Qat. Uni. Sci. Bull. (1984) 4: 7-12

THE SUPERIMPRIMITIVE SUBGROUPS OF THE ALTERNATING GROUP OF DEGREE 8

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Key words: Alternating group, Block system of imprimitivity of length n, Transitivity.

ABSTRACT

A transitive permutation group G is called superimprimitive if it is imprimitive with non-trivial block systems of imprimitivity of lengths all the non-trivial divisors of the degree of G; The superimprimitivity concepts was studied first by Omar (2), and later by the authors (3). In the present paper we shall give some results concerning this concept in part 1, and determine in part 2, all superimprimitive subgroups of the alternating group of degree 8.
We proved the following:

Lemma (1): Let G be a transitive group acting on a set X and m is the number of non-trivial divisors of $|X|$. If G contains m intransitive normal proper subgroups each having different orbit lengths then G is superimprimitive. The orbits of each subgroup form a block system of imprimitivity.

Lemma (2): (a) Let G be a superimprimitive group. For every non-trivial divisor d of the degree of G and for $x \in X$, there exists a group Z which lies properly between G_x and G such that the set $\{x^Z\}$ has length d.

(b) If $G_x \subset Z_i \subset G$ holds, where Z_i, $i=1,\dots,m$ are proper subgroups of G and the sets $\{x^{Z_i}\}$ have different lengths, then G is superimprimitive.

Then we show that, among the 48337 subgroups of A_8, which split into 137 classes there are 4425 superimprimitive subgroups which split into 18 classes, their generators are given.

1. The Superimprimitivity

Let G be a transitive permutation group acting on a set X, and m is the number of non-trivial divisors of $|X|$.

Lemma (1):

If G contains m intransitive normal proper subgroups each having different orbit lengths then G is superimprimitive. The orbits of each subgroup form a block system of imprimitivity.

Proof:

Let N be an intransitive proper subgroup of G. If B is an orbit of N, then B^g, $g \in G$ is an orbit of $g^{-1}Ng=N$. Thus G can only permute the pairwise disjoint orbits of N among each other. These therefore form blocks of G. Because $N \neq \{e\}$ they contain more than one point, because of the intransitivity of N they are proper subsets of X, and because of transitivity of G they are conjugate. Since G contains m intransitive normal proper subgroups then there are m different block systems. i.e. There are a block system of imprimitivity for every divisor of the degree of G. Thus G is superimprimitive.

Lemma (2):

(a) Let G be a superimprimitive group. For every non-trivial divisor d of the degree of G and for $x \in X$, there exists a group Z which lies properly between G_x and G such that the set $\{x^Z\}$ has length d.

(b) If $G_x \subset Z_i \subset G$ holds, where Z_i, $i=1,\ldots,m$ are proper subgroups of G and the sets $\{x^{Z_i}\}$ have different lengths, then G is superimprimitive.

Proof:

(a) Let B be a non-trivial block of G and Z the set of those $z \in G$ for which $B=B^z$, Z is clearly a proper subgroup of G. Since $x^g=x$ where $g \in G_x$ and $x \in B \rightarrow B^g=B$ then G_x is a subgroup of Z. Also $|B|>1$ and the transitivity of G implies that G_x is a proper subgroup of Z, this holds for every non-trivial divisor d of the degree of G, since there is a bijection of the subgroups of G containing G_x and the blocks containing x.

(b) Let $G_x \subset Z_l \subset G$ and $B_l=x^{Z_l}$. For $b \in B_1 \cap B_1^g$ with $g \in G$, then $b=x^z=x^{z'g}$ (with z, $z' \in Z_1$), therefore $z'gz^{-1} \in G_x \subset Z_1$, thus $B_1^g=B_l$ and B_l is a block. Because $G_x \subset Z_l$ does not consist of x alone. Since $B_l=B_1^g$ holds only for $g \in Z_l$, and $Z_1 \subset G$ there is a $g \in G$ with $B_l \neq B_1^g$, therefore $B_l \neq X$, hence B_l is a non-trivial block. Since we have m subgroups of Z_i then we have m blocks of orders $|\{x^{Z_i}\}|=|Z_i:G_x|$. Thus G is superimprimitive.

