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ON A RELATION BETWEEN TWO ABSOLUTE SUMMABILITY METHODS

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حــول علاقــة بــين طريقتــين للتجمــيع المطلــق لمتسلســلة لا نهــائيــة

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فــي هــذا البحـث نبرهــن نظـرية تـدور حـول العـلاقـة بين طريقـتين للتـجمـيع المطلق هي $|R,p_n,q_n|_k$ و $|R,u_n|_k$ للمتسلسلة اللانهائية والتي تعمم نظرية سابقة لـ Bor [2] .

Key Words : Summability, Series, Sequences

ABSTRACT

In this paper we prove a theorem concerning a relation between the summability methods $|R,p_n,q_n|_k$ and $|R,u_n|_k$, $k \geq 1$, which generalizes a result of Bor [2].

1. INTRODUCTION

Let $\sum_1^\infty a_n$ be an infinite series with partial sums s_n. Let σ^δ_n and η_n^δ denote the nth Cesaro mean of order δ ($\delta > -1$) of the sequences $\{s_n\}$ and $\{na_n\}$ respectively. The series $\sum_1^\infty a_n$ is said to be summable $|C, \delta|_k$, $k \geq 1$ if

$$\sum_{n=1}^\infty n^{k-1} \left|\sigma_n^\delta - \sigma_{n-1}^\delta\right|^k < \infty$$

or equivalently $$\sum_{n=1}^\infty n^{-1} \left|\eta_n^\delta\right|^k < \infty \quad .$$

Let $\{p_n\}$ be a sequence of real or complex numbers with

$$P_n = p_0 + p_1 + \dots + p_n \quad , \quad P_{-1} = p_{-1} = 0 \ .$$

The series $\sum_1^\infty a_n$ is said to be summable $|N , p_n|$, if

$$\sum_{n=1}^\infty |t_n - t_{n-1}| < \infty \tag{1}$$

where $$t_n = P_n^{-1} \sum_{n=1}^\infty p_{n-v} s_v \qquad (t_{-1} = 0) \ .$$

We write $p = \{p_n\}$ and

$$M = \left\{ p : p_n > 0 \ \& \ \frac{p_{n+1}}{p_n} \leq \frac{p_{n+2}}{p_{n+1}} \leq 1 \ , n = 0,1,\dots \right\}$$

It is known that for $p \in M$, (1) holds if and only if (Das [3])

$$\sum_{n=1}^\infty \frac{1}{nP_n} \left| \sum_{v=1}^n p_{n-v} v a_v \right| < \infty$$

Definition 1 (Sulaiman [4]). For $p \in M$, we say that $\sum a_n$ is summable $|N , p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^\infty \frac{1}{nP_n^k} \left| \sum_{v=1}^n p_{n-v} v a_v \right|^k < \infty \quad .$$

In the special case in which $p_n = A_n^{r-1}$, $r > -1$, where A_n^r is the coefficient of x^n in the power series expansion of $(1-x)^{-r-1}$ for $|x| < 1$, $|N, p_n|_k$ summability reduces to $|C,r|_k$ summability see [3] .

The series $\sum a_n$ is said to be summable $|R,p_n|_k$ respectively $|\overline{N},p_n|_k, k \geq 1$, (Bor [2] & [1]) if.

$$\sum_{n=1}^{\infty} n^{k-1}|T_n - T_{n-1}|^k < \infty \text{ respectively } \sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|T_n - T_{n-1}|^k < \infty,$$

where

$$T_n = P_n^{-1}\sum_{v=0}^{\infty} p_v s_v$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), then $|R,p_n|_k$ summability is the same as $|C,r|_k$ (resp. $|R,p_n|$) summability.

Let $\{q_n\}$, $\{u_n\}$ be sequences of numbers and denote with

$Q_n = q_0 + q_1 + ... + q_n$, $q_{-1} = Q_{-1} = 0$

$U_n = u_0 + u_1 + ... + u_n$, $u_{-1} = U_{-1} = 0$

$R_n = p_0 q_n + p_1 q_{n-1} + .. + p_n q_0$

$\Delta f_n = f_n - f_{n+1}$, for any sequence $\{f_n\}$.

Here we give the following new definition.

