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GRAVITOHYDRODYNAMIC INSTABILITY OF A STREAMING FLUID CYLINDER

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استقرار الإتزان الهيدروديناميكي
لمائع أسطواني ذا جاذبية ذاتية

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تمت دراسة استقرار مائع إسطواني ينساب بانتظام (تحت تأثير قوي الشد السطحي والجاذبية الذاتية) في وسط مهمل الحركة ذا جاذبية ذاتية وذلك باستخدام طريقة لاجرانج المعتمدة على مبدأ الطاقة الكلية (أي طاقة الحركة وطاقة الوضع المختلفة).

بإستخدام معادلة لإجرانج التفاضلية ذات الرتبه الثانية تم اشتقاق علاقة التشتت العامة ومنها تم الحصول على علاقات تشتت لمسائل مدروسة سابقا كحالات خاصة. من الدراسة والمناقشات التحليلية والعددية لعلاقات التشتت وجدنا أن الإنسياب له تأثير عدم استقراري بصفة عامة أي لايعتمد على نوع الاضطراب الحادث. كل من قوة الشد السطحي وقوة الجاذبية الذاتية لها تأثير عدم استقراري في الأنماط المتماثلة. والعكس صحيح في الأنماط المضطربه الغير متماثلة. ثم تحديد الشروط والمواقع الفاصلة بين مجالات الاستقرار ومجالات عدم الاستقرار .

Key words: Stability, Surface tension, self-gravitation, Energy technique.

ABSTRACT

The capillary-gravitodynamic instability of a self-gravitating fluid cylinder (radius R_o) dispersed in a self-gravitating medium of negligible motion has been developed. General stability criteria are derived, upon utilizing the Lagrangian second order differential equations concerning the energy principle as the fluid is stationary. As the fluid is axially streaming we have used the macroscopic perturbation technique of small increments. The stability eigenvalue relations are discussed analytically and the results are confirmed numerically. Both the capillary and the self-gravitating forces are strongly destabilizing in the axisymmetric mode $m = 0$ as long as the perturbed wavelength λ is longer than the circumference $2\pi R_o$ of the fluid cylinder where m is the azimuthal wavenumber. The model is capillary-gravitodynamic stable in the domains ($\lambda \leq 2\pi R_o$, $m = 0$) of symmetric disturbance and ($0 < \lambda < \infty$, $m \neq 0$) of asymmetric disturbances. The streaming has strong destabilizing influence not only in the $m = 0$ mode but also in the modes $m \neq 0$. The self-gravitating and capillary forces have destabilizing influences on each other for some states in $m = 0$ but they have pure stabilizing influences on each other for all states in $m \neq 0$ modes. In $m = 0$ mode the instability of the model is very fast when the capillary and gravitational forces are acting all together and become more and more pronounced as the fluid is axially streaming. The latter, in addition, decreases the stable domains whether the disturbance is $m = 0$ or/and $m \neq 0$.

INTRODUCTION

The capillary instability of a full fluid cylinder in vacuum has been studied experimentally by Bassat [1] and Plateau [2]. Rayleigh [3] derived its dispersion relation and laid the theoretical foundation for such and related problems. These studies along with several extensions to different problems are documented by Drazin and Reid [4]. Its importance is not only for the academic view but also for its practical applications in the astronomical domains as well as in the industrial fields such as spray drying, fuel atomization and the production of controlled surfaces for heat and mass transfer in industrial processes. The self-gravitating instability of a fluid cylinder in a self-gravitating vacuum was studied for the first time by Chandrasekhar & Fermi [5] by using the method of presenting solenoidal vectors in terms of poloidal and toroidal quantities. This has been done in the axisymmetric mode only. Moreover, they indicated the application of such study and its correlation with the breaking-up of the spiral arms of galaxies. The pioneering works of Rayleigh [3] and Chandrasekhar & Fermi [5] had been carried out with the assumption that the fluid is stationary in the unperturbed state. The self-gravitating instability of a full fluid jet acting upon different forms of external electrodynamic or/and electromagnetic forces has been recently investigated by Radwan [6], [7], [8] and [9].

The purpose of the present work is to investigate the stability of a fluid jet under the combined effects of the capillary and self-gravitating forces, dispersed in a self-gravitating medium of negligible motion. This has been carried out in two categories: in the first we utilized the Lagrangian second order differential energy equations while in the second category we have assumed that the fluid is streaming and used the method of linear perturbation technique for solving this problem.

Capillary gravitodynamic instability of a fluid cylinder

Consider a circular gravitational fluid cylinder (Radius R_o) dispersed in a self-gravitating vacuum in the unperturbed state. The fluid is assumed to be incompressible, inviscid and of uniform density ρ. The forces acting on the present problem are the self-gravitation, pressure gradient and the capillary forces. We shall utilize the cylindrical polar coordinates (r, φ, z) system with the z-axis coinciding with the axis of the fluid cylinder. For an infinitismal departure from the unperturbed state, every physical quantity $\eta\ (r, \varphi, z; t)$ may be expressed in the form

$$\eta\ (r; \varphi, z; t) = \eta_o\ (r) + \varepsilon(t)\ \eta_1(r, \varphi, z) \qquad (1)$$

where $\eta_o(r)$ denotes the value of η in the unperturbed state while $\eta_1(r, \varphi, z)$ is an infinitismal increment due to disturbance. $\varepsilon(t)$ is the amplitude of the perturbation applied along the cylindrical interface at time t, given by

$$\varepsilon(t) = \varepsilon_o \exp(\sigma\ t) \qquad (2)$$

where $\varepsilon o (= \varepsilon(t)$ at $t = 0)$ is the initial amplitude and σ is the temporal amplification of the perturbation, if $\sigma\ (= i\omega$, $i = (-1)^{1/2}$ is imaginary then $\omega/2\pi$ is the oscillation frequency. Thence, the deformation in the interfacial of the fluid cylinder could be written in the form

$$r = R_o + \varepsilon(t)\ R_o \cos(kz + m\varphi) \qquad (3)$$

The second term in the right side of (3) is the elevation of the surface wave normalized with respect to R_o and measured from the unperturbed level surface where k (any real number) is the longitudinal wave number and m (an integer) is the transverse wave number.