2. The subgroups of A_8

To give examples of the superimprimitive groups, we looked through the subgroups of A_n. For $n \leqslant 7$ there is only one superimprimitive subgroup of A_4. For $n=8$, all the subgroups of A_8 can be classified into mutually disjoint classes as follows:

	Subgroup Description	The number of classes	The total number of subgroups
	Intransitive cyclic	11	6973
	Intransitive abelian	3	1225
I	Transitive abelian	2	630
	Intransitive elementary abelian	11	2590
II	Transitive elementary abelian	2	30
	Intransitive nilpotent	9	5985
III	Transitive nilpotent	13	3990
	Intransitive self-normalizing	13	8030
	Transitive self-normalizing	8	2235
	Intransitive simple	3	252
	Transitive simple	2	121
	Intransitive self-normalizing simple	2	240
	Intransitive self-normalizing simple maximal	1	8
	Intransitive self-normalizing maximal	2	84
	Transitive self-normalizing maximal	3	65
IV	Transitive self-normalizing nilpotent	1	315
	Intransitive	38	13189
	Transitive	13	2375
		137	48337

By searching through all the above 137 classes case-by-case check to pick up the superimprimitive subgroups, we have the following table. For the notation of the isomorphism type see (4):-

Class	The generators	Class length	Subgroup order	Isomorphism type
I 1	$\langle A_1,A_2\rangle$	315	8	C_2xC_2
2	$\langle A_1,A_3\rangle$	315	8	C_2xC_4
II 1	$\langle A_4,A_5,A_6\rangle$	15	8	$C_2xC_2xC_2$
2	$\langle A_4,A_7,A_8\rangle$	15	8	$C_2xC_2xC_2$
III 1	$\langle A_1,A_6\rangle$	360	8	D_4
2	$\langle A_1,A_9\rangle$	210	8	Q_4
3	$\langle A_1,A_{10}\rangle$	315	16	Γ_2c_1
4	$\langle A_1,A_{15}\rangle$	315	16	Γ_2c_1
5	$\langle A_1,A_2,A_6\rangle$	315	16	C_2xD_4
6	$\langle A_1,A_2,A_9\rangle$	360	16	Γ_2b
7	$\langle A_1,A_3,A_6\rangle$	315	16	C_2xD_4
8	$\langle A_1,A_{14}\rangle$	315	32	Γ_2e_1
9	$\langle A_1,A_{10},A_{11}\rangle$	315	32	Γ_7a_1
10	$\langle A_1,A_{10},A_{12}\rangle$	315	32	Γ_7a_1
11	$\langle A_1,A_{10},A_{13}\rangle$	105	32	Γ_4a_2
12	$\langle A_1,A_{15},A_{16}\rangle$	105	32	Γ_4a_2
13	$\langle A_1,A_2,A_3,A_9\rangle$	105	32	Γ_5a_1
IV 1	$\langle A_{17},A_{18},A_{19}\rangle$	315	64	$C_2XC_2XC_2//C_2XC_2XC_2$
18		4425		

Where the generators are:

A_1=(1857) (2436) , A_2=(1456) (2837) , A_3=(1253) (4768),
A_4=(15) (23) (46) (78) , A_5=(14) (27) (38) (56) , A_6=(13) (25) (48) (67),
A_7=(12) (35) (48) (67) , A_8=(14) (28) (37) (56) , A_9=(1352) (4768),
A_{10}=(1836) (2457) , A_{11}=(1824) (3756) , A_{12}=(1856) (2437),
A_{13}=(1824) (3657) , A_{14}=(1637) (2854) , A_{15}=(1268) (3475),
A_{16}=(1247) (3685) , A_{17}=(1826) (3754) , A_{18}=(1634) (2857),
A_{19}=(1235) (4768).

So the conclusion is: There are 18 classes of superimprimitive subgroups of A_8.

ACKNOWLEDGEMENT

The authors would like to thank Dr. Volkmar Felsch, for supplying reference (1)

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الزمر الجزئية متعددة غير الأولية من الزمرة A_8

عبد الرؤوف عمر

يقال لزمرة التبديلات الأنتقالية أنها متعددة غير الأولية اذا كانت غير الأولية ولها نظام من البلوكات الفصلية لكل قاسم فعلي من قواسم درجة الزمرة .

ولقد قدم هذا البحث نظريتين لشروط مكافئة للتعريف . للتعرف على الزمر متعددة غير الأولية .

ثم وضحنا أنه بين كل الزمر الجزئية لزمرة التبديلات الزوجية من درجة ثمانية ، A_8 ، وعددهم ٤٨٣٣٧ زمرة جزئية مقسمين الى ١٣٧ فصل تكافؤ . يوجد ٤٤٢٥ زمرة جزئية متعددة غير الأولية مقسمين الى ١٨ فصل تكافؤ .