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HOMOMORPHISMS AND SUBALGEBRAS OF MS-ALGEBRAS

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التشـــابهات والـجبـريات الـجــزئية لـجـبـريات MS

ســـناء العصـــار و عبـد المحســن بـــدوي

نتيجـة لتكـوين وتمثيل جبريـات MS بواسطـة الثلاثي المصـاحب أمكننا تمثيل تشـابهات جبريات MS بواسطة مفهوم الثلاثي وأيضاً أمكن تحديد الثلاثي المصـاحب للجبريات الجزئية وكيفية تكوينها بواسطته . ولقد أمكننا حل مشكلة ملء الفراغ الخاصة بالثلاثي المصاحب .

ABSTRACT

According to the characterization of MS-algebras from the subvariety K_2 by mean of the triple construction, we characterize the homomorphisms and the subalgebras of the MS-algebras. Also, we solve the "Fill-in" problem for the associated triples.

Key Words : MS-algebras, De Morgan algebras, kleene algebras, Varieties, Homomorphisms, Subalgebras.

INTRODUCTION

Blyth and Varlet introduced MS-algebras which are algebras of type (2, 2, 1, 0, 0) abstracting de Morgan and Stone algebras (see [2] and [3]. In [1] and [4] they considered a certain subvariety K_2 of MS-algebras whose members may be thought of as algebras abstracting kleene and Stone algebras. Each member of K_2 contains two simpler substructures, one being a kleene algebra and the other a distributive lattice with unit. They developed the "Chen-Gratzer" style construction theorem for the members of K_2 utilizing methods similar to those employed by Katrinak [6] and [7].

The purpose of this note is to study the properties of the triple dealing with the homomorphisms and the subalgebras of MS-algebras from K_2. The last part deals with fill-in theorems, giving sufficient conditions in order that (K, D, ?) can be filled in to make a triple.

A De Morgan algebra (L;∨, ∧, °, o, 1) is an algebra of type (2, 2, 1, 0, 0) such that (L; ∨, ∧, o, 1) is a bounded distributive lattice and ° is a unary operation satisfying the identities :

$$x = x^{\circ\circ} \quad \text{and} \quad (x \vee y)^{\circ} = x^{\circ} \wedge y^{\circ}.$$

As a direct consequence of the definition, we have, for all x,y ε L, $(x \wedge y)^{\circ} = x^{\circ} \vee y^{\circ}$, $0^{\circ} = 1$, $1^{\circ} = 0$ and the assignment $x \to x^{\circ}$ satisfies $x \leq y$ if and only if $x^{\circ} \geq y^{\circ}$.

A Kleene algebra (K;∨, ∧, °, 0, 1) is a De Morgan algebra on which for every x, y, $x^{\circ} \wedge x \leq y \vee y^{\circ}$ holds. An MS-algebra is an algebra (L;∨, ∧, °, 0, 1) of type (2, 2, 1, 0, 0) such that (L;∨, ∧, 0, 1) is a bounded distributive lattice and $x \to x^{\circ}$ is a unary operation and the following identities are satisfied :

(1) $x \leq x^\circ$,

(2) $(x \wedge y)^\circ = x^\circ \vee y^\circ$,

(3) $1^\circ = 0$.

The class of all MS-algebras forms a variety. The subvariety K_2 is defined by the additional two identities :

(4) $x \wedge x^\circ = x^{\circ\circ} \wedge x^\circ$,

(5) $(x \wedge x^\circ) \vee y \vee y^\circ = y \vee y^\circ$.

For any $L \varepsilon K_2$, we have

(6) $x = x^{\circ\circ} \wedge x^\circ (x \vee x^\circ)$, for every $x \varepsilon L$,

(7) $L^{\circ\circ} = \{ x \varepsilon L : x = x^{\circ\circ} \}$ is a Kleene algebra,

(8) $L^\wedge = \{ x \varepsilon L : x \leqq x^\circ \} = \{ z \varepsilon L : z = x \wedge x^\circ \}$ is an ideal of L and

(9) $L^\vee = \{ x \varepsilon L : x \geqq x^\circ \} = \{ z \varepsilon L : z = x \vee x^\circ \}$ is a filter of L.

For $a \varepsilon L^{\circ\circ}$, denote $d_a = a \vee a^\circ \varepsilon L^\vee$.

