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APPROXIMATING THE DISTRIBUTION OF QUADRATIC FORMS USING ORTHOGONAL POLYNOMIALS

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تقريب توزيع الصيغ التربيعية باستخدام كثيرات الحدود المتعامدة

أحمـــد علـــي الزغـــول

في هذا البحث نقترح بعض الطرق لتقريب توزيع الصيغ التربيعية في المتغيرات العشوائية ذات التوزيع الطبيعي. أول طريقتين تعتمدان كثيرات الحدود المتعامدة وتحديداً كثيراً الحدود هيرمايت ولاجيير.

أما الطريقة الثالثة فهي تعتمد على مزيج من التوزيع الطبيعي وتوزيع جاما المقطوع. سيتم دراسة دقة هذه التقريبات ومناقشتها لعدة أشكال من الصيغ التربيعية.

Key Words : Quadratic forms, Orthogonal polynomials, Hermitte polynomials, Laguerre polynomials, Mixture of normal and gamma

ABSTRACT

In this paper we propose some techniques to approximate the distribution of quadratic forms in normal random variables. The first two techniques will be based on orthogonal polynomials, namely; the Hermitte and Laguerre polynomials. The third technique will be based on a mixture of normal and truncated gamma distributions. The performance of these approximations will be studied and discussed for various quadratic forms.

1. Introduction :

Let $\acute{X}$ be a random vector distributed multivariate normal with mean and covariance matrix Σ, Consider the random variable $Q=\acute{X}\ AX$, where A is a real matrix, Assume that Σ is non-singular, because, otherwise, it is straight forward to transform it into a non-singular matrix without changing the main structure of Q. It can be shown that Q has the same distribution as a linear combination of independent chi-square random variables each with one degree of freedom with coefficients being the eigenvalues of the matrix A Σ, i.e

$$Q \underline{\underline{D}} \sum_{k=1}^{n} \lambda_k \chi_k^2 (1) \qquad (1.1)$$

where χ_k^2 (1), k = 1,2, ... n are independent chi-square random variables each with one degree of freedom, and λ_k, k = 1,2, ... n are the eigenvalues of $A\Sigma$.

These eigenvalues could be positive or negative real numbers, and that depends on whether the matrix A is positive definite or negative definite matrix.

In this paper, we develop three methods to approximate the distribution of Q in (1.1), First, an approximation based on Hermitte polynomials associated with normal distribution will be proposed as an approximation of "symmetric" quadratic forms, Second, for "asymmetric" quadratic forms, approximation based on three parameters gamma distribution associated with Laguerre polynomials will be given.

Then, we will suggest a mixture of normal and gamma distributions as an approximation for Q.

Several methods have been used to approximate the distribution of indefinite quadratic forms. Some authors, like Grenander et'al (1959), simply, used normal distributions or gamma distributions as approximations of quadratic forms, Box (1954) suggested an F distribution as an approximation of ratios of quadratic forms, Some authors approximated the distribution of Q based on its first few moments by fitting certain Pearson type distributions, Amongst these are David and Johnson (1951), Taneja (1976), Aiuppa and Bargment (1977),Other approximations based on numerical inversion of the moment generating function or the characteristic function of Q when the λ's are positive where given by Grad and Solomon (1955), then their results where generalized for all λ's by Johnson and Kotz (1970). Numerical inversion of the characteristic functions were given by Imhof (1961), L'Esperence et'al (1976), Rice (1980), Farebrother (1985), and El-Qasem (1988). Infinite series expansions were given by many authors as exact representations of the distribution of Q. Amongst them are Robins and Pitman (1949), Gurland (1955), Shah (1963), Robinson (1965), Press (1966), Wang (1967), Johnson and Kotz (1967), Provost and Ruduik (1993), and Mathei et'al (1995).

2. Approximation of a distribution function by a series of orthogonal polynomials

A countable set of polynomials {po(x), ...} is said to be orthogonal in the interval (a,b) with weight function w(x) if for any two polynomials, $p_m(x)$ and $p_n(x)$ of the set, we have

$$\int_a^b \omega(x) p_m(x) p_n(x) dx = c_{mn} \delta_{mn},$$

where

$$\delta_{mn} = \begin{cases} 1 \ ; m = n \\ 0 \ ; m \neq n . \end{cases}$$

and w(x) is a non-negative Lebesgue integrable function satisfying $\int_a^b \omega(x) dx > 0$.

