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Lumping Processes of Periodic ARMA Processes

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العمليات التراكمية لعمليات (PARMA)

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في هذا البحث تمت مقارنة ثلاث عمليات تراكمية لعمليات (PARMA) وهي العملية الأمامية، العملية العكسية والعملية الدورية، لقد توصلنا إلى العلاقات الثنائية التي تربط ما بين معلمات كل من هذه العمليات التراكمية كما تم إثبات أن استقرار أي من هذه العمليات يؤدي إلى أن العمليات الأخرى هي أيضاً مستقرة.

Keywords: *PARMA processes, Lumped process, Stationarity, Periodic stationarity.*

ABSTRACT

In this article various lumping processes are defined for periodic ARMA (PARMA) processes, namely, the forward, the backward and the cyclic lumping processes. The inter-relations between the parameters of these lumping processes are obtained. It is then proved that stationarity of any lumped process implies stationarity of all other lumped processes.

1. Introduction

In the literature of time series analysis, there are several methods for modeling seasonal time series. The multiplicative seasonal ARIMA model and classical decomposition method could be the most common among them.

Recently, an extension of ordinary ARMA model has been developed for modeling seasonal time series named as periodic ARMA (PARMA) model. The essence of this model is that if the time series is seasonal with period ω, then the model consists of ω equations, i.e. a separate equation for each season. For a detailed account on PARMA models see, for example, [1] and [2].

An m-variate zero-mean stochastic process $\{X_t\}$ is said to follow $PARMA_{\omega}(p(\nu),q(\nu))$ model iff

$$\Phi (B) X_t = \Theta (B) a_t, \tag{1.1}$$

where $\Phi (B) = 1 - \varphi_1(t)B^{\rho} - \dots - \varphi_{\rho}(t)B^{\rho}$, $\Theta (B) = 1 - \theta_1(t)B - \dots - \theta_q(t)B^q$, $\varphi_i(t)$ $[\theta_j(t)]$ are $m \times m$ periodic AR [MA] coefficient matrices; $i = 1, \dots, p$, $j = 1, \dots, q$ where $p = \max_{\nu} (p(\nu))$ and $q = \max_{\nu} (q(\nu))$, $\nu = 1, \dots, \omega$, $\varphi_i(t) = O_{m \times m}$, $\forall\, i > p(\nu)$, and $\theta_j(t) = O_{m \times m}$, $\forall\, j > q(\nu)$, and $\{a_t\}$ is an m-variate periodic white noise process with zero mean and periodic covariance matrices $\Sigma_a(t)$.

It is usually more realistic to write the time index t in (1.1) in ω modulus, i.e. $t = k\omega+\nu$, so that, for monthly data, say, k refers to the year index and ν represents the season index. Therefore, (1.1) can be rewritten as

$$(1 - \varphi_1(\nu)B - \dots - \varphi_{p(\nu)}(\nu)B^{p(\nu)})X_{k\omega+\nu} = (1 - \theta_1(\nu)B - \dots - \theta_{q(\nu)}(\nu)B^{q(\nu)})a_{k\omega+\nu}. \tag{1.2}$$

PARMA processes are not stationary (in the ordinary weak sense) simply because their parameters vary with time. In fact, PARMA processes are rather examined for a weaker type of stationarity named as periodic stationarity (see, for example [3]).

Many mathematical results concerning PARMA processes have been developed utilizing the fact that any PARMA model can be written as a multivariate ARMA model and vice versa. This is simply done by lumping the ω components in the PARMA process, i.e. $X_{k\omega+\nu}$, $\nu = 1, \dots, \omega$ in one vector, which in turn is an $m\omega$-variate process. Estimation of PARMA models [4, 5], periodic stationarity of PARMA processes [3] and estimation of multivariate ARMA models [6] are important by-products of such lumping relations.

In the next section we will define three lumping processes, namely, the forward, backward and cyclic lumping processes. In this paper we will obtain the parametric relations between those lumping processes. Then we show that if any of the lumping processes is stationary then all other lumping processes are also stationary.

2. Various Lumping Processes

In what follows, we assume that $\{X_{k\omega+\nu}\}$ is a stochastic process which follows the m-variate zero-mean

$PARMA_\omega(p(\nu),q(\nu))$ model, (1.2). We can rewrite the ω-equations in this model in matrix format by lumping the components of this process in one vector. In this article, we will consider three of those which are the forward lumping process (FLP), the backward lumping process (BLP) and the cyclic lumping process (CLP).