Let $L \varepsilon K_2$. $L^\vee$ is a filter of L and hence $L^\vee$ is a distributive lattice with the largest element 1. $F(L^\vee)$, the lattice of all filters of $L^\vee$, is distributive. The map $\phi(L) : L^{\circ\circ} \to F(L^\vee)$ defined in the following way .

$a\phi(L) = \{ x \varepsilon L : x \geqq x^\circ \} = [a^\circ) \bigcap L^\vee, a \varepsilon L^{\circ\circ}$

is a polarization, that is $\phi(L)$ is a (0,1) - homomorphism such that $a\phi(L) = L^\vee$ for every $a \varepsilon L^{\circ\circ\vee}$ and $a\phi(L)$ is a principal filter of $L^\vee$ for every $a \varepsilon L^{\circ\circ\wedge}$. The triple [$L^{\circ\circ}$, $L^\vee$, $\phi(L)$], which we call briefly the triple associated with L, uniquely determines the algebra L.

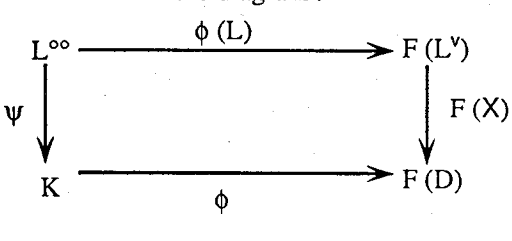
A K_2 - triple (triple) is (K, D, ϕ), where

(i) $K = (K; \vee, \wedge, ^\circ, 0, 1)$ is a Kleene algebra,

(ii) D is a distributive lattice with 1 and

(iii) $\phi : K \to F(D)$ is a polarization .

A K_2 - triple constructs an MS - algebra from K_2 (See [1] and [5] such that $L^{\circ\circ}$ is isomorphic with K, $L^\vee$ is isomorphic with D and the diagram .



is commutative. (ψ, χ are isomorphisms of $L^{\circ\circ}$ and K and of $L^\vee$ and D, respectively and $F(\chi)$ stands for the isomorphism of $F(L^\vee)$ and F (D) induced by χ).

The constructing MS-algebra L is described by

$L = \{ (a, a^\circ\phi \cup [x)), a \varepsilon K, x\gamma \varepsilon a^\circ\phi \} \subset K \times F_d(D)$

where γ is a modal operator on D with

$\operatorname{Im}\gamma = \{ z \varepsilon D : [z) = a\phi$ for some $a \varepsilon K^\wedge \}$.

Let $(a, a^\circ \cup [x)), (b, b^\circ\phi \cup [y)) \varepsilon L$. Then we have

(10) $(a, a^\circ\phi \cup [x)) \wedge (b, b^\circ\phi \cup [y))$
$= (a \wedge b, (a \wedge b)^\circ\phi \cup [x \wedge y))$,

(11) $(a, a^\circ\phi \cup [x)) \vee (b, b^\circ\phi \cup [y))$
$= (a \vee b, (a \vee b)^\circ\phi \cup [t)), t \varepsilon D$,

(12) $(a, a^\circ\phi \cup [x)) \leq (b, b^\circ\phi \cup [y))$ if and only if $a \leqq b$ and $a^\circ\phi \cup [x) \supseteq b^\circ\phi \cup [y)$,

(13) $(0,D) \leqq (a, a^\circ\phi \cup [x)) \leqq (1, [1))$ and

(14) $(a, a^\circ\phi \cup [x))^\circ = (a^\circ, a\phi)$.

MAIN RESULTS

I. Homomorphisms Of Ms-algebras From K_2

Lel L, $L_1 \varepsilon K_2$ and h be a homomorphism of L into L_1, that h is a lattice homomorphism which preserves 0, 1,°.

Definition 1

Let (K, D, ϕ) and (K_1, D_1, ϕ_1) be K_2 - triples (triples). A homomorphism of the triple (K, D, ϕ) into (K_1, D_1, ϕ_1) is a pair (f , g), where f is a homomorphism of K into K_1, g is a homomorphism of D into D_1 such that for every $a \varepsilon k$:

(15) $d_a g = d_{af}$

(16) $a\phi g \subseteq af\phi_1$

holds.