Here, we intend to analyze this problem by utilizing the second order differential equation of Lagrange

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varepsilon}}\right) - \frac{\partial L}{\partial \varepsilon} = 0 \qquad (4)$$

$$\text{with } L = \Omega - V \qquad (5)$$

is the Lagrangian function and ε (dot over it means time derivative) is Lagrangian variable for this problem where Ω is the change in the kinetic energy of the model and V is the change in the total potential energy of the system. The latter is due to the self-gravitational and capillary forces influence on the present model. V can be written as

$$V = V_G + V_T \qquad (6)$$

where V_G is the gravitational potential energy and V_T is that due to the curvature pressure of the capillary force.

Now, suppose that the amplitude of the deformation $\dot{\varepsilon}$ is increased by $\delta\varepsilon$ then, consequent to this infinitismal increase in the amplitude of the deformation, the change δV_G in the gravitational potential energy can be identified by evaluating the work done in redistribution of the fluid required to effect the change in ε. For evaluating this work it is necessary to specify in a quantitative manner the redistribution which does take place.

An arbitrary deformation of an incompressible fluid can be thought of as resulting from Lagrangian displacement ζ applied to each point of the fluid. We propose that the perturbed motion is irrotational, since the irrotational motion of a non-viscous fluid is persistent ([4] p. 16). Therefore, the Lagragian displacement of the fluid could be derived from a scalar function Ψ, say

$$\underline{\zeta} = \underline{\nabla}\, \Psi \quad (7)$$

Equation (7) with the incompressibility condition

$$\underline{\nabla} \cdot \underline{\zeta} = 0 \quad (8)$$

show that the displacement potential Ψ satisfies Laplace's equation

$$\nabla^2 \Psi = 0 \quad (9)$$

From the view point of the (φ, z)-dependence of the deformation [see (3)] and based on the linear perturbation technique, every physical quantity could be expressed as

$$\Psi\,(r, \varphi, z; t) = \varepsilon(t)\, R(r) \cos (kz + m\varphi) \quad (10)$$

By the use of (10), equation (9) yields

$$r^{-1} \frac{d}{dr}\left(r \frac{d}{dr} R(r)\right) - (m^2 r^{-2} + k^2) R(r) = 0 \quad (11)$$

The solution of the ordinary differential equation (11) is given in terms of the ordinary Bessel functions of imaginary argument. Thereafter, under the present circumstances, the non-singular solution of equation (9) is being

$$\Psi = A\, \varepsilon_o\, I_m (kr) \exp(\sigma t) (kz + m\varphi) \quad (12)$$

where $I_m(kr)$ is the modified Bessel function of first kind of order m. The constant A of integration can be determined by applying the boundary condition that the radial component of ζ must reduce to $R_o \cos(kz) + m\varphi)$ at $r = R_0$, thus

$$A = R_o/(k\ I'_m (kRo)) \quad (13)$$

Hence

$$\Psi = \left[\varepsilon R_o /(k\, I'_m (kR_o))\right] \nabla \left\{I_m (kr) \cos (kz + m\varphi)\right\} \quad (14)$$

and consequently the corresponding displacement $\delta\underline{\zeta}$ which must be applied to each point of the fluid in order to increase the amplitude of the deformation by $\delta\varepsilon$ is given by:

$$\delta\underline{\zeta} = \left[R_o \delta\varepsilon /(k\, I'_m (kR_o))\right] \nabla \left\{I_m (kr) \cos (kz + m\varphi)\right\} \quad (15)$$

Now, due to that additional deformation $\delta\varepsilon$, the change in the gravitational energy δV_G (per unit length) can be obtained by integrating the work done by the displacement $\delta\underline{\zeta}$ in the gravitational potential $\delta\Psi$. Thus we have

$$\delta V_G = 2\pi\rho \left\langle\left\langle \int_0^{R_o(1+\cos(kz+m\varphi))} (\delta\zeta \cdot \nabla\delta\Psi)\, r\, dr \right\rangle\right\rangle \quad (16)$$

where the angular brackets signify that the quantity enclosed should be averaged over all φ and z.

The basic equations which govern the gravitational potentials Ψ^{fluid} and Ψ^{vac} are

$$\nabla^2 \Psi^{fluid} = -4\, \pi\, \gamma\, \rho \quad (17)$$

$$\nabla^2 \Psi^{vac} = 0 \quad (18)$$

where γ is the gravitational constant. The solution of these equations, in taking into account the deformation (3), following similar steps as before, is given by

$$\Psi^{fluid} = -\pi\, \gamma\rho\, r^2 + \varepsilon B\, K_m(kR_o)\, I_m m(kr) \cos (kz + m\varphi) \quad (19)$$

$$\Psi^{vac} = 2\, \pi\, \gamma R_0^2\, \rho\, \ln (R_o/R) + \varepsilon C\, I_m (kR_o)\, K_m(kr) \cos (kz) + m\varphi) \quad (20)$$

where $K_m(kr)$ is the modified Bessel function of the second kind of order m. The constants B and C can be determined by applying the condition that the gravitational potential Ψ and its derivative must be continuous across the perturbed surface (3) at $r = R_o$, from which we get

$$B = C = 4\, \pi\, \gamma\, \rho R_0^2 \quad (21)$$

Hence the change in the gravitational potential Ψ^{fluid} is given by

$$\Psi^{fluid} = 4\pi\, \gamma\rho R_0^2\, K_m (kR_o)\, \varepsilon(t)\, I_m(kr) \cos(kz + m\varphi) \quad (22)$$

It is worthwhile to mention here that the solutions (19)-(21) for Ψ^{fluid} and Ψ^{vac} lead to those obtained by Chandrasekhar and Fermi [5].