A useful property of orthogonal polynomials is that for each k, $0 \le k \le n-1$, we have

$$\int_a^b \omega(x) x^k p_n(x) dx = 0. \qquad (2.1)$$

one important class of orthogonal polynomials is the class of classical polynomials, which can be represented by

$$p_n(x) = \omega^{-1}(x) (d \ /dx)^n [\omega(x) \beta^n(x)],$$

where β^n (x) is a polynomial of second degree.

Among the classical polynomials are Hermitte and

Laguerre polynomials which we use to approximate the distribution of Q.

Before discussing using Hermitte and Laguerre polynomials, we need to talk about the cumulants and moments of Q. First, Q can be written as a difference of two positive definite quadratic forms, viz.

$$Q = \sum_{j=1}^{n_1} \lambda_j \chi_j^2(1) - \sum_{j=n_1+1}^{n_2} \lambda_j \chi_j^2(1),$$

where the λ's are reordered and reindexed such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\, 0 > \lambda_{n+1} \geq \lambda_{n1+2} \geq \ldots \geq \lambda_{n2}$ So the expected value and the variance of Q^* are :

$$E[Q] = \sum_{j=1}^{n} \lambda_j, \quad V[Q] = 2\sum_{j=1}^{n} \lambda_j^2.$$

We will be dealing with a standarized quadratic form, that is a quadratic form with mean zero and variance one, So, define Q as:

$$Q^* = \frac{Q - E[Q]}{\sqrt{Var[Q]}}.$$

The cumulants of Q^* are given by :

$\kappa_1 = 0$, and $\kappa_r(Q^*) = (r-1)!\, 2^{\frac{r}{2}-1}\, \eta_2^{-r/2}\, \eta_r$; r = 2,3,...,

where $\eta_r = \sum_{j=1}^{n_1} \lambda_j^r + \sum_{j=n_{1+1}}^{n} (-\lambda_j)^r$.

Approximating Using Hermitte Polynomials

Hermitte polynomials associated with standard normal distribution will be used to approximate the ***d.f.*** of symmetric or slightly skewed quadratic forms. It is natural that a symmetric quadratic form is indefinite, So the approximation in this section is more appropriate for indefinite forms even it may work, as well, for definite forms that are slightly skewed.

Hermitte polynomials are defined by the conditions

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_m(x) dx = n!\, 2^n \sqrt{2\pi}\, \delta_{nm}, \quad -\infty < x < \infty,$$

where

$$\delta_{nm} = \begin{cases} 1 \,;\, n = m \\ 0 \,;\, n \neq m. \end{cases}$$

and where the coefficient of x^n in the nth degree polynomial, $H_n(x)$, is positive, Hermitte polynomials satisfies the second order differential equation

$$y'' - 2xy' + 2ny = 0,$$

and the recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x).$$

Also, Hermitte polynomials have the following representation, which is called Rodrigues representation

$$H_n(x) = (-1)^n e^{x2} (d/dx)^n e^{-x2}, \quad -\infty < x < \infty,$$

To give our approximation a good start, the weight function of the Hermitte polynomial will be chosen to be the density function of the standard normal distribution. Therefore, we have the following Rodrigues representation for the Hermitte polynomial:

$$\phi(x) H_n(x) = (-1)^n \phi^{(n)}(x),$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $-\infty < x < \infty$, and $\phi^{(n)}$ is the nth derivative of $\phi(x)$

The normalization constant of this Hermitte polynomial is, then,n! That is

$$\int_{-\infty}^{\infty} \phi(x) H_n(x) H_m(x) dx = n!\, \delta_{nm}. \tag{2.2}$$

The density function of Q^* will be approximated by the partial sums of the series:

$$f_{Q^*}(x) \approx \phi(x)\left[1 + \sum_{j=1}^{\infty} c_j H_j(x)\right]. \tag{2.3}$$