Lemma 1

Let a , $b \varepsilon K$ and $x, y, t \varepsilon D$. Let γ, γ_1 be modal operators on D and D_1 , respectively. Then

(i) $a\phi \cap [y) = [t)$ and $y\gamma \varepsilon a\phi$ implies $af\phi_1 \cap [yg) = [tg)$ and $yg\gamma_1 \varepsilon af\phi_1$,

(ii) $(a^\circ\phi \cup [x)) \cap (b^\circ\phi \cup [y)) = (a \vee b)^\circ\phi \cup [t)$ and $t\gamma \varepsilon (a \vee b)^\circ\phi$ implies
$((af)^\circ \phi_1 \cup [xg)) \cap ((bf)^\circ\phi_1 \cup [yg))$
$= (a \vee b)^\circ f\phi_1 \cup [tg)$ and $tg\gamma_1 \varepsilon (a \vee b)^\circ f\phi_1$.

Proof

(i) Let $a\phi \cap [y) = [t)$, then $t = x_1 \vee y, x_1 \varepsilon a\phi$ and $x_1 g \varepsilon a\phi g \subseteq af\phi_1$.
If $t_1 \varepsilon af\phi_1 \cap [yg)$, then $t_1 = x_1 g \vee yg$
$= (x_1 \vee y)g = tg$ and $af\phi_1 \cap [yg) \subseteq [tg)$.
Also, $af\phi_1 \cap [yg) \supseteq a\phi g \cap yg)$
$= (a\phi \cap [y))g = [t)g = [tg)$ and $y\gamma \varepsilon a\phi$ implies $yg\gamma_1 \varepsilon af\phi_1$,

(ii) $((af)^\circ\phi_1 \cup [xg)) \cap ((bf)^\circ\phi_1 \cup [yg))$
$= ((af)^\circ\phi_1 \cap ((bf)^\circ\phi_1 \cup [yg))) \cup ([xg) \cap ((bf)^\circ\phi_1 \cup [yg)))$
$= ((af)^\circ\phi_1 \cap (bf)^\circ\phi_1) \cup ((af)^\circ\phi_1 \cap [yg)) \cup ([xg) \cap (bf)^\circ\phi_1) \cup ([xg) \cap [yg))$
$= (a \vee b)^\circ f\phi_1 \cup [tg)$.

and

$((af)^\circ\phi_1 \cap [yg)) \cup (([xg) \cap (bf)^\circ\phi_1) \cup [x \vee y)g)$
$= [t_1 g) \cup [t_2 g) \cup [(x \vee y)g]$
$= [(t_1 \wedge t_2 \wedge (x \vee y)g) = [tg)$,

where

$(af)^\circ\phi_1 \cap [yg) = [t_1 g)$ and $(bf)^\circ\phi_1 \cap [xg) = [t_2 g)$ by (i) and $t = t_1 \wedge t_2 \wedge (x \vee y)$. $\cap$

Now, since $t\gamma \varepsilon (a \vee b)^\circ\phi$, $t\gamma g \varepsilon (a \vee b)^\circ\phi g \subseteq (a \vee b)^\circ f\phi_1$ and $(tg)Y_1 = t\gamma g \varepsilon (a \vee b)^\circ f\phi_1$

Theorem 1

Let L and L_1 be MS-algebras from K_2 , (K, D, ϕ) and (K_1, D_1, ϕ_1) be the associated triples, respectively. Let h be a homomorphism of L into L_1 and h_K, h_D the restrictions of h to K and D, respectively. Then (h_K, h_D) is a homomorphism of the triples. Conversely, every homomorphism (f, g) of triples uniquely determines a homomorphism h of L into L_1 with $h_K = f$, $h_D = g$ by the following rule :

$xh = x^{\circ\circ} f \wedge (x \vee x^\circ) g$ for all $x \varepsilon L$.

(In other words, homomorphisms of MS-algebras from K_2 are the same as homomorphisms of triples).

Proof

To prove the first statement we have to verify (15) and (16) with $g = h_D$ and $f = h_K$. Evidently,

$d_{ah} = ah \vee (ah)^\circ = (a \vee a^\circ)h = d_a h$,

$a\phi h = \{xh : x \varepsilon a\phi\} = \{xh : x \varepsilon [a^\circ) \cap D\}$
$\subseteq \{y : y \varepsilon [(ah)^\circ) \cap D_1\} = ah\phi_1$.

