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THE BACKLUND TRANSFORMATION OF A KP-LIKE CLASS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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تحويلات باكلاند لفصل من المعادلات التفاضلية الجزئية غير الخطية التي تسلك معادلات KP

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تحويلات باكلاند لفصل من المعادلات التي لها نفس سلوك معادلة كادرمشيف بيغياشنلي تشتق بإستخدام تحويلة تربط بين هذا الفصل والفصل المعمم له . كذلك نحسب بعض قوانين الثبات الممكنة لفصل المعادلات المعني .

Key Words: Kp-class, Backlund transformation, Conservation laws.

ABSTRACT

The Backlund transformation of a class that behaves in a manner similar to that of the Kadometsev-Petviashli is derived via the use of a transformation that couples this class with a modified given class of the same order. Some possible conservation laws of the given equation are derived.

1 - INTRODUCTION

In [2] we have seen that if m is multiple root of order four of the algebraic equation

$$a_7 - a_8 k + a_9 k^2 - a_{10} k^3 + a_{11} k^4 = 0 \qquad (1\text{-}1)$$

then the class of nonlinear partial differential equations

$$a_1 u_{xt} + a_2 u_{xx} + a_3 u_{tt} + a_4 uu_{xt} + a_5 uu_{xx} + a_6 uu_{tt} + a_7 u_x u_t + a_8 u_x^2 + a_9 u_t^2 + a_{10} u_{xxxx} + a_{11} u_{xxxt} + a_{12} u_{xxtt} + a_{13} u_{xttt} + a_{14} u_{tttt} = u_{yy} \qquad (1\text{-}2)$$

where a_i (i = 1,2, ... , 14) are real numbers and u(x,t,y) is a real scalar field defined for all (x,t,y) $\in R^3$.

reduces to the class of equations

$$c_1 u_{xt} + c_2 u_{xx} + c_3 u_{tt} + c_4 uu_{xt} + c_5 uu_{xx} + c_6 uu_{tt} + c_7 u_x u_t + c_8 u_x^2 + c_9 u_t^2 + c_{10} u_{xxxx} = u_{yy} \qquad (1\text{-}3)$$

where c_i , i = 1,2, ... , 10 are given interms of a_j , (j = 1, 2, ... , 10). The class (1-2) under the condition (1-1) is called the kp-like class of equations, since it behaves in a manner similar to that of the Kadometsev-Petviashli (kp) equation :

$$\partial / \partial_x (u_t + uu_x + u_{xxxx}) = u_{yy} \qquad (1\text{-}4)$$

in the sense that both of them are conservative equations having a number of conservation laws in the form $\partial/\partial_x X + \partial/\partial_t T + \partial/\partial_y Y = 0$ where X, Y and T are polynomials in the independent variables x, t, y and the dependent variable u and the various derivatives of the latter with respect to the former. [2,3] Furthermore each of them possesses a solitary wave solution of sech2 profile [2].

In this work we consider the kp-like class (1-3) and prove that it possesses a Baclund transformation which is by definition a system of two partial differential equations that generates a new solution from an old one. The derivation of such transformation shall be established via the transformation which we are going to derive in the next section. It couples the kp-like class with its modified class. For this purpose it is convenient to simplify the kp-like class (1-3) acoording to the following lemma.

Lemma 1 : If m is a multiple root of order four of the algebraic equation (1-1) and the coefficients a_i (i = 1,2, ... , 14) of the equation (1-2) satisfy :

i) $a_5-(a_{13}/4a_{14})a_4+(a_{13}/4a_{14})^2=0,$

ii) $a_3^2(2a_5a_2-a_1a_2)=a_6(2a_3a_2-a_1^2).$

then the class (1-1) reduces to the kp-like class in the simple form ;

$$u_t+6uu_x+6uu_t+u_{xxx}+3u_{xxt}+3u_{xtt}+u_{ttt}+3\partial_\xi^{-1}u_{yy}=0 \qquad (1\text{-}5)$$

where $\partial_\xi=\partial/\partial_x+\partial/\partial_t$ and ∂_ξ^{-1} is the integral along the line $\xi=x+t$

Proof : Without loss of generality let $a_5=a_8, a_4=a_7$ and $a_6=a_9$, otherwise there exists a nonsingular linear transformation on the independent variables x,t,y that transforms the given equation to a new one satisfying these equations. Now we apply the nonsingular linear transformation S defined as :

$$S=\begin{vmatrix} 1 & -n/(1-n) & 0 \\ -m/(1-m) & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

on the independent variables x, t, and y where,

$$n=\{a_1-[a_1-4a_3a_2]^{1/2}\},m=\{a_1-[a_1-4a_3a_2]^{1/2}\}$$

Then by direct calculations, the proof of the lemma is obtained.

