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ELLIPTIC SURFACES OVER A GENUS 1 CURVE WITH EXACTLY THE PAIR (I_8, IV) OF SINGULAR FIBERS

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السطوح الناقصية على منحنٍ من الجنس 1 ذات ليفين منفردين بالضبط (I_8, IV)

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نُصنف في هذا البحث جميع السطوح الناقصية الأصغرية ($\pi : E \to C$) على المنحنى C من الجنس 1 ذات مقطع وليفين منفردين بالضبط (I_8, IV).

ABSTRACT

In this paper we classify all minimal elliptic surfaces ($\pi : E \to C$) over a genus 1 curve C, with a section and exactly the pair (I_8, IV) of singular fibers.

1. INTRODUCTION

The serious study of elliptic surfaces was started by Kodaira (see [6]). He listed all possible types of singular fibers, gave their invariants and analyzed an important invariant called the J-map. Beauville (see [2]) has studied elliptic surfaces over P^1, in fact he classified the semi-stable cases (i.e., the cases in which all singular fibers are of type In). He proved that there are 6 semi-stable cases with the minimal number (=4) of singular fibers. In 1985 Schmikler-Hirzebruch wrote Weierstrass equations for all elliptic fibration with at most three singular fibers (see [5]). In 1986, R. Miranda and U. Persson have listed all extremal rational elliptic surfaces (see [8]). In 1988, Stiller (see [12]) has classifed all elliptic surfaces over a genus 1 curve with exactly one singular fiber necessarily of type I_6^*. In 1989, R. Miranda and U. Persson has classified all possible configurations of I_n fibers on elliptic K_3 surfaces (see [9]). In 1990, U. Persson has classifed all possible configurations of singular fibers on rational elliptic surfaces (see [10]). Also, R. Miranda has analyzed the same problem by giving a more combinatorial and less geometric analysis (see [7]).

In this paper we let C denote a genus one curve and study minimal elliptic surfaces $\pi : E \rightarrow C$ with a section and exactly the pair (I_8, IV) of singular fibers.

In this paper the notation $\|[J^{-1}(x)]\| = (n_1, \ldots, n_t)$ will be used to indicate that $J_{-1}(x)$ consists of t points say $\{x_1, \ldots, x_t\}$ such that the multiplicity of J at x_i ($m_{x_i}(J)$) is n_i for all $i \in \{1, \ldots, t\}$.

The plan of the paper goes as follows: First, we review some important ideas which will be used in the text of this paper, then, we give the possible ramification of the J-map and prove its existence, and then we construct the J-map and the required surfaces.

Remark 1.1: To build a minimal elliptic surface with a section and a given number of singular fibers, it is enough to build the J-map associated to this surface (for more details see [7] Section 3).

The following theorem which we call the monodromy theorem is jusst a restatement of Corollary 3.5 of [7].

Theorem 1.2: Let C be a curve and let be a finite subset of P_1 say $|B| = $ n, then there is a one-to-one correspondence between

$$\left\{\begin{array}{c} J : C \rightarrow P^1 \text{ such that} \\ \deg(J) = d \text{ and } J \text{ i} \\ \text{branched at most over } B \end{array}\right\} \text{and} \left\{\begin{array}{c} \text{permutations } \sigma_1, \ldots, \sigma_n \in S_d \\ \text{such that } \sigma_n \ldots \sigma_1 = id, \text{ and the} \\ \sigma_i\text{'s generate a transitive subgroup of } S_d \end{array}\right\}$$

Where the first set is taken up to isomorphism (fixing P^1 and the second set is taken up to conjugation).

2. MAIN RESULTS

There are the following types of singular fibers (see [6]): I_0^*, I_n, I_n^*, II, III, IV, IV^*, III^*, II^* ($n \geq 1$). If $e(F)$ denotes the Euler number of the fiber F, then the Euler numbers of the above list area: 6, n, $n + 6$, 2, 3, 4, 8, 9 and 10 respectively.

