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A FAMILY OF NONRAMIFIED FUNCTIONS, DEFINED IN $Z\bar{Z}<1$

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ABSTRACT

In this article a family of nonramified functtions, defined in $z\bar{z}<1$, is obtained. It is characterized by an achieved distortion theorem; its Bloch constant and the limit of convexity are determined. Peschl's principle is, mainly, used throughout this work.

INTRODUCTION

One of the interesting questions in complex analysis is the study of analytic functions based on the nature of their produced mappings. H. A. Schwartz and C. Carthéodory [6] gave a powerful geometrical development in this line. E. Study [13] considered, firstly, convex domains; that is the tangent at their boundary points of each domain always, turns in the same direction as we move around the boundary, and obtained the lemma: if f(z) maps the units disk. $|z|<1$, in a convex domain, then f(z) maps every disk $|z|<r<1$ onto a domain of this type. Then, he generalized the behaviour of the boundary curvature in theory of conformal mapping.

E. Pewchl [7] investigated the problem of conformal mapping onto regions with boundary curvature which is bounded from below. He continued and succeeded in concluding and publishing in 1955 his principle [8]. Then, one of the invaluable tools in different branches of mathematics (e.g. differential geometrical theory of function of a complex variable [9] has been introduced. It is worth mentioning that applying that principle has led to some interesting results (see e.g. [1,2,4,5,9,10,12,14,15]).

In this article, the first section is devoted to review briefly, Peschl's principle and its role in complex analysis. As a result of applying it, a new family of nonramified functions**) is obtained in the second section. It is characterized, mainly, by the value of its Bloch constant, the limit of convexity and the nature of the image of the unit disk under each function of the family.

(1) Peschl's principle and its Role

Of interest to us here are functions w(z) which are nonramified in the unit disk of the z-plane, $D=\{ z: |z|< 1\}$.

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** i.e. holomorphic functions with nonvanishing derivative

Suppose the hyperbolic metric $ds_Z = |dz|/(1-|z|^2)$ and the euclidean metric $ds_W = |dw|$ are considered in D and w-plane respectively. Recall in addition that:

$\delta_1 v=(1-|z|^2)^2 v_z v_{\bar{z}}$ and $\delta_2 v= (1-|z|^2)^2 v_{z\bar{z}}$

are the Beltrami differential operators, while the associated mixed Beltrami differetial operator is defined as:

$$\delta_1(u,v) = (1-|z|^2)^2 u_z v_{\bar{z}}.$$

Using the differential invariants,

$$\propto = \log |w'| (1-|z|^2)$$

and

$$\beta = (1-|z|^2) \propto_Z$$

$$= \frac{w''}{2w'} (1-|z|^2)-z,$$

we form the function $v = \beta\bar{\beta} - f(\propto)$, where $f(\propto)$ is initially unknown function and differentiable twice at least. Caclulation of $\delta_2 v$ yields:

$$\delta_2 v= \frac{1}{A}\delta_1 v+B\{\delta_1(v,\propto) + \delta_2(\propto,v)\} + D_1 v + D_0, \quad (1)$$

where

$A = \beta\bar{\beta}$ and $B = A(1+f')$;

$D_1 = -f'' - 2$ and $D_0 = -ff'' + f'^2 + 3f' - 2f + 2.$

Now, let $f(\propto)$ is suggested to the following conditions:

$D_0 = 0$, $D_1 \neq 0$ and $v<0$ in a neighbourhood $U(\partial D)$, where ∂D is the boundary of D (2)

Hence it follows that $v \leqslant 0$ at every point $z \in \{D \cup \partial D\}$.

This principle has been obtained by E. Peschl and its proof is sketched as folows (see e.g. [7] , [10]):

Suppose that he function v takes its maximum at a point $z^* \in D$, where $v(z^*) = \max_D . (v) > 0$ and $\delta_2 v \leqslant 0$. Moreover, since at z^*, $\delta_1 v = 0$ and $\delta_1(\propto, v) = \delta_1(v,\propto) = 0$, it follows that equation (1) reduces to $\delta_2 v = D_1 v$.