If the above series is convergent, the approximate equality becomes exact equality. To evaluate the coefficients c_k, we take the kth moment of both sides of **(2.3)**, and get.

$$E\left[Q^{*k}\right] = \int_{-\infty}^{\infty} x^k \phi(x)\left[1 + \sum_{j=1}^{\infty} c_j H_j(x)\right] dx$$

By **(2.1)**, we have

$$E\left[Q^{*k}\right] = \int_{-\infty}^{\infty} x^k \phi(x)\, dx + \int_{-\infty}^{\infty} x^k \phi(x) \sum_{j=1}^{k} c_j H_j(x)\, dx.$$

$$= \int_{-\infty}^{\infty} x^k \phi(x)\, dx + c_k \int_{-\infty}^{\infty} x^k \phi(x) H_k(x) dx + \sum_{j=1}^{k-1} c_j \int_{-\infty}^{\infty} x^k \phi(x) H_j(x)\, dx.$$

If **Z** is a standard normal random variable, then by **(2.2)**, we have

$$c_k = \frac{E\left[Q^{*k}\right] - E\left[Z^k\right] - \sum_{j=1}^{k-1} c_j \int_{-\infty}^{\infty} x^k \phi(x) H_j(x) dx}{k!}. \tag{2.4}$$

The coefficients c_k can be computed iteratively from **(2.3)**

Note that Q^* and **Z** have the same first and second moments, so $c_0 = 0$ and $c_1 = 0$.

Thus c_3 and c_4 are given by

$$c_3 = \frac{E[Q^{*3}]}{6}, \text{ and } c_4 = \frac{E[Q^{*4}] - E[Z^4] - c_3\int_{-\infty}^{\infty} x^4\phi(x)H_3(x)dx}{24}.$$

If $\mathbf{Q}^*$ is symmetric, then its 3rd moment is zero, and hence.

$$c_4 = \frac{E[Q^{*4}] - E[Z^4]}{24}.$$

Integrating **(2.3)**, we get the following approximation for the c.d.f. of Q^*

$$F_{Q^*}(x) = \Phi(x) - \phi(x)\sum_{k=1}^{\infty} c_k H_{k-1}(x).$$

where c_k are given by **(2.4)**.

How accurate is this approximation? Actually, we expect this approximation to be very accurate when Q is symmetric and this accuracy decreases as Q^* becomes more skewed. We associate the standard normal density with the Hermitte polynomial because this density is proportional to the Hermitte weight function.

To show the performance of this approximation for quadratic forms with various skewness, we will make use of Robinson (1965) and Press (1966) results which give an exact representation for a difference of two chisquare variates with even number of degress of freedom.

Examples : In Table (2.1) below, we give few quadratic forms with the first four coefficients, the third and the fourth moments, and the skeweness, Table (2.2) contains the approximate and the exact c.d.f. at some given values.

Table (2.1) : Examples of quadratic forms with their third an forth moments, skewness, and the third and the fourth coefficients of the series in (2.3).

Quadratic Form	$E[(Q_1^*)^3]$	$E[(Q_1^*)^4]$	*skewness*	c_3	c_4
$Q_1^* = \dfrac{\chi_1^2(20)-\chi_2^2(20)}{\sqrt{80}}$	0.000	3.300	0.000	0.0000	0.0125
$Q_2^* = \dfrac{\chi_1^2(12)-\chi_2^2(12)}{\sqrt{48}}$	0.000	3.500	0.000	0.0000	0.0208
$Q_3^* = \dfrac{\chi_1^2(8)-\chi_2^2(8)}{\sqrt{32}}$	0.000	3.750	0.000	0.0000	0.0313
$Q_4^* = \dfrac{\chi_1^2(4)-\chi_2^2(4)}{\sqrt{4}}$	0.000	4.500	0.000	0.0000	0.0625
$Q_5^* = \dfrac{10\chi_1^2(8)-\chi_2^2(4)-76}{\sqrt{1608}}$	0.991	4.485	0.992	0.1653	0.0619
$Q_6^* = \dfrac{10\chi_1^2(2)-\chi_2^2(2)-18}{\sqrt{404}}$	1.968	8.882	1.968	0.3281	0.2451