Conversely, let (15) and (16) hold. We represent the elements of L and L_1 as in Construction Theorem that is

$L = \{(a, a^\circ\phi \cup [x)) : a \varepsilon K, x \varepsilon D, x\gamma e a^\circ\phi\}$,

where γ is a modal operator on D with $Im\gamma = \{z \varepsilon D : [z) = a\phi$ for some $a \varepsilon K^\wedge\}$ and

$L_1 = \{(b, b^\circ\phi_1 \cup [y)) : b \varepsilon K_1, y \varepsilon D_1, y\gamma_1 \varepsilon b^\circ\phi_1\}$,

where γ_1 is a modal operator on D_1 with

$Im\gamma_1 = \{z \varepsilon D_1 : [z) = a\phi_1$ for some $a \varepsilon K_1^\wedge\}$.

Then the definition of h reads :

$(a, a^\circ\phi \cup [x))h = (af, (af)^\circ\phi_1 \cup [xg)), xg\gamma_1 \varepsilon (af)^\circ\phi_1$.

We show that h is well defined. Let

$(a, a^\circ\phi \cup [x)) = (b, b^\circ\phi \cup [y))$.

Then $a = b$ and $a^\circ\phi \cup [x) = b^\circ\phi \cup [y)$.

Hence, $x \geqq x_1 \wedge y$ and $y \geqq y_1 \wedge x$ for some $x_1, y_1 \in a^\circ \phi$. Since g is a homomorphism and (16) holds, then we have $xg \geqq x_1 g \wedge yg$ and $yg \geqq y_1 g \wedge xg$ with $x_1 g, y_1 g \in (af)^\circ \phi_1$. So we obtain $(af)^\circ \phi_1 \cup [xg) = (af)^\circ \phi_1 \cup [yg)$.

Thus, $(a, a^\circ \phi \cup [x))h = (b, b^\circ \phi \cup [y))h$. Therefore, h is a map of L into L_1. Obviously, $h_K = f$ and $h_D = g$. To prove that h is a homomrphism, we have to verify the following three formulae :

(17) $((a,a^\circ \phi \cup [x)) \wedge (b,b^\circ \phi \cup [y)))h$
$= (a,a^\circ \phi \cup [x))h \wedge (b, b^\circ \phi \cup [y))h$;

(18) $((a, a^\circ \phi \cup [x)) \vee (b, b^\circ \phi \cup [y)))h$
$= (a, a^\circ \phi \cup [x))h \vee (b, b^\circ \phi \cup [y))h$;

(19) $(a, a^\circ \phi \cup [x))^\circ h = ((a, a^\circ \phi \cup [x))h^\circ$.

(17) $((a, a^\circ\phi \cup [x)) \wedge (b, b^\circ\phi \cup [y)))h$
$= (a \wedge b, (a \wedge b)^\circ \phi \cup [x \wedge y))h$
$= ((a \wedge b)f, ((a \wedge b)f)^\circ \phi_1 \cup [(x \wedge y)g)))$
$= (af, (af)^\circ \phi_1 \cup [xg)) \wedge (bf, (bf)^\circ \phi_1 \cup [yg))$
$= (a,a^\circ \phi \cup [x))h \wedge (b, b^\circ \phi \cup [y))h$

(18) $((a, a^\circ \phi \cup [x)) \vee (b, b^\circ \phi \cup [y)))h$
$= (a \vee b, (a \vee b)^\circ\phi \cup [t))h$
$= ((a \vee b)f, ((a \vee b)f)^\circ \phi_1 \cup [tg))$
$= ((a \vee b)f, ((af)^\circ \phi_1 \cup [xg))$
$\cap ((bf)^\circ \phi_1 \cup [yg))$
by lemma 1 (ii)
$= (af, (af)^\circ \phi_1 \cup [xg)) \vee (bf, (bf)^\circ \phi_1 \cup [yg))$
$= (a, a^\circ \phi \cup [x))h \vee (b, b^\circ \phi \cup [y))h$.

(19) $(a,a^\circ \phi \cup [x))^\circ h = (a^\circ, a \phi)h = (a^\circ f, af \phi_1)$
$= (af, (af)^\circ \phi_1 \cup [xg))^\circ$
$= ((a, a^\circ\phi \cup [x))h)^\circ$.