But from the preceeding, D_1 is either >0 or <0. On one hand if $D_1>0$, this leads, immediately, to a contradiction ($\delta_2 v$ is not allowed to be positive at a maximum situation). On the other hand, if $D_1<0$, using a modified maximum principle [10] , a similar contradiction follows.

Now, it is of importance to define $f(\alpha)$, such that $D_0=0$, $D_1 \neq 0$ and $v<0$ in a neighbourhood $U(\partial D)$. These functions are obtainable in Ref. [11].

Considering $v = \beta\bar{\beta} - f(\alpha) \leqslant 0$, $\beta = (1-|z|^2)\alpha_z$, it follows:

$$\int \frac{d\alpha}{\sqrt{f(\alpha)}} \leqslant \int \frac{1}{\sqrt{f(\alpha)}} \left|\frac{d\alpha}{dr}\right| dr \leqslant 2 \int \frac{dr}{1-r^2}, \tag{3}$$

where $r = |z|$. Upon integration, one has eastimatin for α, integrating once more one has:

$$H_1(r) \leqslant |w(z) - w(0)| \leqslant H_2(r)$$

where $H_1(r)$ and $H_2(r)$ depend on $f(\alpha)$. Without loss of generality, $w(z)$ is normalized such that $w(0) = 0$. Thus, it is clear that: corresponding to any function, $f(\alpha)$, satisfying the above conditions, (2), and by applying this approach, one obtains estimations for the differential invariant α and $|w(z)|$. So, Peschl's principle and the main method of its application is represented. Its role has been introduced in different works, and results are obtained. For example:

(i) Since the above differential invariant α, satisfies the equation, $\delta_2\alpha = -1$, E. Raupach[12] gave different estimations for α.

(ii) Considering the hyperbolic and elliptic metric in the w-plane, K.-W. Bauer[1] proved the principle and gave estimations for the solutions of the equation, $\delta_2 \alpha = -1 \pm e^{2\alpha}$ where α is the relevent differential invariant. Also distortion theorems are obtained.

(iii) K. J. Wirths [14] considered the principle as a maximum princple and generalized it. Furthermore, using the original principle, he solved the following problem [15,] which was stated in 1965 (see [16 problem N]):

For all $z \in D$, which value of $\beta \in R$ under which $|f'(z)|<(1-|z|^2)^{-2}$ holds?, where $f(z)$ is holmorphic and $|f(z)| <(1-z^2)^{-1}$; also what is the coefficients ranges for the class of these functions?

(iv) In [2] some families of analytic functions are defined and a set of disortion theorems are obtained considering hyperbolic and elliptic metric in addition to the euclidean metric.

(v) In [3] families of ramified and nonramified functions and a relation governing them are obtained. Moreover, in [4] considering hyperbolic metric, and using Julia theorem a covering theorem is deduced.

(vi) Lastly, in [5], a relation governing solutions of $\delta_2 u = -1$ and Joukowski's profile is induced.

(2) A family of nonramified functions

Let $\mathfrak{F}$ denotes the family of all nonramified functions, $w(z)$, which are defined in the unit disk D and are considered in the definition of the above differential invariant α. In correspondance with a selected solution of the nonlinear differential equation $D_0=0$. the family $\mathcal{F}$ is characterized by conditions hold for each $w(z) \varepsilon \mathcal{F}$.

let us select the following solution of $D_0=0$:

$\propto = \log(t+b) - t$, $f = \{(t+b) - 1\}^2$, (b arbitrary and $t>-b$)

which satisfies the conditions (2).