Substituting from (15) and (22) into (16), yields

$$\delta V_G = 2\, \pi^2 R_0^4\, \gamma\, \rho^2 (\varepsilon\, \delta\varepsilon) \left(1 - \frac{2 K_m(x)}{x I'_m(x)}\right) \int_0^x \left[(I'_m(z))^2 + (1 + m^2 z^{-2})(I_m(z))^2\right] z\, dz \quad (23)$$

where x (= kR_o) is the dimensionless longitudinal wavenumber and z = kr.

Using the differential identity (which follows from Bessel equations) [10]

$$\frac{d}{dz}(zI_m(z)\,I'_m(z)) = z\left[(I'_m(z))^2 + (1+m^2z^{\,2})(I_m(z))^2\right] \quad (24)$$

equation (23) gives the change in the fluid gravitational potential energy (per unit length in the z-direction) in the form

$$\delta V_G = -\pi^2\gamma R_0^4\,\rho^2\left[I_m(x)\,K_m(x) - \frac{1}{2}\right](\varepsilon\delta\varepsilon) \qquad (25)$$

By integrating this equation from zero to ε we get

$$V_G = 2\,\pi^2\gamma R_0^4\,\rho^2\,\varepsilon^2\left[\frac{1}{2} - I_m(x)\,K_m(x)\right] \qquad (26)$$

The potential energy of a system arising from the capillary force is simply proportional to the total superficial area. In a cylindrical polar coordinates (r, φ, z) frame, the superficial area S (say) is given by

$$S = 2\,\pi\,\left\langle\left\langle r\left[1+\left(\frac{\partial r}{\partial z}\right)^2 + r^{-2}\left(\frac{\partial r}{\partial\varphi}\right)^2\right]^{1/2}\right\rangle\right\rangle \qquad (27)$$

where the angular brackets signify that the quantity enclosed should be averaged over all φ and z. By a resort to equation (3), the superficial area S per unit length (in the z-direction) is finally given by

$$S = 2\,\pi R_o - (\pi\,R_o/2)\,(1 - m^2 - x^2)\,\varepsilon^2 \qquad (28)$$

The change in the potential energy VT (per unit length in the z-direction) of the fluid cylinder is

$$V_T = -\,(\pi\,R_0^2\;T/2)\,(1 - m^2 - x^2)\,\varepsilon^2 \qquad (29)$$

where T is the coefficient of the surface tension. Thus, the change in the total potential energy (see equation (6), on using (26) and (29), is finally given by

$$V = \left[[4\pi^2\gamma\,R_0^4\rho^2\left(\frac{1}{2} - I_m(x)K_m(x)\right) - (\pi R_0 T/2)(1-m^2-x^2)\right]\,\varepsilon^2 \qquad (30)$$

In order to use the present theoretical technique based on the energy principle and to apply equation (4) we have to find out the change in the kinetic energy of the model under consideration. Since the Lagrangian coordinate ε is a function of time, each element of the fluid will execute motions. These could be derived from the Lagrangian displacement

$$\underline{u} = \frac{\partial\zeta}{\partial t} \qquad (31)$$

so that the velocity vector of the fluid cylinder is

$$u = \left(R_0^2\,/(x\,I'_m(x))\right)\frac{d\varepsilon}{dt}\,\nabla.\left(I_m(kr)\cos(kz+m\varphi)\right) \quad (32)$$

The change in the total kinetic energy Ω (per unit length) of the fluid cylinder associated with the motion specified by (32) is

$$\Omega = \int_0^{R_o}\int_0^{kz=2\pi}\int_0^{2\pi}\tfrac{1}{2}\rho(\underline{u}_1.\underline{u}_1)\,d\varphi\,\frac{d\,kz}{2\pi}\;r\,dr$$

$$= \frac{\pi\rho R_0^2}{2k^2\,(I'_m(x))^2}\left(\frac{d\varepsilon}{dt}\right)^2\int_0^x\left[(I'_m(z))^2 + (1+m^2z^{-2})(I_m(z))^2\right]z\,dz$$

$$= R_0^3\pi\rho\left[\frac{I_m(x)}{2\,k\,I'_m(x)}\right]\dot{\varepsilon}^{\,2} \qquad (33)$$

where use has been made of the differential identity (24).

By substituting from (3) and (33) into (5), the Lagrangian function L could be clearly constructed and then equation (4) along with equation (2), at once, yields

$$\sigma^2 = \frac{T}{R_0^3\rho}\,\frac{x\,I'_m(x)}{I_m(x)}(1 - m^2 - x^2) + 4\pi\,\gamma\,\rho\,\frac{x\,I'_m(x)}{I_m(x)}\left[I_m(x)\,K_m(x) - \frac{1}{2}\right] \qquad (34)$$

Equation (34) is the required general eigenvalue relation for a self-gravitating fluid cylinder endowed with surface tension and dispersed in a self-gravitating medium of negligible motion. It relates the temporal amplification σ with the fundamental quantities $(4\pi\,\gamma\,\rho)^{-1/2}$, $(T/(R_0^3\,\rho))^{-1/2}$ as a unit of time t, the modified Bessel functions $I_m(x)$ and $K_m(x)$ of the first and second kind of order m, the wavenumbers x and m and with the problem parameters ρ, T, R_o and r. By means of this relation the characteristics of the present model could be identified. The neutral (marginal) stability is obtained by just setting σ = 0 in (34). The dispersion relation (34) is a linear combination of the dispersion relations of a self-gravitating fluid cylinder ambient with a self-gravitating vacuum and that of a fluid cylinder submerged in a vacuum acting upon the capillary force.

If we propose that T = 0 and at the same time m = 0, then relation (34) yields

$$\sigma^2 = 4\pi\,\gamma\rho\,\frac{x\,I_1(x)}{I_0(x)}\left(I_0(x)\,K_0(x) - \frac{1}{2}\right) \qquad (35)$$

That coincides with the dispersion relation derived for first time by Chandrasekhar and Fermi [5] in the axisymmetric mode m = 0 of disturbances.

If we neglect the influence of the surface tension, the relation (34) reduces to

$$\sigma^2 = 4\pi\gamma\rho \frac{x I'_m(x)}{I_m(x)} (I_m(x) K_m(x) - \frac{1}{2}) \quad (36)$$

which is valid for all axisymmetric m = 0 and non-axisymmetric $m \neq 0$ modes of disturbance.