Once we have simplified the kp-like class of equations in the form (1-5) we turn to derive a useful transformation that couples (1-5) with the equation :

$$v_t+v_{xxx}+3v_{xxt}+3v_{xtt}+v_{ttt}-3[(1/2)v^2v_x+(1/2)v^2v_t -v_x\partial_\xi^{-1}v_y-v_t\partial_\xi^{-1}v_y+\partial_\xi^{-1}v_{yy}=0 \qquad (1\text{-}6)$$

where $\partial_\xi=\partial/\partial_x+\partial/\partial_t$.

Equation (1-6) is called the modified equation of the equation (1-5). For this purpose the following lemma is introduced.

Lemma 2 : If v evolves according to the modified equation (1-6), then ;

$$u=-(1/4)v^2v_x-(1/2)v_x-(1/2)v_t-(1/2)\partial_\xi^{-1}v_y \qquad (1\text{-}7)$$

evolves according to the kp-like equation (1-5).

Proof : Since u is in the form (1-7) then by direct caculations and making use of the following relations :

$$\partial_\xi^{-1}[v^2v_x]_y+\partial_\xi^{-1}[v^2v_t]_y=\partial_\xi^{-1}[v^2v_\xi]_y$$

$$\partial_\xi^{-1}[v^2v_\xi]_y=v^2v_y,\partial_\xi^{-1}[v_{\xi y}\partial_\xi^{-1}v_y]_y=v_y\partial_\xi^{-1}v_y-\partial_\xi^{-1}v_y^2$$

and

$$\partial_\xi^{-1}v_y\partial_\xi^{-1}v_{xy}+\partial_\xi^{-1}v_y\partial_\xi^{-1}v_{yt}=\partial_\xi^{-1}v_y\partial_\xi^{-1}v_{\xi y}=v_y\partial_\xi^{-1}v_y$$

we obtain

$$u_t+6uu_x+6uu_t+u_{xxx}+3u_{xxt}+3u_{xtt}+u_{ttt}+3\partial_\xi^{-1}u_{yy}=$$
$$[-(1/2)v-(1/2)\partial/\partial_x-(1/2)\partial/\partial_t-(1/2)\partial_\xi^{-1}\partial/\partial_x].\{[v_t+v_{xxx}+3v_{xxt}+3v_{xtt}+v_{ttt}]-3(1/2)v^2v_x+(1/2)v^2v_t-v_x\partial_\xi^{-1}v_y -v_t\partial_\xi^{-1}v_y-\partial_\xi^{-1}v_{yy}]\}.$$

This completes the proof of the lemma.

2 - BACKLUND TRANSFORMATION

As has been pointed out in the last section it is convenient to consider the kp-like class of equations in the form (1-5), i.e.

$$u_t+6uu_x+6uu_t+u_{xxx}+3u_{xxt}+3u_{xtt}+u_{ttt}+3\partial_\xi^{-1}u_{yy}=0 \quad ...(1\text{-}5)$$

This equation is clearly invariant under the transformation : u(x, t, y) → u(2x-t, x, y) - λ where λ is constant. So we could work with u - λ rather than u. Thus the transformation (1-7) reduces to :

$$u=\lambda-(1/4)v^2v_x-(1/2)v_x-(1/2)v_t-(1/2)\partial_\xi^{-1}v_y \qquad (2\text{-}1)$$

where the modified equation (1-6) reduces to

$$v_t+v_{xxx}+3v_{xxt}+3v_{xtt}+v_{ttt}-3[(1/2)(v^2+\lambda)(v_x+v_t) -v_t\partial_\xi^{-1}v_y+\partial_\xi^{-1}v_{yy}=0 \qquad (2\text{-}2)$$