***Table (2.2) :** The exact and the approx. c.d.f. at various values.*

Quadratic Form	*Approx. c.d.f. form*	x	*Exact* $F_{Q^*}(x)$	*Approx.* $F_{Q^*}(x)$
Q_1^*	$\phi(x) - 0.0125\,\phi(x)(x^3-3x)$	1.96	0.97374	0.97384
		2.34	0.98849	0.98869
		3.10	0.99820	0.99825
		4.00	0.99991	0.99985
Q_2^*	$\phi(x) - 0.02083\,\phi(x)(x^3-3x)$	1.96	0.97299	0.97314
		2.34	0.98724	0.98777
		3.10	0.99764	0.99778
		4.00	0.99985	0.99975
Q_3^*	$\phi(x) - 0.03125\,\phi(x)(x^3-3x)$	1.96	0.97200	0.97238
		2.34	0.98568	0.98678
		3.10	0.99694	0.99724
		4.00	0.99978	0.99962

Quadratic Form	*Approx. c.d.f. form*	*x*	*Exact* $F_{Q^*}(x)$	*Approx.* $F_{Q^*}(x)$
Q_4^*	$\phi(x) - 0.0625\,\phi(x)\,(x^3 - 3x)$	1.96	0.97193	0.97064
		2.34	0.98101	0.98452
		3.10	0.99485	0.99584
		4.00	0.99891	0.99916
Q_5^*	$\phi(x) -$ $\phi(x\,[0.1653(x^2 + 1) + 0.0619\,(x^3\,3x)]$	1.96	0.94754	0.95539
		2.34	097125	0.97353
		3.10	0.98916	0.99113
		4.00	0.99881	0.99714
Q_6^*	$\phi(x) -$ $\phi(x\,[0.3281(x^2 + 1) + 0.2451\,(x^3\,3x)]$	1.96	0.89690	0.94844
		2.34	0.91580	0.96481
		3.10	0.96254	0.98361
		4.00	0.99755	0.99337

Looking at Table (2.2) above, we note that the Hermitte polynomial associated with the standard normal distribution gives a fairly good approximation when the quadratic form is symmetric Q_1 to Q_4 or slightly skewed Q_5, but it is worse when the quadratic form is skewed Q_6. This suggests finding another approximation for asymmetric quadratic forms.

Approximation By Laguerre Polynomials

The weight function of the Laguerre polynomial is a gamma type function? The gamma distribution is asymmetric distribution, so we hope that using a series of Laguerre.

polynomials associated with gamma density function would give a good approximation for asymmetric quadratic forms.

Suppose X is a random variable which is distributed gamma with shape, scale, location parameters a, b, and g, respectively. That is,

$$f_X(x) = \frac{\beta^{-\alpha}}{\Gamma(\alpha)}(x-\gamma)^{\alpha-1}e^{-(x-\gamma)/\beta}\,;\; x > \gamma \qquad \textbf{(2.5)}$$

For α>0,Laguerre polynomials, $L_n(x)$, are defined by the conditions

$$\int_0^\infty e^{-x}x^\alpha L_n^\alpha(x)L_m^\alpha(x)\,dx = \Gamma(\alpha+1)\binom{n+\alpha}{\alpha}\delta_{mn}\,.$$

In addition, it is required that the coefficients of x^n in the nth degree polynomial, $L_n(x)$, to have a positive sign if n is even and a negative sign if n is odd. This gives, for n = 0, 1, 2, …,

$$L_n^\alpha(x) = \sum_{j=0}^{n}\binom{n+\alpha}{n-j}\frac{(-x)^j}{j!}\,.$$

Laguerre polynomials are solutions to the second order differential equation

$$xy'' - (a - n + 1)\,y' + ny = 0,$$

and they have the following Rodriguez representation

$$L_n^\alpha(x) = \frac{x^{-\alpha}}{n1}e^x\,(d/dx)^n e^{-x}x^{n+\alpha}\,.$$

The weight function of the Laguerre Polynomials that will be used in our approximation is the truncated gamma density given in (2.5), So, their Rodriguez representation is

$$L_n^{(\alpha-1)}\left(\frac{x-\gamma}{\beta}\right) = \frac{1}{n!}\left[\left(\frac{x-\gamma}{\beta}\right)\right]^{-\alpha+1} e^{(x-\gamma)/\beta}\,(d/dx)^n\,[e^{-(x-\gamma)/\beta}\left(\frac{x-\gamma}{\beta}\right)^{\alpha+n-1} \qquad \textbf{(2.6)}$$

