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TWO NOTES ON MODULAR p-ALGEBRAS

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ABSTRACT

According to the definition of congruence pairs which is given by T. Katriňák, every congruence relation of a modular p-algebra can be uniquely determined by a congruence pair.

In the present paper, we characterize strong extensions of modular p-algebras and permutability of congruences using the congruence pair technique.

INTRODUCTION

The representation of the congruence relations of modular p-algebras as congruence pairs which is given by T. Katrinák [5], enables us to solve the problems concerning the congruence lattices by means of congruence pairs technique. The congruence lattice of a modular p-algebra is characterized by Katrinák in [6]. In the present paper we deal with two problems: The characterization of strong extensions of modular p-algebras and the permutability of the congruences.

PRELIMINARIES

A p-algebra is an algebra $\langle L; \vee, \wedge, {}^*, 0, 1\rangle$, where $\langle L; \vee, \wedge, 0, 1\rangle$ is a bounded lattice and $x \leq a^*$ iff $x \wedge a = 0$ for every $a \in L$. The set $B(L) = \{x \in L: x = x^{**}\}$ of closed elements is a Boolean algebra. $D(L) = \{x : x^* = 0\}$ is the set of dense elements. A p-algebra is said to be distributive (modular) if the lattice $\langle L; \vee, \wedge, 0, 1\rangle$ is distributive (modular).

Let L be a modular p-algebra and let $\theta \in \mathrm{Con}(L)$. Let θ_B and θ_D denote the restrictions of θ to B(L) and D(L), respectively. Evidently

$(\theta_B, \theta_D) \in \mathrm{Con}(B(L)) \times \mathrm{Con}(D(L))$.

An arbitrary pair $(\theta_1, \theta_2) \in \mathrm{Con}(B(L)) \times \mathrm{Con}(D(L))$ will be called a congruence pair if $a \in B(L)$, $u \in D(L)$, $u \geq a$ and $a \equiv 1(\theta_1)$ imply $u \equiv 1(\theta_2)$.

The standard results on p-algebras can be found in [3], [4]

Theorem A [5] Theorem 11

Let L be a modular p-algebra. Then every congruence θ of L determines a congruence pair (θ_B,θ_D). Conversely, every congruence pair (θ_1,θ_2) uniquely determines a congruence θ on L with $\theta_B = \theta_1$ and $\theta_D = \theta_2$ by the following rule:

$x \equiv y\ (\theta)$ iff (i) $x^* \equiv y^*\ (\theta_1)$ and
(ii) $x \vee x^* \equiv y \vee y^*\ (\theta_2)$

STRONG EXTENSIONS

It is known that some classes of algebras satisfy the Congruence Extension Property (CEP in the sequel): an algebra A satisfies the CEP if for every subalgebra B of A and every θ on B,θ extends to A. The class of distributive lattices enjoys the CEP (see[2], [3]). J. Varlet [8] introduced the notion of a strong extension of algebras. An algebra A is said to be a strong extension of the algebra B, if B is a subalgebra of A and every congruence relation on B has at most one extension to A. This notion is important in studying classes of algebras satisfying the CEP. An algebra A satisfying the CEP is a strong extension of the algebra B if for every congruence on B there exists a unique extension to A.

In the following $\triangle$ and ∇ respectively denote the identity and the universal congruences respectively.

Lemma 1 Katriňák,[7]

Let A and B be Boolean algebras. Then A is a strong extension of B if and only if A = B.

Now, we formulate

Theorem 1

Let L_1 and L be modular p-algebras. Then L_1 is a strong extension of L if and only if:
(i) $D(L_1)$ is a strong extension of $D(L)$,
(ii) $B(L_1) = B(L)$.

Proof

Let L_1 be a strong extension of L. Let $\theta_2 \in$ Con (D(L)). We have to show that if θ_2 extends to $D(L_1)$, then the extension is unique (if it exists). Suppose that $\bar{\theta}_2,\theta'_2 \in$ Con $(D(L_1))$ such that

$\bar{\theta}_2|D(L) = \theta'_2|D(L) = \theta_2$

Clearly $(\triangle,\bar{\theta}_2)$ and $(\triangle,\acute{\theta}_2)$ are congruence pairs. By Theorem A, these determine congruence relations $\bar{\theta}$ and $\acute{\theta}$ of Con (L_1). Hence, $\bar{\theta} \mid L = \theta' \mid L = \theta$. By hypothesis most one extension. Thus $\overline{\theta} = \theta'$ which gives $\overline{\theta}_2 = \acute{\theta}_2$, proving (i).

Similarly by considering a congruence relation $\theta_1 \in$ Con (B(L)) the CEP can be proved to hold for the class of Boolean algebras. Thus θ_1 has an extension to $B(L_1)$; we will show that this extension is unique. Let $\overline{\theta}_1$, $\acute{\theta}_1 \in$ Con ($B(L_1)$) satisfy

$\overline{\theta}_1 | B(L) = \acute{\theta}_1 | B(L) = \theta_1$

Clearly $(\overline{\theta}_1, \nabla)$ and $(\acute{\theta}_1, \nabla)$ are congruence pairs. Then there exist two corresponding congruence relations $\overline{\theta}$ and $\acute{\theta}$ of Con (L_1),

$\overline{\theta} | L = \acute{\theta} | L = \theta$.

But θ has at most one extension to L_1. So $\overline{\theta} = \acute{\theta}$. Accordingly, $\overline{\theta}_1 = \acute{\theta}_1$ and $B(L_1)$ is a strong extension of B(L). Using Lemma 1 we get $B(L_1) = B(L)$.