Now, since equation (2-2) is invariant under the transformation v--- - v, y → -y, this suggests that we introduce two functions U_1, U_2 defines by :

$$u_1=\lambda-(1/4)v^2-(1/2)v_x-(1/2)v_t-(1/2)\partial_\xi^{-1}v_y, \qquad (2\text{-}3\text{-}a)$$
$$u_2=\lambda-(1/4)v^2+(1/2)v_x+(1/2)v_t+(1/2)\partial_\xi^{-1}v_y \qquad (2\text{-}3\text{-}b)$$

for given u and λ . These two equations imply that :

$$u_1-u_2=v_x-v_t-\partial_\xi^{-1}v_y, \qquad (2\text{-}4)$$
$$u_1+u_2=2\lambda-(1/2)v^2 \qquad (2\text{-}5)$$

Now, inserting the additional transformation :

$$u_i = \partial w_i / \partial \mu (i=1,2) \qquad (2\text{-}6)$$

(where $\partial / \partial_\mu = \partial / \partial_x + \partial / \partial_t + \partial_\xi^{-1} \partial / \partial_y$). then (2-4) and (2-5) reduce to :

$$w_1 - w_2 = v, \qquad (2\text{-}7)$$

$$-[(w_1+w_2)_x + (w_1+w_2)_t + \partial_\xi^{-1}(w_1+w_2)_y = 2\lambda - (1/2)(w_1 - w_2)^2 \qquad (2\text{-}8)$$

respectively. Equation (2-8) constitutes one part of the Bachlund transformation for w_1 and w_2 which in turn generates solution of the kp-like equation (1-5) via equation (2-6). Furthermore, equation (2-2) can be written as

$$\begin{aligned}&(w_1-w_2)_t + (w_1-w_2)_{xxx} + 3(w_1-w_2)_{xxt} + 3(w_1-w_2)_{xtt} \\ &+(w_1-w_2)_{ttt} - (3/2)\{[(5/4)(w_1-w_2)^2 + (1/2)(w_1-w_2)_x \\ &+(1/2)(w_1-w_2)_t][(w_1-w_2)_x + (w_1-w_2)_t] - [(w_1-w_2)_x \\ &+(w_1-w_2)_t]\partial_\xi^{-1}(w_1-w_2)_y + 3\partial_\xi^{-1}(w_1-w_2)_{yy}\} = 0\end{aligned} \qquad (2\text{-}9)$$

Equations (2-8) and (2-9) constitute the auto Backlund transformation for the kp-like equation (1-5).

3- CONSERVATION LAWS OF A KP-LIKE NONLINEAR PARTIAL DIFFERENTIAL EQUATION

This section is devoted to the derivation of some possible conservation law for equation (1-5) where the nth conservation laws of a given differential equation such as equation (1-1) is defined in the form

$$\frac{\partial}{\partial t} T_n + \frac{\partial}{\partial n} X_n + \frac{\partial}{\partial y} Y_n = 0$$

Where $T_n, X_n \text{and} Y_n$ are polynomials of x, y, t, u and the various derivatives of the latter with respect to the former.

Theorem 1 : The first two independent conservation laws of (1-5) have the form

$$T_1 = u + 3u^2 + 3u_{xt} + u_{tt}, X_1 = 3u^2 + u_{xx} + 3u_{xt}, Y_1 = 3\partial_\xi^{-1} u_y,$$

$$T_2 = \frac{1}{2}u^2 + 2u^3 + 3uu_{xt} + uu_{tt} - \frac{u_x^2}{2} - \frac{3}{2}u_x^2 - \frac{3}{2}(\partial_\xi^{-1}u_y)^2,$$

$$X_2 = 2u^3 + uu_{xx} + 3uu_{xt} - \frac{u_x^2}{2} - \frac{3}{2}u_t^2 - \frac{3}{2}(\partial_\xi^{-1}u_y)^2 \text{and}$$

$$Y_2 = 3u\partial_\xi^{-1}u_y$$

Proof : Equation (1-5) can be rewritten in the conserved form :

$$\frac{\partial}{\partial t}[u + 3u^2 + 3u_{xt} + u_{tt}] + \frac{\partial}{\partial x}[3u^2 + u_{xx} + 3u_{xt}] + \frac{\partial}{\partial y}[3\partial_\xi^{-1}u_y] = 0 \qquad (3\text{-}1)$$

Hence $T_1, X_1 \text{and} Y_1$ follow and the first conservation laws is obtained.