Put y = (x – y) / β, then the first four polynomials are :

$L_0^{(\alpha-1)}(y) = 1;\ L_1^{(\alpha-1)}(y) = -y+\alpha;\ L_2^{(\alpha-1)}(y) = y^2 - 2(\alpha+1)y + \alpha(\alpha+1);$
$L_3^{(\alpha-1)}(y) = -y^3 + 3(\alpha+2)y^2 - 3(\alpha+2)(\alpha+1)y + \alpha(\alpha+2)(\alpha+1);$
$L_4^{(\alpha-1)}(y) = y^4 - 4(\alpha+3)y^3 + 6(\alpha+2)(\alpha+3)y^2 - 4(\alpha+1)(\alpha+2)(\alpha+3)y + \alpha(\alpha+1)(\alpha+2)(\alpha+3).$

We approximate the density function **Q** by the partial sums of the series

$$f_{Q^*}(x) = g(x;\alpha,\beta,\gamma)\left[1 + \sum_{j=1}^{\infty} d_j L_j^{(\alpha-1)}\left(\frac{x-\gamma}{\beta}\right)\right], \qquad \textbf{(2.7)}$$

where $g(x;\alpha,\beta,\gamma)$ is the truncated gamma density defined in (2.5). The cumulants of Q are

$$\kappa_r(Q) = (r-1)!\,2^{r-1}\,\eta_r;\ r = 1, 2, \ldots.$$

and the cumulants of a random variable X having the truncated gamma density is

$$\kappa_r(X) = (r-1)!\,\alpha\beta^r;\ r = 1, 2, \ldots$$

To estimate the gamma parameters equate the first three cumulants of Q to those of X to get : $\eta_1 = (\gamma+\alpha\beta)$, $2\eta_2 = \alpha\beta^2$, and $8\eta_3 = 2\alpha\beta^3$. Solving these equations for α, β, and γ, we obtain

$$\alpha = \eta_2^3 / 2\eta_3^2,\ \beta = 2\eta_3/\eta_2,\ \text{and}\ \gamma = \eta_1 - \left(\eta_2^2/\eta_3\right).$$

The coefficients d_1, d_2, ... will be computed by applying the expectation operator on both sides of**(2.7)**.Having done that, we obtain

$$E[Q^k] = E[X^k] + \sum_{j=1}^{\infty} d_j \int_\gamma^\infty x^k L_j^{(\alpha-1)}\left(\frac{x-\gamma}{\beta}\right) f_X(x)dx,$$

Making use of the orthogonality of the Laguerre polynomials and following steps similar to those in computing the coefficients in the Hermitte expanion, we have the following expression for the d_k's

$$d_k = \frac{E[Q^k] - E[X^k] - \sum_{j=1}^{k-1} d_j \int_\gamma^\infty x^k L_j^{(\alpha-1)}\left(\frac{x-\gamma}{\beta}\right) f_X(x)dx}{\int_\gamma^\infty x^k L_k^{(\alpha-1)}\left(\frac{x-\gamma}{\beta}\right) f_X(x)dx}. \qquad \textbf{(2.8)}$$

Substituting the Rodriguez represention (2.6), in the integrand in the numerator of (2.8), and then integrating by parts j times, one has

$$\int_\gamma^\infty x^k L_j^{(\alpha-1)}\left(\frac{x-\gamma}{\beta}\right) f_X(x)dx = (-1)^j \frac{\Gamma(k+1)}{\Gamma(k-j+1)} E_X\left[X^{k-j}(X-\gamma)^j\right],$$

therefore **(2.8)** reduces to

$$d_k = \frac{\Gamma(\alpha)\left\{E[Q^k] - E[X^k] - \sum_{j=1}^{k-1} (-1)^j d_j \frac{\Gamma(k+1)}{\Gamma(k-j+1)} E_X\left[X^{k-j}(X-\gamma)^j\right]\right\}}{(-1)^k \beta^k \Gamma(k+1)\Gamma(k+\alpha)}$$

Note that $d_1 = d_2 = d_3 = 0$, so that $d_4 = \dfrac{\Gamma(\alpha)\left\{E[Q^4] - E[X^4]\right\}}{24\beta^4\Gamma(\alpha+4)}$.