The FLP is defined by

$$F_k = (X_{k\omega+1}^T, X_{k\omega+2}^T, \ldots, X_{k\omega+\omega}^T)^T. \tag{2.1}$$

This process has been considered, for example, in [1, 7, 8]. In [7], the author proved that a univariate PARMA (or any periodically correlated) process is periodic stationary iff its FLP is stationary. The authors in [1] obtained the parameters and orders of the FLP for univariate PARMA processes. Their result has then been generalized to the multivariate case in [8].

In the following lemma the parametric equation of the model of the FLP is given.

Lemma (2.1): The FLP, F_k, follows the following mω-variate ARMA(p^*,q^*) model

$$A F_k - \sum_{l=1}^{p^*} B_l F_{k-l} = C\varepsilon_k - \sum_{l=1}^{q^*} D_l \varepsilon_{k-l} \tag{2.2}$$

with

$$[A]_{ij} = \begin{cases} I_m, & i=j \\ 0_m, & i<j \\ \varphi_{i-j}(i), & i>j \end{cases}$$

and $[B_l]_{ij} = \varphi_{l\omega+i-j}(i)$ for i, j = 1, …, ω, and C and D_l are the same as A and B_l but with θ's replacing φ's and ε_k is the FLP of the periodic white noise process $\{a_{k\omega+\nu}\}$ (as in (2.1)), with

$$p^* = \max_\nu \left[(p(\nu) - \nu)/\omega\right] + 1, \tag{2.3a}$$

$$q^* = \max_\nu \left[(q(\nu) - \nu)/\omega + 1\right], \tag{2.3b}$$

where [.] stands for the greatest integer.

The Proof follows from [1,2].

Based on the lemma above, it is easily observed that the FLP is stationary iff all the roots of the determinantal equation

$$|\lambda^{p^*} I_{m\omega} - \lambda^{p^*-1} A^{-1} B_1 - \ldots - A^{-1} B_{p^*}| = 0 \tag{2.4}$$

are less than one in modulus, which is nothing but the stationarity condition of a multivariate ARMA process (see, for example, [9]). This fact will be utilized later on.

The BLP contains the seasonal components in reverse order as compared to the FLP. It is defined by

$$Y_k = (X_{k\omega+\omega}{}^T, X_{k\omega+\omega-1}{}^T, \ldots, X_{k\omega+1}{}^T)^T. \quad (2.5)$$

This process has some advantages over the FLP. It is more natural in time series models to express the time series at time t in terms of those at times t-1, t-2, and so on in a backward manner. This is helpful, for example, for forecasting. In PARMA context, the backward representation usually provides some analytical simplicity [3].

The BLP Y_k also follows $m\omega$-variate ARMA(p^*,q^*) model similar to (2.2), where p^* and q^* are the same as for the FLP given in (2.3). This model is given by

$$L\,Y_k - \sum_{\ell=1}^{p^*} M_\ell Y_{k-\ell} = N e_k - \sum_{\ell=1}^{q^*} P_\ell e_{k-\ell} \quad (2.6)$$

where

$$[L]_{ij} = \begin{cases} I_m, & i = j \\ 0_m, & i > j \\ -\varphi_{ij}(\omega + 1 - i), & i < j \end{cases}$$

and $[M_l]_{ij} = \varphi_{l\omega+j-i}(\omega+1-i)$ for i, j = 1, …, ω, and N and P_l are the same as L and M_l but with θ's replacing φ's and e_k is the BLP of the periodic white noise process $\{a_{k\omega+v}\}$[3].

A comparison between the parameters in (2.2) and (2.6) reveals the fact that L in (2.6) contains the same elements as A in (2.2). Considering A in a partitioned form of order ω x ω, if the first row is interchanged with the last row, the second row with the second to the last row, and so on, and then the columns are interchanged in the same way, then the resulting matrix will be L. The same relation holds between the other corresponding parameters in (2.2) and (2.6).

Another type of lumping processes is any cyclic permutation of either the FLP or the BLP. There are 2ω different cyclic permutations resulting from both the FLP and BLP. As it is sufficient to consider one of those, we consider the first cyclic permutation of the FLP process defined as

$$F^{(1)}_k = (X_{k\omega+2}{}^T, \ldots, X_{k\omega+\omega}{}^T, X_{k\omega+1}{}^T)^T. \quad (2.7)$$

which will be abbreviated to the CLP.