The second conservation laws can be obtained by multiplying (3-1) by u, thus we have

$$uu_t + 6u^2u_x + 6u^2u_t + uu_{xxx} + 3uu_{xxt} + 3uu_{xtt} + uu_{ttt} + 3u\partial_\xi^{-1}u_{yy} = 0 \qquad (3\text{-}2)$$

Equation (3-2) can be, clearly, rewritten in the conserved form, i.e.,

$$\begin{aligned}&\frac{\partial}{\partial t}[\frac{u^2}{2} + 2u^3 + 3uu_{xt} + uu_{tt} - \frac{u_t^2}{2} - \frac{3}{2}u_x^2 - \frac{3}{2}(\partial_\xi^{-1}u_y)^2] + \frac{\partial}{\partial x}[2u^3 + uu_{xx} \\ &+3uu_{xt} - \frac{1}{2}u_x^2 - \frac{3}{2}u_t^2 - \frac{3}{2}(\partial_\xi^{-1}u_y)^2] + \frac{\partial}{\partial y}[3\partial_\xi^{-1}u_y] = 0\end{aligned} \qquad (3\text{-}3)$$

Hence T_2, $X_2 \text{and} Y_2$ follow and the theorem is proved .

Theorem 2 : The third and fourth conservation laws of (1-5) have the form:

$$T_3 = \frac{1}{2}u\partial_\xi^{-1}u_y + \frac{1}{2}\partial_\xi^{-1}u_t\partial_\xi^{-1}u_y + 3u^2\partial_\xi^{-1}u_y + u_{xx}\partial_\xi^{-1}u_y + 2u_{xt}\partial_\xi^{-1}u_y$$
$$+u_{tt}\partial_\xi^{-1}u_y - u_xu_y - u_tu_y,$$

$$X_3 = \frac{1}{2}\partial_\xi^{-1}u_t\partial_\xi^{-1}u_y + 3u^2\partial_\xi^{-1}u_y + u_{xx}\partial_\xi^{-1}u_y + 2u_{xt}\partial_\xi^{-1}u_y + u_{tt}\partial_\xi^{-1}u_y$$
$$-u_xu_y - u_tu_y,$$

$$Y_3 = \frac{3}{2}(\partial_\xi^{-1}u_y)^2 - \frac{1}{2}u\partial_\xi^{-1}u_t + u^3 + \frac{1}{2}u_x^2 + u_xu_t + \frac{1}{2}u_t^2,$$

$$T_4 = -u^3 + \frac{1}{2}u_x^2 - \frac{1}{2}u_t^2 - \frac{1}{2}(\partial_\xi^{-1}u_y)^2 + \frac{1}{2}(\partial_\xi^{-1}u_t)^2 + 3u\partial_\xi^{-1}u_t + u_{xx}\partial_\xi^{-1}u_t$$
$$+2u_{xt}\partial_\xi^{-1}u_t + 2u_{xt}\partial_\xi^{-1}u_t + u_{tt}\partial_\xi^{-1}u_t,$$

$$X_4 = \frac{1}{2}(\partial_\xi^{-1}u_t)^2 + 3u^2\partial_\xi^{-1}u_t + u_{xx}\partial_\xi^{-1}u_t + 2u_{xt}\partial_\xi^{-1}u_t + u_{tt}\partial_\xi^{-1}u_t$$
$$-u_xu_t - u_t^2,$$

$$Y_4 = \partial_\xi^{-1}u_y.\partial_\xi^{-1}u_t$$

Proof : Multiplying (1-5) by $\partial_\xi^{-1}u_y$, then we have

$$\begin{aligned}&u_t\partial_\xi^{-1}u_y + 6uu_x\partial_\xi^{-1}u_y + 6uu_t\partial_\xi^{-1}u_y + u_{xxx}\partial_\xi^{-1}u_y + 3u_{xxt}\partial_\xi^{-1}u_y \\ &+3u_{xtt}\partial_\xi^{-1}u_y + u_{ttt}\partial_\xi^{-1}u_y + 3\partial_\xi^{-1}u_{yy}.\partial_\xi^{-1}u_y = 0\end{aligned} \qquad (3\text{-}4)$$