Lemma 2.1: Let C be a genus 1 curve, suppose $\pi : E \rightarrow C$ is a minimal elliptic surface with a section and exactly two singular fibers. If the degree of the line bundel **L** is 1 (i.e., **L** is the conormal bundle to the ection), then there are twenty five possible pairs (F_1, F_2) of singular fiber types such that the sum of the Euler numbers is 12.

Proof. Immediate from the fact that if (F_1, F_2) is a possible pair of singuar fibers, then $e(F_1) + e(F_2) = 12$. ■

Notice that the pair I_8, IV) is one of these possible pairs, and this case cannot occure if the genus of the base curve is 0 (i.e. $C \cong P1$).

Lemma 2.2: Let C be a genus 1 curve, suppose $\pi : E \rightarrow C$ is a minimal elliptic surface with a section and exactly the pair (I_8, IV) of singular fibers. If $J : C \rightarrow P^1$ is the J-map associated to this fibration, then degree $(J) = 12$ and J is ramified as follows: $\|[J^{-1}(0)]\| = (3,3,2)$ $\|[J^{-1}(1)]\| = (2,2,2,2)$ and $\|[J^{-1}(\infty)]\| = (8)$.

Proof. Deg $(J) = \sum_{n\geq 1} n(\neq$ of I_n – fibers + $\neq$ of $I_n^* \neq$ fibers) = 8 (see [7}, p. 194).

Let R = {ramification points of J} and let $m_x(J)$ denote the multiplicity of J at the point x. By Hurwitz's formula for the genus of a curve we have: $16 = \sum_{n\geq 1} (m_x(J) - 1)$. Now over 0 we have the IV fiber, hence the minimum ramification of J over 0 is obtained if $\|[J^{-1}(0)]\| = (3,3,2)$ (see [7], over 1 we have smooth fibers, hence the minimum ramification of J over 1 is obtained if $\|[J^{-1}(1)]\| = (2, 2, 2, 2)$, and over ∞ we have I_8 – fiber, hence $\|[J^{-1}(\infty)]\| = (8)$.

Thus

$$\sum_{x \in J^{-1}(\{0,1,\infty\})} (m_x(J) - 1) = 5 + 4 + 7 = 16,$$

hence $R = \{0, 1, \infty\}$ and there is not other ramification of J. ■

Theorem 2.3: Under the hypothesis of Lemma 2.2 above the degree 8 map $J : C \rightarrow P^1$ (ramified as in Lemma 2.2), exists and is unique. Moreover, the genus 1 curve C is unique.

Proof. To prove this theorem it is enough to find three permutations σ_0, σ_1, and σ_∞ in S_8 reprsenting the monodromy of

J around 0, 1 and ∞ respectively, such that: $\sigma_0\sigma_1 = \sigma_\infty^{-1}$, σ_0, σ_1, and σ_∞ are unique up to conjugation, the triple (σ_0, σ_1, σ_∞) generates a transitive subgroup of S_8 and such that the cycle structure of σ_0 is (3^2, 2), that of σ_1 is (2^4) and that of σ_∞ is (8).

To this end let $\sigma_0 = (1\ 2)\ (3\ 4\ 5)\ (6\ 7\ 8)$ and $\sigma_1 = (a\ b)\ (c\ d)\ (e\ f)\ (g\ h)$. Since 1 has to appear in one of the 2-cycles of s1 we may assume $a = 1$, hence $b \neq 2$ (otherwise we would have a fixed element in the product $\sigma_0\sigma_1$ which is not allowed); therefore we assume that $b = 3$. Now 2 has to appear in σ_1, so assume $c = 2$, and hence $d \neq 1, 2, 3$, hence d = 4 or 5 or we may assumed $d = 6$, but clearly if $d = 4$ or 5, then this forces (7 8) to be in σ_1 which is not valid, hence $d \neq$ 1, 2, 3, 4, 5, so assume $d = 6$. Now it is easy to check that $(e\ f) = (4\ 7)$ or (4 8) and if $(e\ f) = (4\ 8)$, then we get the cycle (5, 8) in $\sigma_0\sigma_1$ which is not allowed, hence $(e\ f) \neq (4\ 8)$; there fore, we get:

$$\sigma_1 = (1\ 3)\ (2\ 6)\ (4\ 7)\ (5\ 8)$$
$$\sigma_0\sigma_1^{-1} = \sigma_\infty^{-1} = (1\ 4\ 8\ 3\ 2\ 7\ 5\ 6)$$

and clearly the permutations σ_0, σ_1, and σ_∞ satisfy all the conditions stated in the begining of the proof; thus $J : C \to P^1$ (ramified as in Lemma 2.2) exists and is unique and the curve C is unique. ■