In the absence of the self-gravitating force, the relation (34) becomes to

$$\sigma^2 = \frac{T}{R_0^3 \rho} \frac{x I'_m(x)}{I_m(x)} (1 - m^2 - x^2) \quad (37)$$

which is the capillary classical dispersion relation, of a full liquid jet in vacuum, derived for the first time by Rayleigh [3]

In order to examine the effect of the capillary or/and the gravitating forces on the instability of the fluid cylinder we have to study some properties of the Bessel functions.

From the viewpoint of the recurrence relations (see [10]).

$$2 F'_m(x) = F_{m+1}(x) + F_{m-1}(x) \quad (38)$$

where $F'_m(x)$ stands for $I'_m(x)$ and $-K'_m(x)$ while $F_m(x)$ stands for $I_m(x)$ and $K_m(x)$, and utilizing the fact, for $x \neq 0$, that

$$I_m(x) > 0, \; K_m(x) > 0 \quad (39)$$

we can observe that

$$I'_m(x) > 0 \;, \quad K'_m(x) < 0 \quad (40)$$

Therefore,

$$\frac{x I'_m(x)}{I_m(x)} > 0 \;, \quad (41)$$

$$I_m(x) K_m(x) > 0 \quad (42)$$

By the use of (41) in (37), we deduce that

$$\sigma_0^2 > 0 \text{ as } 0 < x < 1 \quad \text{in } m = 0 \text{ mode} \quad (43)$$
$$\sigma^2 \leq 0 \text{ as } \infty > x \geq 1$$

$$\sigma_m^2 \leq 0 \text{ as } 0 < x < \infty \quad \text{in } m \neq 0 \text{ modes} \quad (44)$$

where σ^2 in equation (37) is replaced by σ_m^2 in order to distinguish between the different modes of disturbance. This means, in the absence of the self-gravitating forces, that the fluid cylinder is unstable only in the domain $0 < x < 1$ for axisymmetric disturbances m = 0. It is stable as m = 0 in the domain $1 < x < \infty$ and also stable for all disturbed states in the non-axisymmetric modes $m \neq 0$.

Again using the identities (41) and (42) in equation (36), the oscillation and instability states may be identified.

In the non-axisymmetric modes $m \geq 1$ it is well known for $x \neq 0$ that

$$I_m(x) K_m(x) < 1/2 \quad (45)$$

This means that $\sigma^2 \leq 0$ for all $x \neq 0$ values in $m \geq 1$. In other words in neglecting the surface tension effect the self-gravitating fluid jet is purely stable in all disturbance states of modes $m \geq 1$. In the axisymmetric mode m = 0 it is found that the value of Io(x) Ko(x) may be greater or smaller than 1/2 and that depends on $x \neq 0$ values where the characteristic equation $I_o(x)K_o(x)=1/2$ is corresponding to the marginal states as $\sigma = 0$. Therefore, in such kind of perturbation the model is stable or unstable according to certain restrictions. Numerically it is found that the fluid cylinder is gravitationally unstable in the domain of $0 < x < 1.0668$ and stable in all other domains $1.0668 \leq x < \infty$.

From the foregoing results of separate cases as the model is acted upon by the capillary force only or is acted upon by the gravitating forces only, we can deduce the influence of the capillary force on the self-gravitating instability of the fluid cylinder.

In the general case in which the fluid cylinder is under the combined effect of the capillary and gravitating forces we predict the following results:

The fluid cylinder is purely capillary-gravitodynamic stable in the non-axisymmetric modes $m \geq 1$, while in the m = 0 mode it is stable or unstable according to restrictions. The latter could be determined by studying the general stability criterion (34) numerically in the most critical mode m = 0 of disturbances. To carry out such study, it is found convenient to rewrite (34) in the dimensionless form

$$\frac{\sigma^2}{4\pi\gamma\rho} = \frac{x I'_m(x)}{I_m(x)} \left(I_m(x) K_m(x) - 1/2\right) + M \frac{x I'_m(x)}{I_m(x)} (1 - m^2 - x^2) \quad (46)$$

where

$$M = T(4\pi\gamma R_0^3 \rho^2)^{-1} \quad (47)$$

is a dimensionless quantity since both the fundamental quantities $(4\pi\gamma\rho)$ and $T(R_0^3\rho)^{-1}$ have the same unit of $(\text{time})^{-2}$. The eigenvalue relation (46) has been computed numerically for the m = 0 mode for regular values of $x \neq 0$ for different values of M to determine the influence of the curvature pressure on the self-gravitating instability of the fluid cylinder. The numerical data are collected, classified, tabulated and presented graphically, see figure (1). Many characteristics can be deduced from these curves. It is found that the domain of instability is increasing with increasing M, this means that the capillary force increases the self-gravitating instability of the fluid cylinder. This is obvious from the following brief considerations:

(i) Corresponding to M = 0, 0.25, 0.5, 1.0 and 2.0 it is found that the maximum mode temporal amplification values are $(\sigma(4\pi\gamma\rho)^{-1/2})_{max}$ = 0.24529, 0.29657, 0.34199, 0.42828 and 0.54045 at x = 0.60, 0.65, 0.65 and 0.70. This shows how much the area under the instability curves are increasing with increasing M values.

(ii) Corresponding to M = 0, 0.25, 0.5, 1.0 and 2.0, it is found that the unstable domains, respectively, are $0 < x < 1.0668$, $0 < x < 1.03911$, $0 < x < 1.03253$, $0 < x < 1.02604$ & $0 < x < 1.02021$. This shows that the unstable domains are slowly decreasing horizontally and this can be ignored relative to the much vertical increasing, see figure (1). Therefore, there is always capillary-gravitodynamic unstable domains whatever is the value (small or large) of M.

We conclude that the model of a full fluid cylinder ambient with vacuum is always unstable in the axisymmetric mode m = 0 of disturbance whether it is acted upon by the capillary or/and the self-gravitating force. While it is stable in m = 0 mode if $x \geq 1.0668$ and also completely stable in the non-axisymmetric modes $m \geq 1$ of disturbances for all short and long wavelengths of perturbation when $x \neq 0$.

