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On Stability in Multiobjective Integer Linear Programming: A Stochastic Approach

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أسلوب تصادفي لدراسة الأتزان في البرمجة الخطية الصحيحة متعددة دوال الهدف.

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نقدم في هذه الورقة أسلوب تصادفي لدراسة الأتزان في البرمجة الخطية الصحيحة متعددة دوال الهدف حيث تتواجد وسائط عشوائية في الطرف الأيمن لدوال القيود وتتبع هذه الوسائط التوزيع الطبيعي كما نقدم فيها أيضا مفاهيم أتزان حل المسألة محل الدراسة وقد دعمت الدراسة النظرية وأسلوب الحل المقترح بمثال توضيحي استخلصت نتائجه باستخدام حزم الحاسوب الملائمة للبرمجة الصحيحة.

Key Words: *Multiobjective integer linear programming, chance-constrained technique, Branch-and Bound method, Stability.*

ABSTRACT

In this paper we consider a multiobjective integer linear stochastic programming problem individual chance constraints. We assume that there is randomness in the right-hand sides of the constraints only and that the random variables are normally distributed. Some stability notions for such problem are characterized. An auxiliary problem is discussed and an algorithm as well as an illustrative example are presented

Introduction

Decision problems of stocastic or probabilistic optimization arise when certain coefficients of an optimization model are not fixed or known but are instead, to some extent, stochastic (or random or probabilistic) quantities.

In recent years methods of multiobjective stochastic optimization have become increasingly important in scientifically based decision-making involved in practical problems arising in economic, industry, health care, transportation, agriculture, military purposes and technology. We refer the Stochastic programming Web Site (2002)[1] for links to software as well as test problem collections for stochastic programming. In addition, we should point the reader to an extensive list of papers maintained by Maarten van der Vlerk at the Web Site: http:// mally.eco.rug.nl /biblio/ SP list.html.

In literature there are many papers that deal with stability of solutions for stochastic multiobjective optimization problems. Among the many suggested approaches for treating stability for these problems, for example, are those due to [2, 3, 4, 5, 6].

This paper is organized as follows: we start in Section 2 by formulating the model of chance-constrained multiobjective integer linear programming problem (CHMOILP) and the solution concept is introduced. In Section 3, a parametric study is carried out on the problem of concern, where some basic stability notions are characterized for the formulated model.

These notions are the set of feasible parameters; the solvability set, and the stability set of the first kind (SSK1). Moreover, an algorithm is describled to determine the (SSK1) for the (CHMOILP). In Section 4, we provide an example to illustrate our results. Finally, in Section 5, we state some open problems for future work in the area of stochastic multiobjective integer optimization problems.

Problem Formulation And Solution Concept

The chance-constrained multiobjective integer linear programming problem with random parameters in the rihgt-hand side of the constraints can be stated as follows:

(CHMOILP): max F(x),
subject to
$x \in X$,

where

$$X=\left\{ x\in R^n \middle| P\left\{ g_i(x)=\sum_{j=1}^{n} a_{ij}x_j \le b_i \right\} \ge \alpha_i, i=1,2,\dots,m, x_j \ge 0 \text{ andinteger}, j=1,2,..n \right\}.$$

Here x is the vector of integer decision variables and F(x) is a vector of k-linear real-valued objective functions to be maximized. Furthermore, P means probability and α_i is a specified probability value. This means that the linear constraints may be violated some of the time and at most $100(1-\alpha_i)$ % of the time. For the sake of simplicity, we assume that the random parametrs b_i, (i = 1, 2,...m) are distributed normally with known means $E\{b_i\}$ and variances Var $\{b_i\}$ and independently of each other.

Definition 1.

A point $x^* \in X$ is said to be an efficient solution for problem (CHMOILP) if there does not exist another $x \in X$ such that $F(x) \ge F(x^*)$ and $F(x) \ne F(x^*)$ with

$$P\left\{ g_i(x^*)=\sum_{j=1}^{n} a_{ij}x_j^* \le b_i \right\} \ge \alpha_i, i=1,2,\dots,m.$$