Definition 2. Let $\{p_n\}$, $\{q_n\}$ be sequences of positive real constants such that $q \in M$. We say theis $\sum_1^{\infty} a_n$ is summable $|R,p_n,q_n|_k$, $k \geq 1$, if.

$$\sum_{n=1}^{\infty}\frac{1}{n}\left|\frac{np_n}{P_n R_{n-1}}\sum_{v=1}^{n} P_{v-1} q_{n-v} a_v\right|^k < \infty$$

Clearly, $|R,p_n 1|_k$ and $|R,1, p_n|_k$ are equivalent to $|R,p_n|_k$ and $|N,q_n|_k$ respectively . This follows as $\sum a_n$ summable $|R,p_n 1|_k$, iff.

$$\sum_{n=1}^{\infty} n^{k-1}\left|\frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} P_{v-1} a_v\right|^k < \infty$$

Iff $\sum_1^{\infty} a_n$ summable $|R,p_n|_k$. Since $0 \leq Q_n \leq Q_{n+1}$ then $Q_n = O(Q_{n+1})$. As

$$Q_{n+1} = Q_n + q_{n+1} \leq Q_n + q_n \leq 2Q_n,$$

then $Q_{n+1} = O(Q_n)$

Therefor $\sum_1^{\infty} a_n$ summable $|R,1, q_n|_k$, iff

$$\sum_{n=1}^{\infty}\frac{1}{n}\left|\frac{1}{Q_{n-1}}\sum_{v=1}^{n}(v-1)q_{n-v}a_v\right|^k < \infty$$

iff $$\sum_{n=1}^{\infty}\frac{1}{n}\left|\frac{1}{Q_n}\sum_{v=1}^{n} v q_{n-v} a_v\right|^k < \infty$$

iff $\sum a_n$ summable $|N, P_n|_k$

2. MAIN RESULT

We prove the following :

THEORM 1. :

Let $\{P_n\}$, $\{q_n\}$ and $\{u_n\}$ be sequences of positive real numbers such that $q = \{q_n\}$, and assume $q \in M$, $\{n^{1-1/k} P_n / p_n R_{n-1}\}$ nonincreasing for $q_n \neq c$. Let T_n denote the (N,u_n) - mean of the series $\sum a_n$. Let $\{\epsilon_n\}$ be a sequence of constants. If :

$$\sum_{n=v+1}^{m+1}\frac{n^{k-1}p_n}{P_n^k R_{n-1}} q_{n-v-1} = O\left\{\frac{n^{k-1}p_n^{k-1}}{P_n^k}\right\},$$

$$\sum_{n=1}^{\infty} n^{k-1}\left(\frac{p_n}{P_n}\right)^k\left(\frac{P_{n-1}}{R_{n-1}}\right)^k\left(\frac{U_n}{u_n}\right)^k|\epsilon_n|^k|\Delta T_{n-1}|^k < \infty,$$

$$\sum_{n=1}^{\infty} n^{k-1}|\epsilon_n|^k|\Delta T_{n-1}|^k < \infty,$$

$$\sum_{n=1}^{\infty} n^{k-1}\left(\frac{p_n}{P_n}\right)^k\left(\frac{U_n}{u_n}\right)^k|\epsilon_n|^k|\Delta T_{n-1}|^k < \infty,$$

and

$$\sum_{n=1}^{\infty} n^{k-1}\left(\frac{U_n}{u_n}\right)^k|\Delta \epsilon_n|^k|\Delta T_{n-1}|^k < \infty,$$

then the series $\sum a_n$ is summable $|R,p_n,q_n|_k$, $K \geq 1$.

In [5], we proved that if $q \in M$, then for $0 < r \leq 1$,

$$\sum_{n=v+1}^{\infty}\frac{q_{n-v-1}}{n^r Q_{n-1}} = O(v^{-r}).$$

Proof of the Theorem :

Denote :

$$\phi_n = \sum_{v=1}^{n} P_{v-1} q_{n-v} a_v \,\epsilon_v.$$

Since $$T_n = U_n^{-1}\sum_{v=0}^{n} u_v \sum_{r=0}^{v} a_r = U_n^{-1}\sum_{v=0}^{n}(U_n - U_{v-1})a_v,$$

then $$-\Delta T_{n-1} = \frac{u_n}{U_n U_{n-1}}\sum_{v=1}^{n} U_{v-1} a_v.$$

By means of the Abel's transformation, one gets :

$$\phi_n = \sum_{v=1}^{n} U_{v-1} a_v (P_{v-1} q_{n-v} U_{v-1}^{-1} \epsilon_v)$$