By similar analogy to (2.6) for the BLP, the model of $F^{(1)}_k$ can easily be obtained. Its orders, on the other hand, are the same as that of the BLP and the FLP, (2.3). Also, the parametric equation of the model is very close to (2.2). For instance, in place of A in (2.2), we will have another matrix which is a slight modification of A. That is, considering A in a partitioned form of order ω x ω, the first row is removed, the other rows are shifted up, then it is inserted in place of the last row (this is nothing but a cyclic permutation of order 1 of the rows of A). Then, the columns of the resulting matrix are manipulated in the same manner. The same relation carries over other corresponding parameters in (2.2).

One practical situation in which we may need cyclic lumping processes, such as the CLP, is that we may have in practice a time series which starts in a season other than the first season. For example, a monthly time series may start in February rather than in January. In this case, using such representations will enable us to use all the time series for inference and modeling purposes rather than, possibly, ignoring some data in our time series.

Another general reason to prefer one lumping process to another is the analytic simplicity in parameters, such as, for instance, triangular matrices or so forth.

3. Parametric Relations Between Various Lumping Processes

In this section we will obtain parametric relations between various lumping processes discussed in the previous section. Then, we will see implications of those relations on stationarity of various lumping processes.

We now give some results concerning permutation matrices that will be useful later. Such results can be found in most standard books in matrix theory [10].

Definition (3.1): A ω x ω permutation matrix is any matrix resulting from re-ordering rows of the ω x ω identity matrix.

Lemma (3.1): Permutation matrices are orthogonal, i.e. if P is a permutation matrix then $P^{-1} = P^T$.

We now consider two special cases of permutation matrices, defined as

$$\mathbf{P}_{rev} = \begin{bmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{bmatrix}$$

and

$$\mathbf{P}_{cyc} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Thus, as $\mathbf{P}_{rev}$ is symmetric, then, if $\mathbf{P}_{rev}$ is pre (post)-multiplied by any matrix then its rows (columns) are rearranged in reverse order. Also, if $\mathbf{P}_{cyc}$ is pre-multiplied by any matrix then the first row is inserted in place of the last row and other rows are shifted up. The same function applies on the columns of any matrix if post-multiplied by the transpose of $\mathbf{P}_{cyc}$.

In the next theorem we assume that the permutation matrices $\mathbf{P}_{rev}$ and $\mathbf{P}_{cyc}$ are mω x mω partitioned matrices with m x m sub-matrices. Also, the null and identity ω x ω matrices replace 0 and 1 in these matrices, respectively. Note that the functions of Prev and Pcyc explained above continue but rather on partitioned matrices.

Theorem (3.1): Let $\{X_{k\omega+v}\}$ be an m-variate $\text{PARMA}_{\omega}(p(v),q(v))$ process following (1.2).

(i) The relation between the corresponding parameters of the FLP model, (2.2), and those of the BLP, (2.6), given in terms of A and L, is

$$L = \mathbf{P}_{rev} \, A \, \mathbf{P}_{rev} \tag{3.1}$$

(ii) The parameter of the CLP, defined in (2.7), corresponding to A in the FLP model, (2.2), is

$$\mathbf{P}_{cyc} \, A \, (\mathbf{P}_{cyc})^{T}, \tag{3.2}$$

which is the same relation between other corresponding parameters.

Proof: Equations (3.1) and (3.2) are valid in view of the relations between the parameters of the FLP, the BLP and the CLP explained in the previous section and the functions of permutation matrices $\mathbf{P}_{rev}$ and $\mathbf{P}_{cyc}$ given above.

Corollary (3.1): If any of the FLP, the BLP or the CLP is stationary then the other two are also stationary.

Proof: It is sufficient to show that if the FLP is stationary then the BLP and the CLP are also stationary. The FLP is stationary iff the roots of the determinantal equation (2.4) are less than one in modulus. The BLP is stationary iff the same condition applies on the determinantal equation

$$|\lambda^{p^*} I_{m\omega} - \lambda^{p^*-1} L^{-1} M_1 - \ldots - L^{-1} M_{p^*}| = 0,$$

where L and M_i's are the parameters of the BLP in (2.6). Now, utilizing (3.1), this equation simplifies to

$$|\lambda^{p^*} I_{m\omega} - \lambda^{p^*-1} (\mathbf{P}_{rev} \mathbf{A} \mathbf{P}_{rev})^{-1} (\mathbf{P}_{rev} \mathbf{B}_1 \mathbf{P}_{rev}) - \ldots - (\mathbf{P}_{rev} \mathbf{A} \mathbf{P}_{rev})^{-1} \mathbf{P}_{rev} \mathbf{B}_{p^*} \mathbf{P}_{rev})| = 0$$