The basic idea in treating problem (CHMOILP) is to convert the probabilistic nature of this problem into a deterministic form. In this section, the idea of employing deterministic version will be illustrated by using the interesting technique of chance-constrained programming [7]. In this case, the set of constarints X of problem (CHMOILP) can be rewritten in the deterministic form as:

$$X'=\left\{ x\in R^n \middle| \sum_{j=1}^{n} a_{ij}x_j \le E\{b_i\}+K_{\alpha_i}\sqrt{Var\{b_i\}},\ i=1,2,\dots,m, x_j \ge 0 \text{ and integer}, j=1,2,..n \right\},$$

where K_{α_i} is the standard normal value such that $\Phi(K_{\alpha_i}) = 1 - \alpha_i$; and $\Phi(a)$ represents the "cumulative distribution function" of the standrad normal distribution evaluted at a. Thus, problem (CHMOILP) can be understood as the following deterministic version of a multiiobjective integer linear programming problem:

(MOILP): $\max\ [f_1(x), f_2(x),\ldots, f_k(x)]$,
subject to
$$x \in X'.$$

Now it can be obsereved, from the nature of problem (MOILP) above, that a suitable scalarization technique for treating such problems is to use the $\in$- constraint method [8]. For this purpose, we consider the following integer linear programming problem with a single-objective function as:

$P_s(\varepsilon)$: $\max\ f_s(x)$,
subject to
$$X(\varepsilon) = \{x \in R^n \mid f_r(x) \geq \varepsilon_r, r \in K - \{s\}, x \in X'\},$$
where $s \in K = \{1, 2, \ldots, k\}$ which can be taken arbitrary.

It should be stated here that an efficient solution x* for problem (CHMOILP) can be found by solving the scalar problem $P_s(\varepsilon)$ and this can be done when the minimum allowable levels ($\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{s-1}, \varepsilon_{s+1}, \ldots, \varepsilon_k$) for the (k-1) objectives ($f_1, f_2,\ldots, f_{s-1}, f_{s+1},\ldots, f_k$) are determined in the feasible region of solutions $X(\varepsilon)$.

It is clear from [8] that a systematic variation of ε_i's will yield a set of efficient solutions. On the other hand, the resulting scalar problem $P_s(\varepsilon)$ can be solved easily at a certain parameter $\varepsilon = \varepsilon^*$ using the branch-and bound method [9]. If $x^* \in X(\varepsilon^*)$ is a unique optimal integer solution of problem $P_s(\varepsilon^*)$, then x* becomes an efficient solution to problem (CHMOILP) with a probability level α_i^*, $(i = 1, 2,\ldots m)$.

A parametric Study on Problem (Chmoilp)

In this section, before we go further, we can rewrite problem $P_s(\varepsilon)$ in the following scalar relaxed subproblem which may occur in the branch-and-bound process as:

$P_s'(\varepsilon)$: $\max\ f_s(x)$,
subject to
$$x \in X_s(\varepsilon),$$
where

$$X_s(\varepsilon) = \left\{ \begin{array}{l} x \in R^n \mid\ f_r(x) \geq \varepsilon_r, r \in K - \{s\}, \\ \qquad g_i(x) = \sum_{j=1}^{n} a_{ij} x_j \leq C_i,\ i = 1, 2, \ldots, m, \\ \qquad \gamma_j \leq x_j \leq \beta_j\ , j \in J \subseteq \{1,2,..n\} \\ \qquad \text{and } x_j \text{ integer.} \end{array} \right\},$$

where the constraint $\gamma_j \leq x_j \leq \beta_j\ , j \in J \subseteq \{1,2,..n\}$ is an additional constraint on the decision variable x_j and that has been added to the set of constraints of problem $P_s(\varepsilon)$ for obtaining its optimal integer solution x* by the branch-and-bound algorithm [9].

In addition, it is supposed that:

$$C_i = E\{b_i\} + K_{\alpha_i}\sqrt{Var\{b_i\}}\ , (i = 1,2,....m).$$

In what follows, definitions of some basic stabiltiy notions are given for the relaxed problem $P_s^{'}(\varepsilon)$ above. We shall be essentially concerned with three basic notions: the set of feasible parameters; the solvability set, and the stability set of the first kind (SSK1). The qualitative and quantitative analysis of these notions have been introduced in details by Osman [10,11] for different classes of parametric optimization problems. Moreover, stability results for such problems have been derived.
The feasibility condition for problem $P_s^{'}(\varepsilon)$ is given in the following.

The Set of Feasible Parameters
Definition 2.