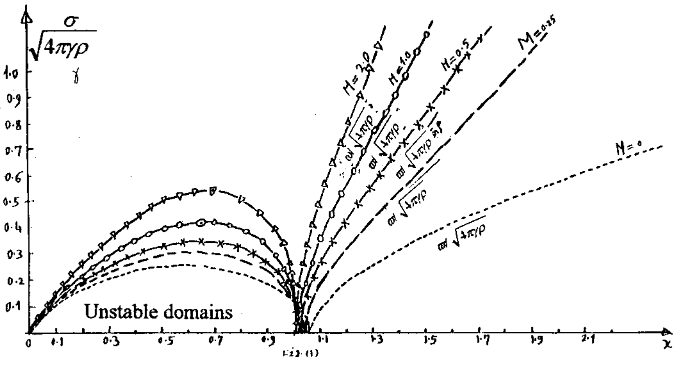

Fig. (1)

Gravitodynamic instability of a streaming fluid cylinder

Basic equations

Consider a uniform infinite cylinder (radius R_o) of an incompressible inviscid fluid. In the initial unperturbed state the fluid is assumed to be streaming uniformly with the velocity

$$u_o = (0, 0, U)$$

along the cylindrical coordinates (r, φ, z) with the z-axis coinciding with the axis of the fluid cylinder.

The self-gravitating fluid cylinder (density ρ) is ambient with self-gravitating medium of negligible motion. The fluid jet is acted upon by the capillary, self-gravitating, inertia and pressure gradient forces. The fundamental equations describing the motion of the fluid particles are a combination of the ordinary hydrodynamic equations together with those of Newtonian's gravitational field theory. For the problem under consideration they are: the vector gravitohydrodynamic equation of motion, continuity equation, of the curvature pressure due to the capillary force. Under the present circumstances these equations can be written as

$$\rho\left(\frac{\partial u}{\partial t} + (\underline{u}\,.\,\nabla)\underline{u}\right) = -\nabla p + p\nabla\Psi \quad (3.1)$$

$$\nabla\,.\,\underline{u} = 0 \quad (3.2)$$

$$\nabla^2\Psi^{int} = -4\pi\gamma\rho \quad (3.3)$$

$$\nabla^2\Psi^{ext} = 0 \quad (3.4)$$

$$p_s = T(\nabla.N_s) \quad (3.5)$$

Here p and u are the fluid kinetic pressure and velocity vector. Ψ^{int} and Ψ^{ext} are the gravitational potentials interior and external to the fluid, γ is the gravitational constant, p_s is the curvature pressure due to the capillary force (surface tension coefficient T) influence and N_s is a unit outward vector normal to the fluid boundary surface given by

$$N_s = \nabla F(r, \varphi, z; t) / |\nabla F(r, \varphi, z; t)| \quad (3.6)$$

where $F(r, \varphi, z; t) = 0$ is the boundary surface equation at time t.

The unperturbed basic state is studied and the fundamental quantities in this state are given by

$$\Pi_o = \Psi^{int} + (p_o/\rho) = \text{const} \quad (3.7)$$

$$p_{os} = T/R_o \quad (3.8)$$

$$\Psi_0^{int} = -\pi\gamma\rho r^2 \quad (3.9)$$

$$\Psi_0^{ext} = -\pi\gamma\rho R_0^2\left(1 + 2\ln(r/R_o)\right) \quad (3.10)$$

where the conditions that Ψ_o and its derivative are continuous across the unperturbed boundary surface r = R_o were applied to obtain (3.9) and (3.10) due to the solution of the differential equations (3.3) and (3.4) with zero derivative with respect to and z. Moreover, applying the condition that the total pressure must be balanced across the boundary surface r = R_o, we obtain

$$p_o = (T/R_o) + \pi\ \gamma\ \rho^2 (R_0^2 - r^2) \tag{3.11}$$

where the physical restrictions $p_o > 0$ at $r = R_o$ is identically satisfied due to the presence of the curvature pressure influence in this problem.

Perturbation analysis

Since we consider departures from an unperturbed right-cylinder shape of an incompressible fluid, a normal mode can be expressed uniquely in terms of the deformed surface. Suppose that the deformed interface is described by

$$r = R_o + \varepsilon(t)\, R_1 \tag{3.12}$$

with $$R_1 = R_o \exp\,[i(kz + m\varphi)] \tag{3.13}$$

where k (a real number) is the longitudinal wavenumber and m (an integer) is the transverse wavenumber, note that R_1 is the elevation of the surface wave measured from the unperturbed position and normalized with respect to Ro. $\varepsilon(t)$ is the amplitude of the perturbation given by

$$\varepsilon(t) = \varepsilon_o \exp(\sigma\, t) \tag{3.14}$$

where $\varepsilon_o = \varepsilon(0)$ and σ is the temporal amplification of instability, if σ (= $i\omega$, $i = (-1)^{1/2}$) is imaginary then $\omega/2\pi$ is the oscillation frequency of the stability states. For small departure from the unperturbed streaming state, every variable quantity Q (r, φ, z; t) may be expressed as

$$Q(r, \varphi, z; t) = Q_o(r) + \varepsilon(t)\, Q_1(r, \varphi, z) \tag{3.15}$$

where Q with suffix o indicates the unperturbed quantity while that with a suffix of 1 is an increment due to disturbances, Q stands for p, u, Ψ^{int}, Ψ^{ext} and N_s. In view of these expansions, the perturbation equations are given by

$$(\sigma + ikU)\, \underline{u}_1 = -\nabla \Pi_1 \tag{3.16}$$

$$\Pi_1 = -\Psi_1^{int} + (p_1/\rho) \tag{3.17}$$

$$\nabla .\, \underline{u}_1 = 0 \tag{3.18}$$

$$\nabla^2 \Psi_1^{int} = 0 \tag{3.19}$$

$$\nabla^2 \Psi_1^{ext} = 0 \tag{3.20}$$

$$p_{1s} = -\frac{T}{R_O}(R_1 + \frac{\partial^2 R_1}{\partial \varphi^2} + R_0^2 \frac{\partial^2 R_1}{\partial z^2}) \tag{3.21}$$