Which can be rewritten in the form :

$$\frac{\partial}{\partial t}[\frac{1}{2}u_t\partial_\xi^{-1}u_y+\frac{1}{2}\partial_\xi^{-1}u_t\partial_\xi^{-1}u_y+3u^2\partial_\xi^{-1}u_y+u_{xx}\partial_\xi^{-1}u_y+2u_{xt}\partial_\xi^{-1}u_y$$
$$+u_{tt}\partial_\xi^{-1}u_y-u_xu_y-u_tu_y]+\frac{\partial}{\partial x}[\frac{1}{2}\partial_\xi^{-1}u_t\partial_\xi^{-1}u_y+3u^2\partial_\xi^{-1}u_y+u_{xxx}\partial_\xi^{-1}u_y$$
$$+2u_{xt}\partial_\xi^{-1}u_y+u_{tt}\partial_\xi^{-1}u_y-u_xu_y-u_tu_y]+\frac{\partial}{\partial y}[\frac{3}{2}(\partial_\xi^{-1}u_y)^2-\frac{1}{2}u\partial_\xi^{-1}u_t$$
$$-u^3+\frac{1}{2}u_x^2+u_xu_y+\frac{1}{2}u_t^2].$$

Hence by the definition 1 of conservation laws T_3, X_3 and Y_3 follow .

Similarly, the fourth conservation laws of (1-5) can be obtained by multiplying (1-1) by $\partial_\xi^{-1}u_t$, then we have :

$$u_t\partial_\xi^{-1}u_t+6uu_x\partial_\xi^{-1}u_t+6uu_t\partial_\xi^{-1}u_t+u_{xxx}\partial_\xi^{-1}u_t+3u_{xxt}\partial_\xi^{-1}u_t$$
$$+3u_{xtt}\partial_\xi^{-1}u_t+u_{ttt}\partial_\xi^{-1}u_t+3\partial_\xi^{-1}u_{yy}.\partial_\xi^{-1}u_t=0 \qquad (3\text{-}5)$$

then (1-6) can be rewritten in the conserved form :

$$\frac{\partial}{\partial t}[-u^3+\frac{1}{2}u_x^2-\frac{1}{2}u_t^2-\frac{1}{2}(\partial_\xi^{-1}u_y)^2+\frac{1}{2}(\partial_\xi^{-1}u_t)^2+3u^2\partial_\xi^{-1}u_t+u_{xx}\partial_\xi^{-1}u_t$$
$$+2u_{xt}\partial_\xi^{-1}u_t+u_{tt}\partial_\xi^{-1}u_t]+\frac{\partial}{\partial x}[\frac{1}{2}(\partial_\xi^{-1}u_t)^2+3u^2\partial_\xi^{-1}u_t+u_{xx}\partial_\xi^{-1}u_y$$
$$+2u_{xt}\partial_\xi^{-1}u_t+u_{tt}\partial_\xi^{-1}u_t-u_xu_t-u_t^2]+\frac{\partial}{\partial y}[\partial_\xi^{-1}u_y.u\partial_\xi^{-1}u_t].$$

Again by the Definition of conservation laws it is clear that the fourth conservation laws exists and T_4, X_4 and Y_4 follow. This completes the proof of the theorem.

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CONCLUSION

We have established the Backlund transformation for a class of two spatial dimensions of differential equations and derived four possible conservation laws. It is thought that the class has infinite number of conservation laws of the same equation. We are now investigating this conjecture and shall report on it in the near future.

REFERENCES

[1] **El-Sherbiny, H. M., 1993,** conservation laws of a class of differential equations (ii), J. Appl. Math. letters, 6: 99-103.

[2] **El-Sherbiny, H. M.,** and **K. A., Mustafa, 1995,** A reduction into less order disjoint subclasses of a two dimensonal class of nonlinear partial differentail equations, Bull, Fac. Sci, Assiut, Univ., 24, (1-c): 13-30.

[3] **Infeld, E.,** and **P., Frycz, 1983,** Infinite sequence of conservation laws for the Kadmotsev Petiashvili equation Acta Phys., Polonica, 1, B(14): 129-134.

[4] **Wahlquist, H. D.** and **F. B., Estabrook, 1973,** Backlund transformation for solution of the Korteweg de Vries equation, Phys. Rev. Lett., 31: 1386-1392.