Integrating **(2.7)** we get the following approximate form for the distribution function of Q

$$F_Q(x) = G_X(x;\alpha,\beta,\gamma) + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\beta\, g(x;\alpha+1,\beta,\gamma)\sum_{j=1}^{\infty} d_k L_{j-1}^{(\alpha-1)}\left(\frac{x-\gamma}{\beta}\right). \qquad \textbf{(2.9)}$$

Examples : In Table (2.3) below, we give few quadratic forms with the first four coefficients, the third and the fourth moments, and the skeweness. Table (2.4) contains the approximate and the exact c.d.f. at some given values.

Table (2.3) : Examples of quadratic forms with the estimated gamma parameters, skewness, and the fourth coefficient of the series in (2.7).

Quadratic Form	α	β	γ	Skewness	d_4
$Q_1 = 10\chi_1^2(2) - \chi_2^2(2)$	1.03237	19.78218	-2.42242	1.968	0.0125
$Q_2 = 10\chi_1^2(6) - \chi_2^2(2)$	3.03212	19.92691	-2.42081	1.149	0.0208
$Q_3 = 2\chi_1^2(6) - \chi_2^2(6)$	7.65306	2.80000	-15.42857	0.723	0.0313

***Table (2,4)** : The exact and the approx. c.d.f. using (2.9) at various values.*

Quadratic Form	x	*Exact* $F_{Q^\cdot}(x)$	*Approx.* $F_{Q^\cdot}(x)$
Q_1	1.96	0.94844	0.94827
	2.34	0.96481	0.96473
	3.10	0.98361	0.98362
	4.00	0.99337	0.99340
Q_2	1.96	0.94628	0.95362
	2.34	0.97160	0.97160
	3.10	0.98975	0.98976
	4.00	0.99708	0.99710
Q_3	1.96	0.95948	0.94799
	2.34	0.97657	0.97694
	3.10	0.99246	0.99359
	4.00	0.99819	0.99872

Form table **(2.4)**, we note that the approximation for asymmetrical quadratic forms using Laguerre polynomials associated with truncted gamma density is a very accurate approximation. The reasons behind the accuracy of this approximation are: First, we associate a truncated gamma density, witch is proportional to the weight function of the Laguerre polynomal, with the Laguerre series expansion. Second, the estimates of the gamma parameters are obtained by equating the first three moments of Q to those of the gamma density. Third, the way that the coefficients d_k are computed, by equating higher moments, further improves the accuracy.

We note that $\eta_3 \rightarrow 0$, the quadratic form tends to be symmetric, and at the same time $\alpha \rightarrow \infty$, It is known that as $\alpha \rightarrow \infty$ the gamma density approaches the normal density, This supports our choice of approximating the symmetric quadratic forms by Hermitte polynomials associated with normal distribution.

A final note about these series approximations is that we do not have to concern ourselves much about their convergence as far as we are getting good approximations using only the first few terms of these series, Because the series may be a convergent series, but does not give a good approximation without using too many terms of the series, and therefore, the convergence is of no practical significance.

3. *Approximating By a Mixture of Normal and Gamma Distribution*

In the previous sections we noted that Hermitte polynomials associated with normal distributions give a good approximation for the distribution of symmetric quadratic forms, and the Laguerre polynomials associated with gamma distributions give a good approximation for the distribution of indefinite quadratic forms when the it is not symmetric, Consequently, one may consider a mixture of normal and gamma distributions to be an approximation for the distribution of general quadratic forms. The approximation that we propose is of the form

$$f_Q(x) \cong p\, n(x; \mu, \sigma^2) + (1-p)\, g(x; \alpha, \beta, \gamma) \qquad \textbf{(3.1)}$$

where $n(x; \mu, \sigma^2)$ is the normal density and $g(x; \alpha, \beta, \gamma)$ is the truncated density, and where $0 \le p \le 1$.