then writing $I_{m\omega}$ as $\mathbf{P}_{rev} \mathbf{P}_{rev}$, this will simply lead to (2.4). By similar argument, making use of (3.2), stationarity condition of the CLP simplifies to the same condition for the FLP applied on

$$|\lambda^{p^*} - \lambda^{p^*-1} (\mathbf{P}_{cyc} A (\mathbf{P}_{cyc})^{T})^{-1} (\mathbf{P}_{cyc} B_1 (\mathbf{P}_{cyc})^{T}) - \ldots$$
$$- (\mathbf{P}_{cyc} \mathbf{A} (\mathbf{P}_{cyc})^{T})^{-1} (\mathbf{P}_{cyc} \mathbf{B}_{p^*} (\mathbf{P}_{cyc})^{T})| = 0,$$

then writing $I_{m\omega}$ as $\mathbf{P}_{cyc}(\mathbf{P}_{cyc})^T$, this again leads to (2.4).
In the next example, various lumping processes discussed above, are given for a simple PARMA model. The relations between their parameters provided in theorem (3.1) are also verified.

Example (3.1): Consider the zero-mean, m-variate $PARMA_\omega(p(\nu),q(\nu))$ model, (1.2), with $\omega = 4$ and $p(\nu) = 1$ and $q(\nu) = 0$, for $\upsilon = 1, 2, 3, 4$, i.e., the m-variate $PAR_4(1)$ model, written explicitly as

$$X_{4k+1} = \varphi_1(1)\, X_{4(k-1)+4} + a_{4k+1}$$

$$X_{4k+2} = \varphi_1(2)\, X_{4k+1} + a_{4k+2}$$

$$X_{4k+3} = \varphi_1(3)\, X_{4k+2} + a_{4k+3}$$

$$X_{4k+4} = \varphi_1(4)\, X_{4k+3} + a_{4k+4}.$$

The FLP, defined in (2.1), is $F_k = (X_{4k+1}^T, X_{4k+2}^T, X_{4k+3}^T, X_{4k+4}^T)^T$ which follows the 4m-variate AR(1) model, deduced from (2.2), given by

$$A\,F_k - B_1 F_{k-1} = \varepsilon_k, \tag{3.3}$$

where

$$A = \begin{bmatrix} I & 0 & 0 & 0 \\ -\varphi_1(2) & I & 0 & 0 \\ 0 & -\varphi_1(3) & I & 0 \\ 0 & 0 & -\varphi_1(4) & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 & 0 & -\varphi_1(1) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with I and 0 m x m.

The BLP model, (2.6), simplifies here to an equation very similar to (3.3). The corresponding parameters of A and B_1 are

$$L = \begin{bmatrix} I & -\varphi_1(4) & 0 & 0 \\ 0 & I & -\varphi_1(3) & 0 \\ 0 & 0 & I & -\varphi_1(2) \\ 0 & 0 & 0 & I \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\varphi_1(1) & 0 & 0 & 0 \end{bmatrix}$$

respectively.

Finally, the CLP model will also be similar to (3.3) and corresponding to A and B_1 above we will have, respectively,

$$\begin{bmatrix} I & 0 & 0 & -\varphi_1(2) \\ -\varphi_1(3) & I & 0 & 0 \\ 0 & -\varphi_1(4) & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\varphi_1(1) & 0 \end{bmatrix}$$

It can easily be shown that the parameters above satisfy (3.1) and (3.2).

4. Summary and Conclusions

Three lumping processes of PARMA processes have been considered. The parametric relations among them are obtained. It is shown that stationarity of any of them implies stationarity of the others.

Beside the FLP, the BLP and the CLP, we can define other lumping processes. If we do not care about the order of the time index among the components of a lumping process, then the total number of different lumping processes is $\omega!$ For instance, we may define a lumping process by interchanging the first two components in the FLP.

The parametric relations among such lumping processes can easily be obtained via suitable permutation matrices as (3.1) and (3.2). Hence, we conclude that stationarity of any lumping process implies stationarity of all other lumping processes.

Although such lumping processes may not have apparent applications, one application could be in the context of multivariate time series. As a vector of time series is much the same as a lumping process, then if we are interested in the time series model of some rearrangements of such vector then the results concerning lumping processes observed in this article apply directly. Another close application to this could be in multi-channel filtering in signal processing, see for example [11].

Finally, the results obtained in this article regarding stationarity of a PARMA process and its lumping processes extend directly to invertibility of a PARMA process. All what we need to recall is that invertibility conditions are the same as stationarity conditions with θ's replacing φ's.

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