Based on the linear perturbation technique and in view of the (φ, z)-dependence (3.13) and also from the linearized theory used for solving the stability problems of cylindrical configurations, we may express Q_1 (r, φ, z) in the form

$$Q_1(r, \varphi, z) = Q_1^*(r) \exp[i(kz + m\varphi)] \tag{3.22}$$

By the use of the expansions (3.22), equations (3.19) and (3.20) turn into the total ordinary second order differential equation

$$r^{-1} \frac{d}{dr}(r \frac{dQ_1^*(r)}{dr} - (k^2 + m^2 r^{-2}) Q_1^*(r) = 0$$

(3.23)

where $Q_1^*(r)$ stands for $\Psi_1^{int}(r)$ and $\Psi_1^{ext}(r)$. The solution of this equation is given in terms of the ordinary Bessel functions of imaginary argument. For the problem under consideration, apart from the singular solution, the solution of equations (3.19) and (3.20) are given by

$$\Psi_1^{int} = \varepsilon_o A\, K_m(k R_o) I_m(kr) \exp(i(kz + m\varphi) + \sigma t) \tag{3.24}$$

$$\Psi_1^{ext} = \varepsilon_o B\, I_m(k R_o) K_m(kr) \exp(i(kz + m\varphi) + \sigma t) \tag{3.25}$$

where I_m and K_m are the modified Bessel functions of first and second kind of order m; A and B are constants of integration to be determined. Combining equations (3.16) and (3.18), we obtain

$$\nabla^2 \Pi_1 = 0 \tag{3.26}$$

In view of the expansion (3.22), using similar techniques as has been done for equation (3.19) and (3.20), the finite (as $r \to 0$) solution of equation (3.26) is given by

$$\Pi_1 = \varepsilon_o\, C\, I_m(kr)\ (i(kz + m\varphi) + \sigma\, t) \tag{3.27}$$

from which, with the aid of equation (3.16), we can easily write down the components of the perturbed velocity vector u_1 (= ($\underline{u}_{1r}$, $u_{1\varphi}$, u_{1z})) e.g.

$$u_{1r} = \frac{-\varepsilon_o C k}{(\sigma + ikU)} I'_m(kr) \exp(i(kz + m\varphi) + \sigma\, t) \tag{3.28}$$

Eigenvalue relation

The solution of the relevant perturbation equations (3.16)-(3.21) must satisfy appropriate boundary conditions across the perturbed interface (3.12) at the unperturbed boundary r = Ro. Under the present circumstances they are the following:

(i) The gravitational potential Ψ and its derivative must be continuous on the boundary (12) at $r = R_o$. These conditions yield

$$A = B = 4\pi\,\gamma\,\rho\,R_0^2 \tag{3.29}$$

where use has been made of the Wronskian relation

$$x\,(I'_m(x)K_m(x) - I_m(x)K'_m(x)) = 1 \tag{3.30}$$

(ii) The radial component of the velocity vector $\underline{u}$ must be compatible with the velocity of the boundary surface (12) at $r = R_o$. This condition reads

$$\underline{N}\,.\,\underline{u} = \frac{\partial r}{\partial t} \quad \text{at } r = R_o \tag{3.31}$$

with

$$\underline{u} = (0, 0, U) + \varepsilon\,(u_{1r}, u_{1\phi}, u_{1z}) \tag{3.32}$$

$$\underline{N} = \underline{N}_o + \varepsilon\,\underline{N}_1 = (1, 0, 0) - \varepsilon\,(0, im, ik R_o)\,R_1 \tag{3.33}$$

from which we obtain

$$C = \frac{-(\sigma + ikU)^2 R_o}{x\,I'_m(x)} \tag{3.34}$$

where x(=kRo) is the dimensionless longitudinal wavenumber.

(iii) The jump of the pressure across the surface (3.12) must be discontinuous by the curvature pressure p_{1s} at the unperturbed initial position $r = R_o$. This condition gives, at once, the eigenvalue relation

$$(\sigma + ikU)^2 = 4\pi\gamma\rho\,\frac{x\,I'_m(x)}{I_m(x)}\left(I_m(x)K_m(x) - 1/2\right) + \frac{T}{R_0^3\rho}\,\frac{x\,I'_m(x)}{I_m(x)}(1 - m^2 - x^2) \tag{3.35}$$

DISCUSSION

The eigenvalue relation (3.35) is a simple linear combination of the relations of a streaming fluid cylinder endowed with surface tension and of a self-gravitating streaming fluid cylinder dispersed in a self-gravitating medium of negligible motion. It contains the most fundamental information about the oscillation and instability of the present model of fluid jet.

The neutral (marginal) stability could be obtained from equation (3.35) such that $\sigma = 0$. The stability criteria (2.34)-(2.37) which are deduced in the first part of the present work on using the second order differential equation of Lagrange can be recovered from the general relation (3.35) with appropriate simplifications.

Moreover, if we postulate that $\gamma = 0$ but $T \neq 0$, the relation (3.35) reduces to

$$(\sigma + ikU)^2 = \frac{T}{\rho R_0^3}\left(\frac{x\,I'_m(x)}{I_m(x)}\right)(1 - m^2 - x^2) \tag{3.36}$$

which is the capillary eigenvalue relation of a streaming fluid cylinder ambient with a medium of negligible pressure.

If we assume that $T = 0$ but $\gamma \neq 0$, the relation (3.35) reduces to

$$(\sigma + ikU)^2 = 4\pi\gamma\rho\,\frac{x\,I'_m(x)}{I_m(x)}\left[I_m(x)K_m(x) - 1/2\right] \tag{3.37}$$

which is the eigenvalue relation of a self-gravitating streaming fluid cylinder submerged in a self-gravitating medium of negligible motion.

The relations (3.36) and (3.37) with $U = 0$ are clearly discussed in the first part of this paper.