Corollary 2.4: If C is the unique genus 1 curve of Theorem 2.3 above, then there is a unique (up to analytic isomorphism) minimal elliptic surface $\pi : E \to C$ with a section and exactly the pair (I_8, IV) of singular fibers.

Proof. This is clear since the J-map exists and is unique, and this guarantees the existance and uniqueness of the desired surface. ■

3. THE J-MAP AND THE SURFACE

Next we construct the J-map ($J : C \to P^1$) associated to a minimal elliptic surface $\pi : E \to C$ with a section and exactly the pair (I_8, IV) of singular fibers, where J and C are the unique J-map and the unique curve of Theorem 2.3 above.

The plan here is to realize this J-map as a comosition of two maps: a degree map $f : C \to P^1$, and a degree 4 map $J_1 : P^1 \to P^1$ (i.e., $J = J_1 \circ f$), now we proceed with this construction.

Remark 3.1: If C is a genus 1 curve, then clearly a degree 2 map $f : C \to P^1$ exists, and by Hurwitz's formula for the genus of a curve f must be branched over exactly 4 points of P^1, in fact by a suitable change of coordinates in P^1 we may assume that these four brach points to be any four points of P^1. Moreover, we may assume that the curve C is given by : $y^2 = (x - a_1)\ (x - a_2)\ (x - a_3)$, and the map $f : C \to P^1$ is given by $f(x, y) = x$, hence a_1, a_2, a_3 and ∞ are the four ramification points of f.

In the next remark we give the degree 4 map $J_1 : P_1 \to P_1$, in fact we have

Remark 3.2: Let $P : S \to P^1$ be the rational elliptic surface which has the following singular fibers: the pair (III, III) over 1, a fiber of type II over 0, and a fiber of type I_4 over ∞. This surface has a geometric realization $M(1, 1, 1, 0)$ (see [10], page 10), and this surface is constructed on page 36 of [10]. If $J_1 : P^1 \to P^1$ is the J-map associated to this surface, then deg $(J_1) = 4$ and $J_1 : P_1 \to P_1$ is ramified as follows (see [7], page 207):

$$|[J_1^{-1}(\infty)]| = (4),\ |[J_1^{-1}(1)]| = (1, 1, 2) \textit{and} \quad (3.2.1)$$
$$|[J_1^{-1}(0)]| = (1,3).$$

Moverover we may assume that $J_1 : P^1 \to P^1$ is given by $J_1(x) = 4x^3 - x^4$, and to clarify this moreover statement, let $x = 0$ be the point multiplicity 3 over 0, and $x = \infty$ be the point of multiplicity 4 over ∞, hence $J_1 : P^1 : P^1 \to P^1$ must be of the form $J_1(x) = cx^3\ (x - 1)$, where a and c are constants. Now $J_1(x)$ must have a critical point over 1 (since $|[J_1^{-1}(1)]| = (1, 1, 2)$, and clearly $J_1'(x) = 0$ if and only if $x = \frac{3a}{4}$; thus $J_1(\frac{3a}{4}) = 0$, and hence $27a^3c + 256 = 0$. Clearly $a = \frac{3}{4}$ and $c = -3$ is a solution of this equation; therefore, $J_1(x) = 4x^3 - x^4$.

Lemma 3.3: The degree 4 map $J_1 : P^1 \to P^1$ ramified as in (3.2.1) above is unique.