Conversely, suppose the validity of the conditions (i) and (ii) and let $\theta \in$ Con (L). If θ extends to L_1 we show that this extension is unique. Assume that $\overline{\theta}, \acute{\theta} \in$ Con (L_1) where $\overline{\theta} | L = \acute{\theta} | L = \theta$. By Theorem A, these can be represented by congruence pairs as

$$\overline{\theta} = (\overline{\theta}_1, \overline{\theta}_2) \text{ and } \acute{\theta} = (\acute{\theta}_1, \acute{\theta}_2)$$

and $\theta = (\theta_1, \theta_2)$, where $\overline{\theta}_1 \mid B(L) = \acute{\theta}_1 \mid B(L) = \theta_1$ and $\overline{\theta}_2 \mid D(L) = \acute{\theta}_2 \mid D(L) = \theta_2$. By the conditions (i) and (ii) we get $\overline{\theta}_1 = \acute{\theta}_1$ and $\overline{\theta}_2 = \acute{\theta}_2$.

Corollary 1

Let L, L_1 be two distributive p-algebras. If L_1 is a strong extension of L, then Con (L) $\simeq$ Con (L_1).

Proof

Distributive p-algebras satisfy the CEP. Then every congruence of L has an extension on L_1 and this extension is unique.

PERMUTABILITY OF CONGRUENCES

An algebra A is said to have permuting congruences whenever the usual relational product $\theta \circ \varphi$ of any pair of congruences θ, φ on A commutes; that is $a \equiv t(\theta)$ and $t \equiv b(\varphi)$ implies $a \equiv w(\varphi)$ and $w \equiv b(\theta)$, for some $w \in a$.

It is well known that Boolean algebras have permuting congruences. Hence the permutability of p-algebras depends on the permutability of their lattices of dense elements. This was shown by Berman [1] for distributive p-algebras. In case of modular p-algebras, we have.

Theorem 2

Let L be a modular p-algebra. Then the following conditions are equivalent

(i) L has permuting congruences,
(ii) D(L) has permuting congruences.

Proof

For any congruence $\theta \in$ Con (L) the restriction θ_D on D is a congruence on D. Also for any congruence $\theta_2 \in$ Con (D(L)), (Δ, θ_2) is a congruence pair. This means that there exists a congruence $\theta \in$ Con (L) with $\theta_D = \theta_2$. To prove the equivalence of the conditions (i) and (ii) we have to show that two congruences $\theta, \varphi \in$ Con (L) are permutable if and only if their restrictions θ_D and φ_D are permuting congruences on D(L). Let θ, φ permute and suppose that a,b,c $\in$ D(L) are such that $a \equiv b(\theta_D)$ and $b \equiv c(\varphi_D)$. Then $a \equiv b(\theta)$ and $b \equiv c(\varphi)$.

But θ and φ permute. So there exists $x \in L$ with $a \equiv x(\varphi)$ and $x = c(\theta)$. By Theorem A we have $a \vee a^* \equiv x \vee x^* (\varphi)$ and $x \vee x^* \equiv c \vee c^* (\theta)$. Now $a \in D(L)$ implies that $a^* = O$, and so since $x \in L$ we should have $x \vee x^* = d \in D(L)$. Hence $a \equiv d(\varphi)$, $d \equiv c(\theta)$, a,d,c $\in$ D(L) implies $a \equiv d(\varphi_D)$ and $d \equiv c(\theta_D)$. Thus θ_D, φ_D are permutable congruences.

Conversely. Let 0, $\varphi \in$ con (L) be such that θ_D, φ_D are permutable. Consider the elements x,y,z $\in$ L with $x \equiv y(\theta)$ and $y \equiv z(\varphi)$. Using Theorem A, we get $x^{**} \equiv y^{**} (\theta_B)$, $y^{**} \equiv z^{**} (\varphi'_B)$ and $x \vee x^* \equiv y \vee y^* (\theta_D)$, $y \vee y^* \equiv z \vee z^* (\varphi_D)$. But any two congruences of a Boolean algebra permute. So there exists $w^{**} \in L$ with $x^{**} \equiv w^{**} (\varphi_B)$ and $w^{**} \equiv z^{**} (\theta_B)$. Since θ_D, φ_D are permutable, then there exists $d \in D(L)$ with $x \vee x^* \equiv d(\varphi_D)$, $d \equiv z \vee z^* (\theta_D)$. Then $x^{**} \equiv w^{**} (\varphi)$, $w^{**} \equiv z^{**} (\theta)$ and $x \vee x^* \equiv d(\varphi)$, $d \equiv z \vee z^* (\theta)$, yielding $x^{**} \wedge (x \vee x^*) \equiv d \wedge w^{**} (\varphi)$, d $w^* \equiv z^{**} \wedge (z \wedge z^{**}) (\theta)$. But this means that $x \equiv d \wedge w^{**} (\varphi)$, $d \wedge w^{**} \equiv z(\theta)$ and θ, φ are permutable as to be proven.

Corollary 2

A modular p-algebra L has permutable congruences if D(L) is a relatively complemented lattice.

Proof

It is known that relatively complemented lattices have permutable congruence relations (see [4]). Thus 0_D, $_D$ permute for every 0, Con (L), which means that 0, are permutable.

Corollary 3

A distributive p-algebra L has permuting congruences if and only if D(L) is relatively complemented.

Proof

The proof is clear since a distributive lattice has permuting congruences if and only if it is relatively complemented.

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