Since the streaming has a strong destabilizing influence for all $x \neq 0$ in all non-axisymmetric modes $m \neq 0$ of perturbation and also in the axisymmetric (sausage) mode $m = 0$, therefore we conclude the following:

The streaming has the effect of increasing the capillary unstable domain ($0 \leq x < 1$ in $m = 0$) and decreasing the stable domains ($1 \leq x < \infty$, in $m = 0$) & ($0 < x < \infty$ in $m \geq 1$)

The discussions of the relation (3.37) show that the streaming has the influence of increasing the self-gravitating unstable domain ($0 \leq x < 1.0668$ in $m = 0$) and decreasing the stable domains ($1.07 \leq x < \infty$ in $m = 0$) & ($0 < x < \infty$ in $m \geq 1$).

These results may help in discussing the general stability criterion (3.35). As the model is acted upon by the combined effect of the self-gravitating and capillary forces, we predict that the streaming influence will increase the capillary-gravitational unstable domains and decrease those of stability. This could be proved by numerical evaluation of the dimensionless relation

$$\frac{(\sigma + ikU)^2}{4\pi \gamma \rho} = \left\{ (I_m(x)\, K_m(x) - 1/2) + M(1 - m^2 - x^2) \right\} \frac{x\, I'_m(x)}{I_m(x)} \tag{3.38}$$

for different values of the streaming parameters. U* (= - ikU $(4\pi\ \gamma\ \rho)^{-1/2}$) and M (see equation (2.47) in the most critical mode m = 0. The numerical data, for $\sigma^2/(4\pi\ \gamma\ \rho) > 0$ corresponds to the unstable states while those of stability corresponds to $\sigma^2/(4\pi\ \gamma\ \rho) < 0$. The results are tabulated and presented graphically. See figures (2) - (6). The numerical analysis and discussions reveal the following characteristics and features.

For the same values of M, the unstable domains are fastly increasing while those of stability are simultaneously decreasing with increasing U* values. This confirms the analytical results that the streaming is strongly destabilizing for all different states, see figures (2) - (6) and this can be realized from the following data.

(i) When M = 0.25 it is found that for U* = 0, 0.3, 0.6, 0.8 and 1.0 the unstable domains, respectively, are $0 < x < 0.93911$, $0 < x < 1.18495$, $0 < x < 1.46012$, $0 < x < 1.65136$ and $0 < x < 1.8392$ while those of stability are $0.93911 \le x < \infty$, $1.18495 \le x < \infty$, $1.46012 \le x < \infty$, $1.65136 \le x < \infty$ and $1.8392 \le x < \infty$. See figure (3).

(ii) When M = 0.5 it is found that for U* = 0, 0.3, 0.6, 0.8 and 1.0 the unstable domains, respectively, are $0 < x < 1.03252$, $0 < x < 1.18105$, $0 < x < 1.33274$, $0 < x < 1.47735$ and $0 < x < 1.62021$ while those of stability are $1.03252 \le x < \infty$, $1.18105 \le x < \infty$, $1.33274 \le x < \infty$, $1.47735 \le x < \infty$ and $1.62021 \le x < \infty$. See figure (4).

(iii) When M = 1.0: corresponding to U* = 0, 0.3, 0.6, 0.8 and 1.0 it is found that the unstable domains, respectively, are $0 < x < 1.02604$, $0 < x < 1.09035$ $0 < x < 1.22503$, $0 < x < 1.3308$ and $0 < x < 1.43745$. While those of stability are $1.02604 \le x < \infty$, $1.09035 \le x < \infty$, $1.22503 \le x < \infty$, $1.3308 \le x < \infty$ and $1.43745 \le x < \infty$. See figure (5).

(iv) When M = 2.0: corresponding to U* = 0, 0.3, 0.6, 0.8 and 1.0 it is found that the unstable domains are $0 < x < 1.0202$, $0 < x < 1.070112$, $0 < x < 1.14309$, $0 < x < 1.22569$ and $0 < x < 1.32269$ while those of stability are $1.0202 \le x < \infty$, $1.07011 \le x < \infty$, $1.14309 \le x < \infty$, $1.22569 \le x < \infty$,. See figure (6).

For the same values of U* the unstable domains are fastly vertically increasing (slowly increasing horizontally and could be neglected relative to the vertical increasing) and simultaneously the stable domains are decreasing with increasing M values. This can be clarified through the following data and results:

(1) When U* = 0, we may refer to the first part of this paper.

(2) When U* = 0.3: corresponding to M = 0.25, 0.5, 1.0 and 2.0 it is found that the unstable domains are $0 < x < 1.18105$, $0 < x < 1.09035$ & $0 < x < 1.070112$ and their maximum mode of instability are 0.59621, 0.63958, 0.71741 & 0.84045 at x = 0.6, 0.6, 0.7 and 0.7 respectively. The stable domains are $1.18495 \le x < \infty$, $1.18108 \le x < \infty$, $1.09035 \le x < \infty$ & $1.001 \le x < \infty$.

(3) When U* = 0.6: corresponding to M = 0.25, 0.5, 1.0 and 2.0 it is found that the unstable domains are $0 < x < 1.46014$, $0 < x < 1.33274$, $0 < x < 1.22503$ & $0 < x < 1.14309$ and their maximum mode of instability are 0.89621, 0.93958, 1.01741 & 1.14045. The stable domains are $1.46012 \le x < \infty$, $1.33274 \le x < \infty$, $1.22503 \le x < \infty$ & $1.14309 \le x < \infty$.

(4) When U* = 0.8: corresponding to the same values of M as above it is found that the unstable domains are $0 < x < 1.65136$, $0 < x < 1.33274$, $0 < x < 1.3308$ & $0 < x < 1.2257$ and their maximum mode of instability are 1.0962, 1.1396, 1.21741 & 1.24045. The stable domains neighbouring to the foregoing unstable domains are $1.65136 \le x < \infty$, $1.33274 \le x < \infty$, $1.3308 \le x < \infty$ & $1.22569 \le x < \infty$ where the equalities are corresponding to the marginal stabilities.