Proof : To prove this, it is enough to find a set of three premutations σ_0, σ_1, and σ_∞ S_4 representing the monodromy of J around 0, 1, and ∞ respectively such that $\sigma_0\sigma_1 = \sigma_\infty^{-1}$, the triple ($\sigma_0$, σ_1, σ_∞) generates a transitive subgroup of S_4, σ_0, σ_1, and σ_∞ are unique up to conjugation, and such that the cycle structure of σ_0 is (3), that of σ_1 is (2) and that of σ_∞ is (4).

To this end assume that $\sigma_0 = (2\ 3\ 4)$ and $\sigma_1 = (a\ b)$. Notice that if $(a\ b)$ consists of two elements of σ_0, then the product $\sigma^0\sigma_1$ must have a fixed element which is not allowed; hence we may assume $a = 1$ and $b = 2$; therefore, we get $\sigma_0\sigma_1 = \sigma_\infty^{-1} = (1\ 3\ 4\ 2)$. Moreover, it is clear that these permutations satisfy all the conditions stated in the begining of the proof, hence $J_1 : P^1 \to P^1$ is unique. ■

Lemma 3.4: Let C be a genus 1 curve, let $f : C \to P^1$ be a degree 2 map, let $J_1 : P^1 \to P$ be the degree 4 map defined in Remark 3.2 above. If $J : C \to P^1$ is defined by $J = J_1 \circ f$, then deg(J) = 8, and by a suitable change of coordinates in P^1 (= range of f) J is ramified as in Lemma 2.2, and hence is the unique J-map of Theorem 2.3.

Proof : $Deg(J) = deg(J_1). deg(f) = 9$. Let $S_{\infty 4}$ be the point whose J_1-value is ∞, let S_{11} and t_{11} be the points whose J_1-value is 1, and let S_{01} be the point whose J_1-value is 0, where the second subscript is used to indicate the multiplicity of J_1 and these points. Now change coordinats in P^1 so that $S_{\infty 4}$, S_{11}, t_{11}, and S_{01} are the four branch points of f, hence it is easy to see that $J = J_1 \circ f$ is ramified as required, and hence it is the unique J-map of Theorem 2.3. ■

Theorem 3.5: Given the unique genus 1 curve C of Theorem 2.3, and if $J : C \rightarrow P^1$ is the unique J-map defined in Lemma 3.4 above, then this data can be used to build the unique minimal elliptic surface (see Corollary 2.4). $\pi : E \rightarrow C$ with a section and exactly the pair (I_8, IV) of singular fibers.

Proof : Let $J = J_1 \circ f$ (see Lemma 3.4), then it is clear that f is just a base change of order 2, let $J : C \rightarrow P^1$ be the pull-back of the srface $P : S \rightarrow P^1$ (see Remark 3.2.) via f, then $\pi : E \rightarrow C$ is a minimal elliptic surface with a section and the following singular fibers: two I_0^* fibers over 1, a fiber of type IV over 0, and a fiber of type I_8 over ∞ (see [8], Table 7.1.). Now using the process of deflating two *'s, we deflate the two *'s from the two I_0^*-fibers (see [7], Section 3), so that they becomes smooth fiberes, and notice that the rest of the fiberes remains unchanged, and hence the resulting surface $\pi : E \rightarrow C$ is minimal elliptic surface with a section and exactly the pair (I_8, IV) of singular fibers; thus $\pi : E \rightarrow C$ must be the required (up to analytic isomorphism) surface. ■

We end this paper with the following remark:

Remark 3.6: another way to get our surface is to consider the rational elliptic surface $\alpha : S \rightarrow P^1$, whose Weierstrass equation is given by: $y^2 = x^3 + 3t(t-1)^3x + 2t(t-1)^5$. This surface has J-map given by $J(t) = t$, and it has exactly three singular fiberes (see [7], page 203): a fiber of type II over t - 0, a fiber of type III* over $t = 1$, and a fiber of type I_1 over $t = \infty$.

Let $\pi : E \rightarrow C$ be the pull-back of the rational elliptic surface a : $S \rightarrow P^1$ via $J = J_1 \circ f$, then use Table 7.1 of [8] to get the required surface.

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