 (479) | 140 (310) |

Now, we can show from this table that the new algorithm, for this set of low dimensionality test functions, improve the classical CG-algorithm is about (26%) NOI and (18%) (NOF). Also, the new algorithm improve the (T/S) algorithin about 17%) NOI and (28%) NOF. Finally, the new method improves the H/N method in about (29%) NOI and (35%) NOF.

**Table (2)**

**Comparison of different methods for non-quadratic models.**

**$20 \leq n \leq 80$**

| Test function | n-dimension | CG-NOI(NOF) | T/S NOI(NOF) | H/N NOI(NOF) | New NOI (NOF) |
|---|---|---|---|---|---|
| Non-dig. | 20 | 24 (61) | 24 (61) | 24 (61) | 23 (58) |
| Wood | 20 | 52 (107) | 43 (90) | 34 (70) | 31 (64) |
| Powell | 20 | 34 (78) | 45 (102) | (45110) | 38 (82) |
| Miele | 20 | 46 (114) | 64 (160) | (42105) | 22 (55) |
| Rosen | 20 | 23 (56) | 23 (57) | 22 (52) | 22 (54) |
| Powell | 40 | 72 (158) | 71 (159) | (54120) | 28 (66) |
| OSP | 40 | 25 (62) | 18 (51) | 23 (65) | 19 (56) |
| Powell | 60 | 92 (198) | 75 (167) | 95 (194) | 28 (66) |
| Wood | 80 | 69 (140) | 43 (90) | 76 (154) | 36 (74) |
| Powell | 80 | 112 (239) | 75 (167) | 95 (194) | 28 (66) |
| Total | | 549 (1347) | 481 (1104) | 510 (1026) | 275 (641) |

Clearly the new algorithms beats CG-method in (50%) NOI and (52%) NOF; T/S method in (43%) NOI and (42%) NOF; H./N method in (46%) NOI and (38%) NOF.

**Table (3)**

**Comparison of different methods for non-quadratic models.**

**$100 \leq n \leq 400$**

| Test function | n-dimension | CG-NOI(NOF) | T/S NOI(NOF) | H/N NOI(NOF) | New NOI (NOF) |
|---|---|---|---|---|---|
| Powell | 100 | 129 (63) | 73 (167) | 95 (194) | 28 (66) |
| Non-dig. | 100 | 25 (62) | 25 (62) | 25 (59) | 24 (60) |
| Wood | 100 | 69 (140) | 43 (90) | 76 (154) | 36 (74) |
| Miele | 100 | 110 (257) | 104 (240) | 59 (174) | 71 (158) |
| Wood | 200 | 69 (140) | 47 (98) | 48 (98) | 36 (74) |
| Miele | 200 | 209 (472) | 154 (351) | 59 (174) | 37 (92) |
| Rosen | 400 | 23 (56) | 23 (57) | 22 (51) | 23 (56) |
| Miele | 400 | 404 (897) | 156 (355) | 74 (221) | 33 (83) |
| Total | | 1038 (2287) | 625 (1420) | 458 (1125) | 288 (663) |

In the following table, taking the NOI and NOF 100%, we can determine the performance of the new algorithm according to others used in this paper.

| CG | T/S | H/N | NEW |
|---|---|---|---|
| NOI = 100 | 40 | 56 | 72 |
| NOF = 100 | 38 | 51 | 71 |

Clearly the new algorithm has (72%) improvements in the NOI and it has about (71%) NOF. We conclude that our new proposed rational logarithmic model is superior on some models in both quadratic and non-quadratic models.

## APPENDIX

**1. Rosenbrock Function:**

$$F(x)=\sum_{i=1}^{n/2}\left[100\,(x_{2i}-x_{2i-1}^2)^2+(1-x_{2i-1})^2\right],$$

$x_0=(-1.2;1.0;\ldots)^T$.

**2. Generalized Powell Quadratics Functions:**

$$F(x)=\sum_{i=1}^{n/4}\left[(x_{4i-3}+10x_{4i-2})^2+5\,(x_{4i-1}-x_{4i})^2+(x_{4i-2}-2x_{4i-1})^4+10(x_{4i-3}-x_{4i})^4\right],$$

$x_0=(-3.0;-1.0;0.0;1.0)^T$

**3. Wood Function:**

$$F(x)=\sum_{i=1}^{n/4}100\left[\begin{array}{l}(x_{4i-2}+x_{4i-3}^2)^2+(1-x_{4i-3})^2+90(x_{4i}-x^2{}_{4i-1})^2+1-x_{4i-1})^2+10.1(x_{4i-2}-1)^2\\+(x_{4i}-1)2+19.8\,(x_{4i-2}-1)\,(x_{4i}-1)\end{array}\right]$$

$x_0=(-3.0;\ -1.0;\ -3.0;\ -1.0\ \ldots\ldots)^T,$

**4. Miele Function:**

$$F(x)=\sum_{i=1}^{n/4}\left[\exp\,(x_{4i-3})-x_{4i-2}\right]^2+100\,(x_{4i-2}-x_{4i-1})^6+$$
$$[\tan\,(x_{4i-1}-x_{4i})]^4+x_{4i-3}^8+(x_{4i}-1)^2$$

$x_0=(1.0;\ 2.0,\ 2.0\ \ 2.0\ \ldots\ldots)^2.$

**5. Non-Diagonal Variant of Rosenbrock Function:**

$$F(x)=\sum_{i=2}^{n}[100\,(x_i-x_i^2)^2+(1-x_i)]^2\quad;\qquad n>1,$$

$x_0=(-1.0;\ldots)^T$

**6. OSP Oren and Spedicato Powell Function:**

$$F(x)=\left[\sum_{i=1}^{n} i\,x_i^2\right]^2,\qquad x_0=(-1.0;\ldots)^T$$

**7. Dixon Function:**

$$F(x)=(1-x_1)^2+(1-x_{10})^2+\sum_{i=2}^{9}(x_i-x_{i+1}),$$

$x_0=(-1.0;\ldots)^T$

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