The set of feasible parameters of problem $P_s^{'}(\varepsilon)$, which is denoted by A, is defined by:

$$A = \{\varepsilon \in R^{k-1} \mid X_s(\varepsilon) \neq \Phi\}.$$

The Solvability Set
Definition 3.

The solvability set of problem $P_s^{'}(\varepsilon)$, which is denoted by B, is defined by:

$$B = \{\varepsilon \in A \mid \text{Problem } P_s(\varepsilon) \text{ has an optimal integer solution}\}.$$

The Stability Set of the First Kind
Definition 4.

Suppose that $\varepsilon^* \in B$ with a corresponding optimal integer solution x*, then the stability set of the first kind of problem $P_s^{'}(\varepsilon)$ corresponding to x*, which is denoted by S(x*), is defined by:

$$S(x^*) = \{\varepsilon \in B \mid x^* \text{ remains optimal integer solution of problem } P_s^{'}(\varepsilon)\}.$$

Utilization of the Kuhn-Tucker Necessary Optimality Conditions for $P_s^{'}(\varepsilon)$.

Now, given an optimal point x*, which may be found as described in Section 2, the question is: For what values of the vector ε the Kuhn-Tucker condtions for the subproblem $P_s^{'}(\varepsilon)$ are satisfied?

In the following, the Kuhn-Tucker necessary optimality conditions corresponding to problem $P_s^{'}(\varepsilon)$ will have the form:

$$\left.\begin{array}{lr}
\frac{\partial f_s(x)}{\partial x_j}+\sum_{\substack{r=1\\ r\neq s}}^{k}\mu_r\frac{\partial f_s(x)}{\partial x_j}-\sum_{i=1}^{m}\delta_i\frac{\partial g_i(x)}{\partial x_j}-u_j+v_j=0, & (j=1,2,\dots,n)\\
f_r(x)\geq\varepsilon_r, & r\in K-\{s\},\\
g_i(x)\leq C_i, & (i=1,2,\dots m),\\
x_j\geq\beta_j, & j\in I\subseteq\{1,2,\dots,n\},\\
x_i\leq\gamma_j, & j\in J\subseteq\{1,2,\dots,n\},\\
\mu_r[-f_r(x)+\varepsilon_r]=0, & r\in K-\{s\},\\
\delta_i[g_i(x)-C_i]=0, & (i=1,2,\dots m),\\
u_j[-x_j+\beta_j]=0, & j\in I\subseteq\{1,2,\dots,n\},\\
v_j[x_j-\gamma_j]=0, & j\in J\subseteq\{1,2,\dots,n\},\\
\mu_r\geq 0, & r\in K-\{s\},\\
\delta_i\geq 0, & (i=1,2,\dots m),\\
u_j\geq 0, & j\in I\subseteq\{1,2,\dots,n\},\\
v_j\geq 0, & j\in I\subseteq\{1,2,\dots,n\},
\end{array}\right\}\quad (*)$$

where $I\cup J\subseteq\{1,2,\dots n\}$, $I\cap J=\Phi$ and all the relations of system (*) above are evaluated at the optimal integer solution x*. The variables μ_r, δ_i, u_j, v_j are the langrangian multipiers.

The first and last four relations of the system (*) above represent a Polytope in $\mu\delta$ u v – space for which its vertices can be determined using any algorithm based upon the simplex method, for example, Balinski [12]. According to whether any of the variables μ_r, $r\in K-\{s\}$, δ_i, $(i=1,2,\dots m)$, $u_j,(j\in I)$ and v_j, $(j\in J)$ is zero or positive, then the set of parameters ε's for which the Kuhn-Tucker necessary optimality conditions are utilized will be determined. This set is denoted by T(x*).

Determination of the Set T(x*)

In this section we propose an algorithm in series of steps to find the set of possible ε which will be denoted by T(x*). For the set T(x*) , the point x* remains efficient for all values of the vector ε. Clearly, $T(x^*)\subseteq S(x^*)$

The suggested algorithm can be summarized in the following mannar.

Step 1.
Determine the means $E\{b_i\}$ and $Var\{b_i\}$ $(i=1, 2,\dots m)$.

Step 2.
Convert the original set of constraints X of problem (CHMOILP) into the equivalent set of constraints X'.

Step 3.
Formulate the deterministic multiobjective integer linear problem (MOILP) corresponding to problem (CHMOILP).