To make use of Peschl's principle, inequality (3) is applicable for t $[1-b,\infty]$. Since

$$\int_{\propto(0)} \frac{d\propto}{-vf(\propto)} = -\int_{t_1}^{t} \frac{dt}{t+b} = \log(t_1+b)/(t+b), \tag{4}$$

relevent claculations show that (t_1 is determined through $w'(o) = (t_1+b)e^{-t_1}$):

$QR - b \leqslant t \leqslant Q/R - b,$

where $Q = t_1 + b$ and $R = \dfrac{1-r}{1+r}.$

The above two intervals of t are consistent iff:

1. $1 - b \leqslant QR - b \Rightarrow r \leqslant \dfrac{Q-1}{Q+1} := r^*$

2. $\dfrac{Q}{R} - b \leqslant \infty$ (trivial).

Since $e\propto$ is a monotonic decreasing funcion of t then, by applying (4), one has:

$\exp\{b - Q/R\}(QR) \leqslant e \quad \leqslant \exp\{b - Qr\}(Q/R),$

in this and in the next inequalities the R.H.S. holds only under the restriction $r \leqslant r^*$.

Since $e\propto = |w'|(1-|z|^2)$, the above inequality implies:

$$((1-r^2)^{-1} \exp\{\propto(t_1) - \frac{2Qr}{1-r}\} \leqslant |w'| \leqslant (1+r^2)^{-1}\exp\{(t_1) + \frac{2Qr}{1+r}\}$$

which yields by integration:

$$\tfrac{1}{2}\exp(-t_1)\{1-\exp(-\frac{2Qr}{1-r})\} \leqslant |w| \leqslant \tfrac{1}{2}\exp(-t_1)\{\exp(\frac{2Qr}{1+r}) - 1\}$$

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*) Bloch constant of $\mathcal{F}$ is defined [6] as the infimum of the radii of the largest one sheeted disks lying on the Riemann surface, onto which $w(z) \varepsilon \mathcal{F}$ map D dijectively.

The obtained inequalities hold for each w $(z) \varepsilon \mathcal{F}$, corresponding to the selected solution of $D_0=0$.

Hence Bloch contant*) $B_{\mathcal{F}}$ of the family is claculated from the last inequality by taking the limit of the L.H.S. as $r \to 1$ which gives $B_{\mathcal{F}} = \frac{1}{2}\exp(-t_1)$.

Furthermore, as a conclusion from Peschl's principle [10], the family $\mathcal{F}$ is characterized by the fact that:

the boundary curvature of the image of the unit disk D under $w \varepsilon \mathcal{F}$,

namely $\frac{1}{|W'|} \mathrm{Re}\, \{1 + a\frac{W''}{W'}\}$, is bounded from below by k (say),

where k is given by $\lim\limits_{\substack{\alpha \to -\infty \\ z \to z\, \varepsilon \partial D}}{}^{\dagger} (1 - \beta\bar{\beta})\, \bar{e}^{\dot{\alpha}} := 2e^{-b}$ (>0). This means

that the image of the unit disk D under each $w(z)\varepsilon\mathcal{F}$ is, always convex. Furthermore, w(z) maps every disk $|z|<r<1$ onto a convex domain [13] The suprimum of the radii r of these disks yields the limit of convexity for the family [6]; it can be shown that it is equal to unity.

Hence, the family $\mathcal{F}$, which is defined in correspondance with the above selecte solution of the diffrential equation $D_0=0$, is characterized by the following distortion theorem.

Theorem

For each $w(z) \varepsilon \mathcal{F}$ the following conditions are satisfied:

1. w(z) is nonramified in the unit disk D and w(o)=0.
2. $\frac{e\alpha(t_1)}{1-r^2} \operatorname{esp}\left(\frac{2Qr}{1-r}\right) \leq |w'| \overset{r \leq r^*}{\leq} \frac{e\alpha(t_1)}{1+r^2} \exp\left(\frac{2Qr}{1+r^2}\right)$
3. $\frac{1}{2}\exp(-t_1)\left\{1 - \exp\left(-\frac{2Qr}{1-r}\right)\right\} \leq |w| \overset{r \leq r^*}{\leq} \frac{1}{2}\exp(-t_1)\left\{\exp\left(\frac{2Qr}{1+r}\right) - 1\right\}$

   where t_1, Q and r^* are defined as above.
4. bloch constant $B_{\mathcal{F}}$ is equal to $\frac{1}{2}\exp(-t_1)$ and the limit of convexity is unity.
5. The boundary curvature of the image of the unit disk D at leats equal to $\exp(-2b)$.