Thus, h is a homomrphism of L into L_1. It is easy to see the uniqueness of h with $h_K = f$ and $h_D = g$.

II. Subalgebras of MS - Algebras from K_2

According to the characterization of MS-algebras in K_2 by means of the triple (K, D, ϕ), we characterize the subalgebras and solve the "Fill-in" problem for their associated triples.

Theorem 2

Let L_1 be a subalgebra of an MS-algebra L from K_2, then $L^{\circ\circ}{}_1 = L_1 \cap L^{\circ\circ}$ is a subalgebra of $L^{\circ\circ}$ and $L_1{}^{\vee} = L_1 \cap L^{\vee}$ is a sublattice of $L^{\vee}$ containing 1. The triple associated with L_1 is $(L_1{}^{\circ\circ}, L_1{}^{\vee}, \phi_1)$, where ϕ_1 is given by $a\phi_1 = a\phi \cap L_1{}^{\vee}$, for $a \in L_1{}^{\circ\circ}$.

Proof

Let $x,y \in L_1{}^{\circ\circ}$, clearly $x \vee y$, $x \wedge y$ are elements in $L_1{}^{\circ\circ}$, $L_1{}^{\circ\circ}$ is a sublattice of $L^{\circ\circ}$. Since L_1 is bounded and the bounds 0,1 are squelette elements, then
$0,1 \in L_1{}^{\circ\circ}$ and $1^\circ = 0$
also $(x \vee y)^\circ = x^\circ \wedge y^\circ$
and $x \wedge x^\circ \leqq y \vee y^\circ$ for every $x,y \in L_1{}^{\circ\circ}$.
Thus $L_1{}^{\circ\circ}$ is a subalgebra of $L^{\circ\circ}$.

Since $L^{\vee}$ is a sublattice of L containing 1, then $L_1{}^{\vee} = L_1 \cap L^{\vee}$ is a sublattice of $L^{\vee}$ containing 1.

Now, we define $\phi_1 : L_1{}^{\circ\circ} \to F(L_1{}^{\vee})$ by $a\phi_1 = a\phi \cap L_1{}^{\vee}$, $a \in L_1{}^{\circ\circ}$, we show that ϕ_1 is a polarization $0\phi_1 = 0\phi \cap L_1{}^{\vee} = [1)$, $1\phi_1 = 1\phi \cap L_1{}^{\vee} = L_1{}^{\vee}$ and
$(a \vee b)\phi_1 = (a \vee b)\phi \cap L_1{}^{\vee} =$
$= (a\phi \cup b\phi) \cap L_1{}^{\vee} = (a\phi \cap L_1{}^{\vee}) \cup (b\phi \cap L_1{}^{\vee})$
$= a\phi_1 \cup b\phi_1$,
$(a \wedge b)\phi_1 = (a \wedge b)\phi \cap L_1{}^{\vee} = (a\phi \cap bf) \cap L_1{}^{\vee}$
$= (a\phi \cap L_1{}^{\vee}) \cap (b\phi \cap L_1{}^{\vee})$
$= a\phi_1 \cap b\phi_1$, $a, b \in L_1{}^{\circ\circ}$

which means that ϕ_1 is a $\{0,1\}$ - homomorphism of $L_1^{\circ\circ}$ into $F(L_1^{\vee})$.

For all $a \in L_1^{\circ\circ\vee}$, $a = a_1 \vee a_1^{\circ}$, $a_1 \in L_1^{\circ\circ}$ we have $a\phi_1 = (a_1 \vee a_1^{\circ})\phi_1 = (a_1 \vee a_1^{\circ})\phi \cap L_1^{\vee}$ $= [d_{a1}) \cap L_1^{\vee} = L_1^{\vee}$ (ϕ is a polarization)

For all $a \in L_1^{\circ\circ\wedge}$, $a\phi_1$ is a principal filter of $F(L_1^{\vee})$. Then $(L_1^{\circ\circ}, L_1^{\vee}, \phi_1)$ is the K_2-triple associated with L_1.