$$= \sum_{v=1}^{n-1} \Big(\sum_{r=1}^{v} U_{r-1} a_r\Big) \Delta_v (P_{v-1} q_{n-v} U_{v-1}^{-1} \epsilon_v) +$$

$$+\Big(\sum_{r=1}^{n} U_{r-1} a_r\Big) P_{n-1} q_0 U_{n-1}^{-1} \epsilon_n = \sum_{v=1}^{n-1} \left\{ -\frac{U_{v-1} U_v}{u_v} \Delta T_{v-1} \right\} \Big\{$$

$$\left\{ P_{v-1} \Delta_v q_{n-v} U_{v-1}^{-1} \epsilon_v \right\} + P_{v-1} q_{n-v-1} \frac{u_v}{U_{v-1} U_v} \epsilon_v$$

$$- p_v q_{n-v-1} U_v^{-1} \epsilon_v + P_v q_{n-v-1} U_v^{-1} \Delta \epsilon_v \Big\} -$$

$$- P_{n-1} q_0 U_n u_n^{-1} \epsilon_n \Delta T_{n-1}$$

$$= \sum_{v=1}^{n-1} \Big\{ - P_{v-1} \Delta q_{n-v} \frac{U_v}{u_v} \epsilon_v \Delta T_{v-1}$$

$$- P_{v-1} q_{n-v-1} \epsilon_v \Delta T_{v-1} - p_v q_{n-v-1} \frac{U_{v-1}}{u_v}$$

$$\times \epsilon_v \Delta T_{v-1} + P_v q_{n-v-1} \frac{U_{v-1}}{u_v} \Delta \epsilon_v \Delta T_{v-1} \Big\} -$$

$$- P_{n-1} q_0 \frac{U_n}{u_n} \epsilon_n \Delta T_{n-1}$$

$$= \phi_{n,1} + \phi_{n,2} + \phi_{n,3} + \phi_{n,4} + \phi_{n,5}\,,$$ where

$$\phi_{n,1} = \sum_{v=1}^{n-1} - P_{v-1} \Delta_v q_{n-v} \frac{U_v}{u_v} \epsilon_v \Delta T_{v-1}$$

$$\phi_{n,2} = \sum_{v=1}^{n-1} - P_{v-1} q_{n-v-1} \epsilon_v \Delta T_{v-1}$$

$$\phi_{n,3} = \sum_{v=1}^{n-1} - p_v q_{n-v-1} \frac{U_{v-1}}{u_v} \epsilon_v \Delta T_{v-1}$$

$$\phi_{n,4} = \sum_{v=1}^{n-1} P_v q_{n-v-1} \frac{U_{v-1}}{u_v} \Delta \epsilon_v \Delta T_{v-1}$$

$$\phi_{n,5} = - P_{n-1} q_0 \frac{U_n}{u_n} \epsilon_n \Delta T_{n-1}$$

In order to prove the theorem, it is sufficient, by Minkowski's inequality, to show that.

$$\sum_{n=1}^{\infty} n^{-1} \left| \frac{n p_n}{P_n R_{n-1}} \phi_{n,r} \right|^k < \infty, \quad r = 1,2,3,4,5 \;.$$

Applying Holder's inequality ,

$$\sum_{n=2}^{m+1} n^{-1} \left| \frac{n p_n}{P_n R_{n-1}} \phi_{n,1} \right|^k =$$

$$= \sum_{n=2}^{m+1} n^{-1} \left| \frac{n p_n}{P_n R_{n-1}} \sum_{v=1}^{n-1} P_{v-1} \Delta_v q_{n-v} \frac{U_v}{u_v} \epsilon_v \Delta T_{v-1} \right|^k$$

$$\le \sum_{n=2}^{m+1} n^{-1} \left(\frac{n p_n}{P_n R_{n-1}} \right)^k \sum_{v=1}^{n-1} P_{v-1}^k |\Delta_v q_{n-v}| \left(\frac{U_v}{u_v} \right)^k$$

$$\times |\epsilon_v|^k |\Delta T_{v-1}|^k \times \left\{ \sum_{v=1}^{n-1} |\Delta_v q_{n-v}| \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} P_{v-1}^k \left(\frac{U_v}{u_v} \right)^k |\epsilon_v|^k |\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1} \frac{n^{k-1} p_n^k}{P_n^k R_{n-1}^k} |\Delta_v q_{n-v}|$$