CORROLLARY

It can be proved that the function $w(z) = \frac{1}{2}\{\exp \frac{2Qz}{1+z} - 1\}$ belongs to the family $\mathcal{F}$. This function is the majorant function of the inequality 3 in the theorem.

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† The branch $+\sqrt{f}$ and the parameter t lies on $[-b, 1-b]$.

REFERENCES

1. **K.W. Bauer, 1960.** 'Ueber Abschaetzungen von Loesungen gewisser parteller Differentialgleichugen von elliptischen Typus', Bonner Math. Schr. Nr. 10.

2. **M.K. Gabr, 1974.** 'Ueber die Anwendung einer Abschaetzungsmethode von E. Peschl auf verschiedence Funkitionenfamilien', Diss. Bonn.

3. **M.K. Gabr, 1979.** 'On Peschl's principle', The 4th Int. Congr. on St., Comp. Sc., Soc. and Dem. Res., Cairo.

4. **M.K. Gabr, 1982.** 'On some analytic function', Delta j. of sc.

5. **M.K. Gabr. 1983.** A Type of Equipotential lines related to Joukowski's- profile', Proc. Ind. nat. Sc. Acad. 49, A. No. 6.

6. **G.M. goluzin, 1969.** 'Geometric theory of functions of a complex variable', Trans. of Math. Mon. Vol. 26.

7. **E. Peschl, 1932.** 'Ueber die Kruemmung von Niveaukurven bei der konformen Abbilding einfach-zusammenhaengender Gebiete auf das Innere eines Kreises', Math. Ann 106.

8. **E. Peschl. 1955.** 'Les invariant differentiels non holomorphes et leur rôle dans la théorie de fonctions', Rand. Sem. Math. Messina.

9. **E. Peschl, 1963.** 'Ueber die Verwendung von Differentialinvarianten bei gewissen Funktionenfamilien und die Uebertragung einer darauf gegeruendeten Methods auf partielle Differentialgleichungen von elliptischen Typus', ann. Acad. sc. fennicie, No A. 1.336/6.

10. **E. Peschl, 1974.** 'Ueber die Weiterentwickling einer Majorisierungus-methode, die auf Differentialinvarianten aufbaut', Proc. of the C. Cartheodory Intr. Symp. (Athens, 1973). Greek Math. Soc., Athens.

11. **E. Peschl, K.W. Bauer, 1963.** 'Ueber eine nichtlineare Differentialgleichung 2. Ordnung, die bei einem gewissen Abschaetzungsverfahren eine besondere Rolle spielt'. Forschungsber. d. L. Nordrh. - Westf. 1306.

12. **E. Raupach., 1960.** 'Eine Abschaetzungsmethode fuer reellwertrige Loesungender Differentialgeichung $\triangle \propto = \frac{-4}{1-|z|^2}$, Bonner Math. Schr. Nr. 9

13. **E. Study, 1919.** 'Konforme Abbildungen einfach zusammenhaengender Bereiche', Leipzig.

14. **K.J. Wirths, 1971.** 'Verallgemeinerung eines Maximumprinzips', Bonner Math. Schr. Nr. 43.

15. **K.J. Wirths. 1978.** 'Ueber holomorphe Funktionen, die einer Wachstumsbeschraenkung unterliegen, Arch. Math... (Basel) 30.

عائلة من دوال متفرعة ومعرفة في قرص الوحدة

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في هذا البحث حصلنا على عائلة من دوال غير متفرعة ومعرفة في قرص الوحدة . ميزت هذه العائلة بنظرية تشويه مع تحديد ثابت بلوخ ونهاية التحدب لها . تم استخدام مبدأ بشل بصفة أساسية في هذا البحث .