Theorem 3

Let $L \in K_2$, $L_1^{\circ\circ}$ be a subalgebra of $L^{\circ\circ}$, $L_1^{\vee}$ a sublattice of $L^{\vee}$ containing 1. We can fill - in $(L_1^{\circ\circ}, L_1^{\vee}, ?)$ such that it will become the triple associated with a subalgebra of L iff

(1) $a\phi_1 \cup a^{\circ}\phi_1 = L_1^{\vee}$ for $a \in L_1^{\circ\circ}$

(2) $a \vee a^{\circ} \in L_1^{\vee}$ for $a \in L_1^{\circ\circ}$.

Proof

If $(L_1^{\circ\circ}, L_1^{\vee}, \phi_1)$ is the triple associated with a subalgebra L_1 of L, then $a\phi_1 = a\phi_L \cap L_1^{\vee}$. Hence

$$(a\phi_1 \cup a^{\circ}\phi_1) = (a\phi \cap L_1^{\vee}) \cup (a^{\circ}\phi \cap L_1^{\vee})$$
$$= (a\phi \cup a^{\circ}\phi) \cap L_1^{\vee}$$
$$= L^{\vee} \cap L_1^{\vee} = L_1^{\vee}.$$

Now, let $a \in L_1^{\circ\circ}$, then $a \vee a^{\circ} \in L_1 \cap L^{\vee} = L_1^{\vee}$.

Conversely, assume (1) and (2). Let $K = L^{\circ\circ}$, $D = L^{\vee}$ and $\phi = \phi(L)$. Represent the elements of L as in the Construction Theorem, that is,

$L = \{ (a, a^{\circ}\phi \cup [x)) : a \in K, x \in D,\ x\gamma \in a^{\circ}\phi$ and γ is a modal operator on $D \}$

$L_1 = \{ (a, a^{\circ}\phi \cup [x)) : a \in K_1,\ x \in D_1, x\gamma_1 \in a^{\circ}\phi$ and γ_1 is the restriction of γ to $D_1 \}$.

We show that L_1 is a subalgebra of L. It is clear that $0_L = (0,D)$ and $1_L = (1,[1))$ belong to L_1 and if $(a, a^{\circ}\phi \cup [x)) \in L_1$,

$((a, a^{\circ}\phi \cup [x)))^{\circ} = (a^{\circ}, a\phi) \in L_1$.

Now, let $(a, a^{\circ}\phi \cup [x))$, $(b, b^{\circ}\phi \cup [y))$ be elements of L_1. We have

$(a, a^{\circ}\phi \cup [x)) \wedge (b, b^{\circ}\phi \cup [y))$
$= a \wedge b, (a \wedge b)^{\circ}\phi \cup [x \wedge y)) \in L_1$ and
$(x \wedge y)\gamma_1 = (x \wedge y)\gamma \in (a \wedge b)^{\circ}\phi$,
$(a, a^{\circ}\phi \cup [x)) \vee (b, b^{\circ}\phi \cup [y)) = (a \vee b, (a^{\circ}\phi \cup [x)) \cap (b^{\circ}\phi \cup [y)))$
$= (a \vee b, (a \vee b)^{\circ}\phi \cup [t)) \in L_1$

where

$(a^{\circ}\phi \cup [x)) \cap (b^{\circ}\phi \cup [y)) = ((a^{\circ}\phi \cup [x)) \cap b^{\circ}\phi) \cup ((a^{\circ}\phi \cup [x)) \cap [y))$
$= (a^{\circ}\phi \cap b^{\circ}\phi) \cup ([x) \cap b^{\circ}\phi) \cup (a^{\circ}\phi \cap [y)) \cup [x \vee y)$
$= (a \vee b)^{\circ}\phi \cup [t_1) \cup [t_2) \cup [x \vee y)$
$= (a \vee b)^{\circ}\phi \cup [t_1 \wedge t_2 \wedge (x \vee y))$
$= (a \vee b)^{\circ}\phi \cup [t), t \in D_1,\ t\gamma_1 \in (a \vee b)^{\circ}\phi$.

Since $b^{\circ}\phi \cap [x) = [t_1) = [x \vee x_1)$, $x_1 \in b^{\circ}\phi$ and $a^{\circ}\phi \cap [y) = [t_2) = [y \vee y_1)$, $y_1 \in a^{\circ}\phi$ (by Lemma 1, [1]),

then $t_1, t_2 \in D_1$ and so $t = t_1 \wedge t_2 \wedge (x \vee y) \in D_1$.