Step 4.
Formulate the integer linear problem with a single-objective function $P_s(\varepsilon)$.

Step 5. Solve k-individual integer linear problem P_r, (r =1,2,...,k) where

P_r: max $f_r(x)$, (r=1,2,...,k),
subject to
$x \in X'$,

to find the optimal integer solutions of the k-objectives.

Step 6.
Construct the payoff table and determine n_r, M_r (the smallest and the largest numbers in the r^{th} column in the payoff table).

Step 7.
Determine the ε_i's from the formula:

$$\varepsilon_r = n_r + \frac{t}{N-1}(M_r - n_r), r \in K - \{s\}$$

where t is the number of all partitions of the interval $[n_r, M_r]$.

Step 8.
Find the set $\Im = \{ \varepsilon \in R^{k-1} | \ n_r \le \varepsilon_r \le M_r, r \in K - \{s\} \}$

Step 9.
Choose $\varepsilon_r^* \in \Im$ and solve the integer linear problem $P_s(\varepsilon^*)$ using the branch-and-bound method [9] to find its optimal integer solution x*.

Step 10.
Determine the set If $T_1(x^*)$ by utilizing the Kuhn-Tucker necessary optimality conditions (*) corresponding to problem $P_s'(\varepsilon)$.

Step 11.
If $T_2(x^*)$ is a singleton, go to step 12. Otherwise, go to step 13.

Step 12.
Define $T_2(x^*) = \{ \varepsilon \in R^{k-1} | \ \varepsilon_r^* - \Delta \le \varepsilon_r^* \le M_r, r \in K - \{s\} \}$, where Δ is any small prespecfied positive real number.

Step 13.
Determine $\Im - T_2(x^*)$. If $\Im - T_2(x^*) = \phi$, stop. Otherwise, go to step 14.

Step 14.
Choose another $\varepsilon_r = \bar{\varepsilon}_r \in \Im - T_2(x^*)$ and go to step 9.

The above algorithm terminates when the range of $\Im$ is fully exhausted. Then, the stability set of the first kind S(x*) is given as:

$$S(x^*) = \bigcup_{i=1}^{k-1} T_i(x^*).$$

An Illustrative Example

Here , we provide a numerical example to clarify the developed theory and the proposed algorithm. The problem under consideration is the following bicriterion integer linear programming problem involving random parameters in the right-hand side of the constraints (CHBILP).

(CHBILP): $\max F(x) = [f_1(x), f_2(x)]$,

subject to

$$P\{x_1+x_2 \le b_1\} \ge 0.90,$$
$$P\{-x_1+x_2 \le b_2\} \ge 0.95,$$
$$P\{3x_1+x_2 \le b_3\} \ge 0.90,$$
$$x_1, x_2 \ge 0 \text{ and integers.}$$

where

$$f_1(x) = 2x_1 + x_2,\ f_2(x) = x_1 + 2x_2.$$

Suppose that b_i, $(i = 1,2,3)$ are normally distributed random parameters with the following means and variances.

$$E\{b_1\} = 1, \quad E\{b_2\} = 3, \quad E\{b_3\} = 9,$$
$$Var\{b_1\}=25, \quad Var\{b_2\}=4, \quad Var\{b_3\}=4,$$

From standard normal tables, we have:

$$K_{\alpha_1} = K_{\alpha_3} = K_{0.90} \cong 1.285,\ K_{\alpha_2} = K_{0.95} \cong 1.645$$

For the first constraint, the equivalent determinitic constraint is given by:

$$x_1 + x_2 \le C_1 = E\{b_1\} + K_{\alpha_1}\sqrt{Var\{b_1\}} = 1+1.285(5) = 7.425$$

For the second constraint:

$$-x_1 + x_2 \le C_2 = E\{b_2\} + K_{\alpha_2}\sqrt{Var\{b_2\}} = 3+1.645(2) = 6.29$$

For the third constraint:

$$3x_1 + x_2 \le C_3 = E\{b_3\} + K_{\alpha_3}\sqrt{Var\{b_3\}} = 9+1.285(2) = 11.57$$

Therefore, problem (CHBILP) can be understood as the corresponding deterministic bicriterion integer linear programming problem in the form:

(BILP): $\max [f_1(x) = 2x_1 + x_2,\ f_2(x) = x_1 + 2x_2]$,

subject to

$$x_1 + x_2 \le 7.425,$$
$$-x_1 + x_2 \le 6.29,$$
$$3x_1 + x_2 \le 11.57$$
$$x_1, x_2 \ge 0 \text{ and integers.}$$

Using the ε-constraint method [8], then problem (BILP) above with a single-objective function becomes:

$P_1(\varepsilon)$: $\max f_1(x) = 2x_1 + x_2$,

subject to

$$x_1 + 2x_2 \ge \varepsilon_2,$$
$$x_1 + x_2 \le 7.425,$$
$$-x_1 + x_2 \le 6.29,$$
$$3x_1 + x_2 \le 11.57$$
$$x_1, x_2 \ge 0 \text{ and integers.}$$

It can be shown easily that $12.7775 \leq \varepsilon_2 \leq 14.2825$.
Problem $P_1(\varepsilon)$ can be solved at $\varepsilon_2 = \varepsilon_2^* = 13$ using the branch-and-bound methode [9] and its optimal integer solution is found $(x_1^*, x_2^*) = (1, 6)$.

Furthermore, problem $P_1(\varepsilon)$ can be rewritten in the following parameteric form as:

$P_1'(\varepsilon)$: $\max f_1(x) = 2x_1 + x_2$,

subject to

$$\begin{aligned} x_1 + 2x_2 &\geq \varepsilon_2, \\ x_1 + x_2 &\leq 7.425, \\ -x_1 + x_2 &\leq 6.29, \\ 3x_1 + x_2 &\leq 11.57 \\ 0 \leq x_1 &\leq 1, \\ 0 \leq x_2 &\leq 6 \end{aligned}$$

Therefore, the Kuhn-Tucker necessary optimality conditions corresponding to problem $P_1'(\varepsilon)$ will take the form:

$$\left.\begin{aligned} 2+\mu_1-\delta_1+\delta_2-3\delta_3-u_1&=0, \\ 1+2\mu_1-\delta_1-\delta_2-\delta_3-u_2&=0, \\ x_1+2x_2&\geq\varepsilon_2, \\ x_1+x_2&\leq 7.425, \\ -x_1+x_2&\leq 6.29, \\ 3x_1+x_2&\leq 11.57, \\ 0\leq x_1&\leq 1, \\ 0\leq x_2&\leq 6, \\ \mu_1(-x_1-2x_2+\varepsilon_2)&=0, \\ \delta_1(x_1+x_2-7.425)&=0, \\ \delta_2(-x_1+x_2-6.29)&=0, \\ \delta_3(3x_1+x_2-11.57)&=0, \\ u_1(x_1-1)&=0, \\ u_2(x_2-6)&=0, \\ \mu_1,\delta_1,\delta_2,\delta_3,u_1,u_2&\geq 0 \end{aligned}\right\} \quad (\#)$$

where all the above expressions of system (#) are evaluated at the optimal integer solution
$(x_1^*, x_2^*) = (1, 6)$. In addition, it can be shown that:

$$\delta_1 = \delta_2 = \delta_3 = 0,\ u_1, u_2 > 0,\ \mu_1 \geq 0.$$

Therefore, the set $T_1(1, 6)$ is given by:

$$T_1(1, 6) = \{\varepsilon \in R \mid 12.7775 \leq \varepsilon_2 \leq 13\}.$$

A systematic variation of $\varepsilon_2 \in R$ and $12.775 \leq \varepsilon_2 \leq 13$ will yield another stability set $T_2(1, 6)$.

Conclusions

The general purpose of this paper was to investigate stability of the efficient solution for chance-constrainted multiobjective integer linear programming problem. A parametric study has been carried out on the problem under consideration, where some basic stability notions have been defined and characterized for the formulated problem.

Many aspects and general questions remain to be studied and explored in the field of multiobjective integer optimization problems under randomness. This paper is an attempt to establish underlying results which hopefully will help others to answer some or all of these questions.

There are however several unsolved problems, in our opinion, to be studied in futuer. Some of these problems are:

(i) An algorithm is rquired for solving multiobjective integer linear programming problems involving random parameters in the left-hand side of the constraints,

(ii) An algorithm is needed for treating large-scale multiobjective integer linear nonlinear programming problems under randomness,

(iii) An algorithm should be handled for solving integer linear and integer nonlinear goal programs involving random parameters.

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