In this approximation we have six parameters to be computed, this could be done by equating the first six moments of both sides of **(3.1)**, But this involves too much computational work, we will avoid that by standarizing Q and use a mixture of standard normal and normalized density. That is :

$$f_Q(x) \cong \frac{p}{\sqrt{2\pi}} e^{-x^2/2} \delta_{(-\infty,\infty)}(x) + (1-p) \frac{\sqrt{\alpha}}{\Gamma(\alpha)} (\sqrt{\alpha}\, x + \alpha)^{\alpha-1} e^{-(\sqrt{\alpha} x + \alpha)} \delta_{(-\sqrt{\alpha},\infty)}(x) \qquad \textbf{(3.2)}$$

The first and second moments of both sides of **(3.2)** are already equal and are free of parameters, So, to compute p and α we equate the third and fourth moments of both sides of **(3.2)** above. Having done that, the following two equations are obtained

$$E[Q^3] = 2(1-p) / \sqrt{\alpha}$$

$$E[Q^4] = 3p + (1-p)\left(1 + \frac{2}{\alpha}\right)$$

Solving for p and α, we have

$$p = 1 - \left(\eta_3^2 / \eta_2 \eta_4\right)$$

$$\alpha = \left(\eta_2 \eta_3^2 / 2\eta_4^2\right).$$

It is clear that $a > 0$ as it should be, for p to be between zero and one, we have to restrict our selves to the class of quadratic forms that satisfy $\eta_3^2 \le \eta_2\, \eta_4$ We note that if the distribution of Q is symmetric, then $\eta_3 = 0$, and hence $p=1$, That is, the distribution of Q is approximated by the normal distribution. If the skewness of Q which is given by

$$S = \left(4\eta_3 / \eta_2 \sqrt{2\eta_2}\right),$$

increases, the p decreases, which means that the approximation tends to take its values from the gamma distribution. This support our suggestions to approximate the distribution of symmetric quadratic forms by the Hermitte polynomials associated with normal distribution, and the Laguerre polynomials associated with the gamma distribution to approximate the distribution of asymmetric quadratic forms.

To show the performance of this approximation, we will give examples of three quadratic forms. One example if for p close to one, another is for p in the middle, and a third example is for p near zero.

***Table 5,1** : Approximation of general quadration forms using mixture of normal and gamma distributions.*

Quadratic Form	*p*	*a*	*x*	*Exact* $F_Q.(x)$	*Approx.* $F_Q.(x)$
$Q_1 = \frac{(\chi_1^2(4) - \chi_2^2(2))}{\sqrt{12}}$	*0.89*	*0.33*	1.96	0.95207	0.95274
$E[Q_1^3] = 0.385$			2.34	0.96839	0.96906
$E[Q_1^4] = 0.385$			3.10	0.98829	0.98851
			4.00	0.99897	0.99629
$Q_2 = \frac{(0.5\,\chi_1^2(2) - 0.3\,\chi_2^2(2) - 0.40)}{\sqrt{136}}$	0.48	0.556	1.96	0.95666	0.96081
$E[Q_2^3] = 1.398$			2.34	0.97002	0.97576
$E[Q_2^4] = 13.173$			3.10	0.98541	0.98921
			4.00	0.99362	0.99511
$Q_3 = \frac{(3\,\chi_1^2(4) - \chi_2^2(2) - 10)}{\sqrt{76}}$	0.09	2.009	1.96	0.95207	0.95274
$E[Q_3^3] = 1.280$			2.34	0.96838	0.96906
$E[Q_2^4] = 3.881$			3.10	0.98829	0.98855
			4.00	0.99897	0.99629

In conclusion, it is noticed that all three methods give good approximations for the distribution of general quadratic forms. We recommend using the Hermitte approximation for symmetric quadratic forms, Laguerre approximation for highly skewed quadratic forms, and the mixture of normal and trucated gamma for moderately skewed quadratic forms.

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