Also, we have.

$(a, a^{\circ}\phi \cup [x)) \leq (a, a^{\circ}\phi \cup [x))^{\circ\circ} = (a, a^{\circ}\phi)$,

$((a, a^{\circ}\phi \cup [x)) \wedge (b, b^{\circ}\phi \cup [y)))^{\circ}$
$= (a \wedge b, (a \wedge b)^{\circ}\phi \cup [x \wedge y))^{\circ}$
$= ((a \wedge b)^{\circ}, (a \wedge b)\phi)$
$= (a^{\circ} \vee b^{\circ}, a\phi \wedge b\phi)$
$= (a^{\circ}, a\phi) \vee (b^{\circ}, b\phi)$
$= (a, a^{\circ}\phi \cup [x))^{\circ} \vee (b, b^{\circ}\phi \cup [y))^{\circ}$.

and

$(a, a^{\circ}\phi \cup [x)) \wedge (a, a^{\circ}\phi \cup [x))^{\circ}$
$= (a, a^{\circ}\phi \cup [x)) \wedge (a^{\circ}, a\phi)$
$= (a \wedge a^{\circ}, (a^{\circ} \vee a)\phi \cup [x))$
$= (a, a^{\circ}\phi) \wedge (a^{\circ}, a\phi)$ (Since $(a^{\circ} \vee a)\phi = D$)
$= (a, a^{\circ}\phi \cup [x))^{\circ\circ} \wedge (a, a^{\circ}\phi \cup [x))^{\circ}$.

Similarly,

$(x \wedge x^\circ) \vee (y \vee y^\circ) = y \vee y^\circ \ \forall \ x, y \in L_1$.

Thus L_1 is a subalgebra of L. We show that $L_1^{\circ\circ} \simeq K_1$ and $L_1^{\vee} \simeq D_1$.

$$L_1^{\circ\circ} = \{ (a, a^\circ\phi) \ : \ a \in K_1 \}$$

$$L_1^{\vee} = \{ (a, a^\circ\phi \cup [x)) : a \in K_1^{\vee} \}$$

Define $\psi : K_1 \to L_1^{\circ\circ}$ by $a\psi = (a, a^\circ\phi)$, $a \in K$ and $\chi : D_1 \to L_1^{\vee}$ by $x\chi = (d, d^\circ\phi \cup [x))$, $d \in K_1^{\vee}$.

By easy computations, we can prove that ψ and χ are isomorphisms. Hence we can fill-in $(K_1, D_1, ?)$ by $\phi_1 = \phi_{L_1} = \phi_L \cap D_1$ such that it will become the triple associated with a subalgebra of L.

III. Fill-in Theorems (Fill-in problems)

Fill - in problems are statements containing the answer to the question : for a given kleene algebra K, a distributive lattice D with 1, when does there exist a ϕ such that (K, D, ϕ) is a K_2 - triple ?

Theorem 4

$(K, D, ?)$ can always be filled in to make it a K_2 - triple if K is a Kleene algebra and D a distributive lattice with 1, provided $|K| > 1$. If $|K| = 1$ then $|D| = 1$.

Proof

Take an arbitrary prime ideal P of K. Define

$$\phi : K \to F(D)$$

$$x\phi = D \quad \text{for} \quad x \notin P$$

$$x\phi = [1) \quad \text{for} \quad x \in P.$$

It is easy to check that ϕ is a polarization.

Consider the fill-in problem given by the following diagram

$$\begin{array}{ccc} (K & , D & , \phi) \\ f\downarrow & g\downarrow & \\ (K_1, & D_1 & , ?) \end{array}$$

where f and g are onto homomorphisms. We can formulate

Theorem 5

Let (K, D, f) be a given K_2-triple, $(K_1, D_1, ?)$ a defective triple and a pair of onto homomorphisms $f : K \to K_1$ and $g : D \to D_1$. There exists a ϕ_1 making (K_1, D_1, ϕ_1) a K_2-triple, and (f, g) a homomorphism of (K, D, ϕ) and (K_1, D_1, ϕ_1) iff $(a\phi)\, g = [1)$ for all $a \in of^{-1}$.