(5) When U* = 1.0: corresponding to M = 0.25, 0.50, 1.0 and 2.0 it is found that the unstable domains are $0 < x < 1.83916$, $0 < x < 1.62021$, $0 < x < 1.43745$ & $0 < x < 1.32269$ and their maximum mode of instability are 1.29621, 1.33958, 1.4174 & 1.54045. The neighbouring stable domains to those unstable domains are $1.83916 \le x < \infty$, $1.62021 \le x < \infty$, $1.43745 \le x < \infty$ & $1.32269 \le x < \infty$.

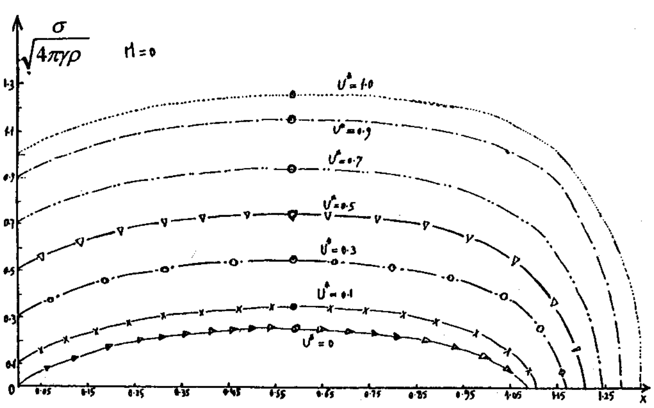

Fig. (2)

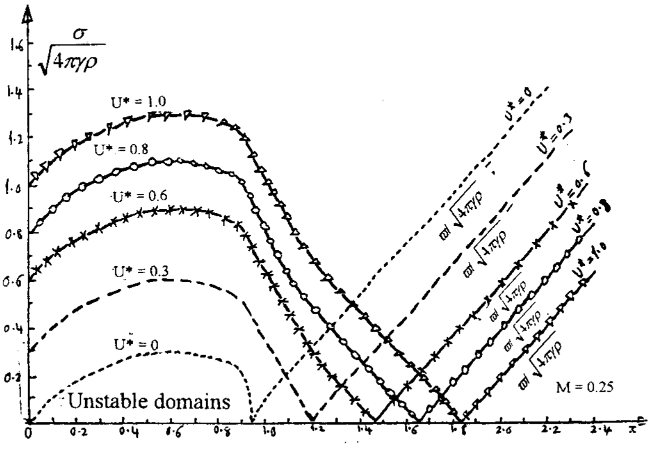

Fig. (3)

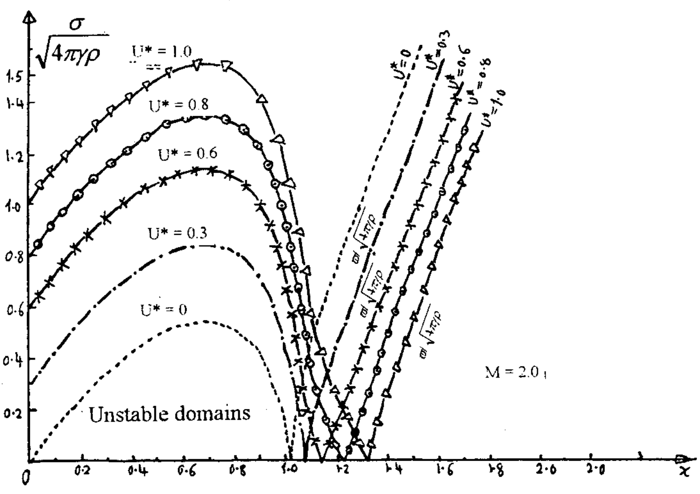

Fig. (6)

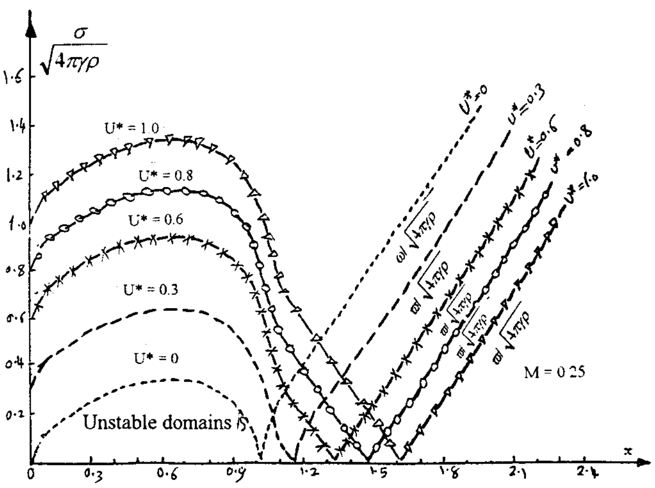

Fig. (4)

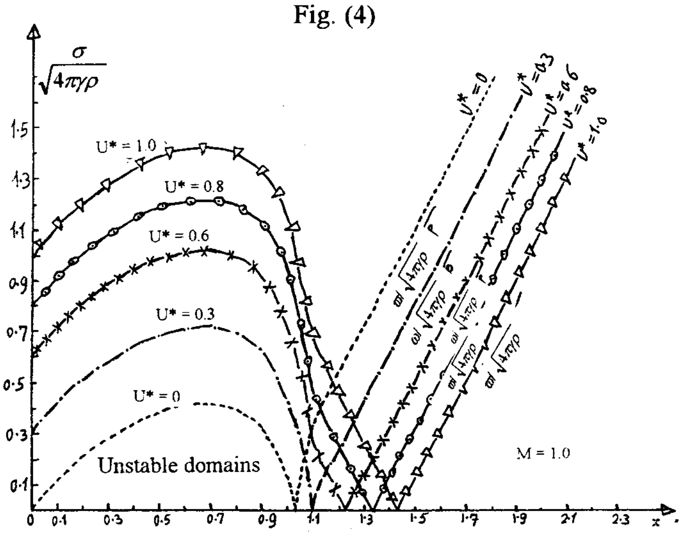

Fig. (5)

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