$$= O(1) \sum_{v=1}^{m} v^{k-1} \left(\frac{p_v}{P_v} \right)^k \left(\frac{P_{v-1}}{R_{v-1}} \right)^k \left(\frac{U_v}{u_v} \right)^k |\epsilon_v|^k |\Delta T_{v-1}|^k$$

$$\sum_{n=2}^{m+1} n^{-1} \left| \frac{n p_n}{P_n R_{n-1}} \phi_{n,2} \right|^k =$$

$$= \sum_{n=2}^{m+1} n^{-1} \left| \frac{n p_n}{P_n R_{n-1}} \sum_{v=1}^{n-1} \frac{P_{v-1}}{p_v} p_v q_{n-v-1} \epsilon_v \Delta T_{v-1} \right|^k$$

$$\le \sum_{n=2}^{m+1} \frac{n^{k-1} p_n^k}{P_n^k R_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_{v-1}}{p_v} \right)^k p_v q_{n-v-1} |\epsilon_v|^k |\Delta T_{v-1}|^k \times$$

$$\times \left\{ \sum_{v=1}^{n-1} \frac{p_v q_{n-v-1}}{R_{n-1}} \right\}^{k-1} \le O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^k p_v |\epsilon_v|^k |\Delta T_{v-1}|^k$$

$$\times \sum_{n=v+1}^{m+1} \frac{n^{k-1} p_n^k}{P_n^k R_{n-1}} q_{n-v-1} = O(1) \sum_{v=1}^{m} v^{k-1} |\epsilon_v|^k |\Delta T_{v-1}|^k$$

$$\sum_{n=2}^{m+1} n^{-1} \left| \frac{n p_n}{P_n R_{n-1}} \phi_{n,3} \right|^k =$$

$$= \sum_{n=2}^{m+1} n^{-1} \left| \frac{n p_n}{P_n R_{n-1}} \sum_{v=1}^{n-1} p_v q_{n-v-1} \frac{U_{v-1}}{u_v} \epsilon_v \Delta T_{v-1} \right|^k$$

$$\le \sum_{n=2}^{m+1} \frac{n^{k-1} p_n^k}{P_n^k R_{n-1}} \sum_{v=1}^{n-1} p_v q_{n-v-1} \left(\frac{U_{v-1}}{u_v} \right)^k$$

$$\times |\epsilon_v|^k |\Delta T_{v-1}|^k \times \left\{ \sum_{v=1}^{n-1} \frac{p_v q_{n-v-1}}{R_{n-1}} \right\}^{k-1}$$

$$\le O(1) \sum_{v=1}^{m} p_v \left(\frac{U_v}{u_v} \right)^k |\epsilon_v|^k |\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1} \frac{n^{k-1} p_n^k}{P_n^k R_{n-1}} q_{n-v-1}$$

$$= O(1) \sum_{v=1}^{m} v^{k-1} \left(\frac{p_v}{P_v} \right)^k \left(\frac{U_v}{u_v} \right)^k |\epsilon_v|^k |\Delta T_{v-1}|^k$$

$$\times \sum_{n=2}^{m+1} n^{-1} \left| \frac{n p_n}{P_n R_{n-1}} \phi_{n,4} \right|^k =$$

$$= \sum_{n=2}^{m+1} n^{-1} \left| \frac{n p_n}{P_n R_{n-1}} \sum_{v=1}^{n-1} P_v q_{n-v-1} \frac{U_{v-1}}{u_v} \Delta \epsilon_v \Delta T_{v-1} \right|^k$$

$$\le \sum_{n=2}^{m+1} \frac{n^{k-1} p_n^k}{P_n^k R_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v} \right)^k p_v q_{n-v-1} \left(\frac{U_{v-1}}{u_v} \right)^k$$

$$\times |\epsilon_v|^k |\Delta T_{v-1}|^k \times \left\{ \sum_{v=1}^{n-1} \frac{p_v q_{n-v-1}}{R_{n-1}} \right\}^{k-1}$$

$$\le O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^k p_v \left(\frac{U_v}{u_v} \right)^k |\Delta \epsilon_v|^k |\Delta T_{v-1}|^k$$