Proof

Assume that (K, D, ϕ) and (K_1, D_1, ϕ_1) be K_2 - triples with a pair of onto homomorphisms $f : K \to K_1$ and $g : D \to D_1$, (f, g) is a homomorphism of the two triples.

Then $a\phi g \subseteq af\phi_1$. Let $a \in Of^{-1}$ then $af = 0$ and $(a\phi)\, g \subseteq (af)\, \phi_1 = 0\, \phi_1 = [1)$, but $[1)$ is the smallest element of $F(D_1)$, then $(a\phi)\, g = [1) \ \forall \ a \in Of^{-1}$. Conversely, let $(a\phi)\, g = [1) \ \forall \ a \in Of^{-1}$. Define $\phi_1 : K_1 \to F(D_1)$ as $b\phi_1 = a\phi g$, where $b = af$, $a \in K$, that is $(af)\, \phi_1 = (a\phi)g$. We have to show that ϕ_1 is a well defined map. Let $af = bf$, $a, b \in k$, then $(af)^\circ = (bf)^\circ$,

$$\begin{aligned} a\phi g &= \{xg : x \in a\phi\} = \{ xg : x \in [a^\circ) \cap D\} \\ &= \{ y : y = xg \in [(af)^\circ) \cap D_1 \} \\ &= \{ y : y \in [(bf)^\circ) \cap D_1\} \\ &= \{ xg : x \in [b^\circ) \cap D \} = b\phi g. \end{aligned}$$

and ϕ_1 is a well defined map.

Since f is a Kleene homomorphism of K onto K_1, then $0\,\phi_1 = (0f)\, \phi_1 = 0\,\phi g = [1)$ which is the zero of $F(D_1)$. Also, $1\,\phi_1 = (1f)\,\phi_1$ is a $\{0,1\}$ - map. Now, let $x, y \in K_1$, then $af = x$, $bf = y$ for some $a, b \in K$.

$$\begin{aligned} (x \vee y)\phi_1 = (af \vee bf)\phi_1 &= (a \vee b) f \phi_1 \\ &= (a \vee b)\phi g \\ &= (a\phi \cup b\phi)\, g \end{aligned}$$

$= a\phi g \cup b\phi g$

$= (af)\phi_1 \cup (bf)\phi_1$

$= x\phi_1 \cup y\phi_1$

and

$(x \wedge y)\phi_1 = (af \wedge bf)\phi_1 = (af \wedge bf)\phi_1$

$= (a \wedge b)\phi g$

$= a\phi g \cap b\phi g$

$= (af)\phi_1 \cap (bf)\phi_1$

$= x\phi_1 \cap y\phi_1$,

then ϕ_1 is a lattice homomorphism.

Also, for all $x \in K_1^{\vee}$, $x = af$, then $x = x_1 \vee x_1^{\circ}$, $x_1 = a_1 f$

$x\phi_1 = (x_1 \vee x_1^{\circ})\phi_1 = (a_1 f \vee (a_1 f)^{\circ})\phi_1$

$= (a_1 \vee a_1^{\circ}) f\phi_1$

$= (a_1 \vee a_1^{\circ})\phi g$

$= Dg = D_1$

and for all $x \in K_1^{\wedge}$, $x = x_1 \wedge x_1^{\circ}$, $x_1 = a_1 f$

$x\phi_1 = (x_1 \wedge x_1^{\circ})\phi_1 = (a_1 f \wedge (a_1 f)^{\circ})\phi_1$

$= (a_1 \wedge a_1^{\circ}) f\phi_1$

$= (a_1 \wedge a_1^{\circ})\phi g$ (since $a_1 \wedge a_1^{\circ} \in K^{\wedge}$)

$= [d)g = [dg)$, $d \in D$

and $x\phi_1$ is a principal filter in $F(D_1)$, for all $x \in K_1^{\wedge}$. Hence ϕ_1 is a polarization and (K_1, D_1, ϕ_1) is a K_2-triple, we have to show that (f, g) is a triple homomorphism. By definition

$a\phi g = af\phi_1$ and

$[d_a g) = [d_a)g = [a \vee a^{\circ})g = ([a \vee a^{\circ}) \cap D)g$

$= [(a \vee a^{\circ})f) \cap Dg$

$= [af \vee (af)^{\circ}) \cap D_1$

$= [af \vee (af)^{\circ})$

$= [d_a f)$

completing the required proof.

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