$$\times \sum_{n=v+1}^{m+1} \frac{n^{k-1} p_n^k}{P_n^k R_{n-1}} q_{n-v-1} = O(1) \sum_{v=1}^{m} v^{k-1} \left(\frac{U_v}{u_v} \right)^k |\Delta \epsilon_v|^k |\Delta T_{v-1}|^k$$

$$\sum_{n=2}^{m+1} n^{-1} \left| \frac{n p_n}{P_n R_{n-1}} \phi_{n,5} \right|^k = \sum_{n=2}^{m+1} n^{-1} \left| \frac{n p_n}{P_n R_{n-1}} P_{n-1} q_0 \frac{U_n}{u_n} \epsilon_n \Delta T_{n-1} \right|^k$$

$$= O(1) \sum_{v=1}^{m} n^{k-1} \left(\frac{p_n}{P_n} \right)^k \left(\frac{P_{n-1}}{R_{n-1}} \right)^k \left(\frac{U_n}{u_n} \right)^k |\epsilon_n|^k |\Delta T_{n-1}|^k$$

3. APPLICATIONS

Throughout the rest of the paper, we may assume that $\{p_n\}$, $\{q_n\}$, and $\{u_n\}$ are sequences of positive real constants such that P_n, Q_n, and U_n are all tends to ∞.

THEOREM 2 (Bor [2]) : A necessary condition that $\Sigma\alpha_n$ is summable $|R,p_n|_k$, whenever it is summable $|R,u_n|_k$, $k \geq 1$ is :

$$p_n U_n = O(p_n u_n) \quad . \tag{2}$$

If

$$\sum_{n=v}^{\infty} \frac{n^{k-1} p_n^k}{P_n^k P_{n-1}} = O\left\{\frac{n^{k-1} p_n^{k-1}}{P_n^k}\right\}, \tag{3}$$

then (2) is also sufficient.

Proof : Necessity. This follows on the lines of [2] as for $\epsilon_n = 1$ and $q_n = 1$,

$$\frac{np_n}{P_n R_{n-1}} \phi_{n,5} = \frac{p_n U_n}{P_n u_n} \Delta T_{n-1}.$$

Sufficiency. Follows from theorem 1 by putting $\epsilon_n = 1$ & $q_n = 1$,

THEOREM 3 : Sufficient conditions that Σa_n is summable $|R,p_n|_k$, whenever it is summable $|\bar{N}, u_n|_k$, $k \geq 1$, are (2), (3) & $nu_n = O(U_n)$.

Proof : Follows from theorem 1 by putting $\epsilon_n = 1$ & $q_n = 1$.

THEOREM 4 : Sufficient conditions that Σa_n is summable $|N, p_n|_k$, whenever it is summable $|\bar{N}, u_n|_k$, $k \geq 1$, are.

$$n = O(Q_n) \ , \ nu_n = O(U_n) \ \& \ U_n = O(nu_n).$$

Proof : Follows from theorem 1 by putting $\epsilon_n = 1$ & $q_n = 1$, and making use of lemma 1 .

COROLLARY 1 : Sufficient conditions that Σa_n is summable $|R,p_n|_k$, whenever it is summable $|C,1|_k$, $k \geq 1$, are (3) & $np_n = O(P_n)$.

Proof : Follows from theorem 3 by putting $u_n = 1$.

REMARK : If $P_n = O(np_n)$, then $|R,p_n|_k \Rightarrow |\bar{N},p_n|_k$.

COROLLARY 2 : Sufficient conditions that Σa_n is summable $|\bar{N}, p_n|_k$, whenever it is summable $|C,1|_k$, $k \geq 1$, is.

$$np_n = O(P_n) \ \& \ P_n = O(np_n). \tag{4}$$

Proof :

$$\sum_{n=v}^{\infty} \frac{n^{k-1} p_n}{P_n^k P_{n-1}} = O(1) \sum_{n=v}^{\infty} \frac{p_n}{P_n P_{n-1}} = O(\frac{1}{P_v}) =$$

$$= O\left\{\frac{P_v^{k-1}}{P_v^k}\right\} = O\left\{\frac{v^{k-1} P_v^{k-1}}{P_v^k}\right\}.$$

The proof follows from corollary 2 and the remark.

THEOREM 5 (Bor [1]) : If (4) is satisfied, then the series Σa_n is $|\bar{N}, p_n|_k$, if and only if it is $|C,1|_k$, $k \geq 1$.

Proof : Follows from theorem 4 with $q_n = 1$, and corollary 2.

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