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Application of homotopy-perturbation method to fractional IVPs

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Abstract

Fractional initial-value problems (fIVPs) arise from many fields of physics and play a very important role in various branches of science and engineering. Finding accurate and efficient methods for solving fIVPs has become an active research undertaking. In this paper, both linear and nonlinear fIVPs are considered. Exact and/or approximate analytical solutions of the fIVPs are obtained by the analytic homotopy-perturbation method (HPM). The results of applying this procedure to the studied cases show the high accuracy, simplicity and efficiency of the approach.

Keywords: Fractional IVPs; Homotopy-perturbation method; Caputo's fractional derivative

1. Introduction

In recent years, fractional differential equations (fDEs) have successfully modelled many physical and engineering phenomena such as seismic analysis, viscous damping, viscoelastic materials and polymer physics [25,22,13]. One very important class of fDEs is the fractional initial-value problems (fIVPs) written in the form:

$$D^{\alpha}y(t) = f(t, y(t)), \quad y^{(k)}(0) = y_0^{(k)}, \;\; k = 0, 1, \ldots, n-1, \tag{1}$$

where f is an arbitrary function, D^{α} denotes the fractional differential derivative in the sense of Caputo, $y^{(k)}(t)$ is the kth derivative of y and $y_0^{(k)}$ are the specified initial conditions.

Finding accurate and efficient methods for solving (1) has become an active research undertaking. Several numerical methods for solving (1) have been presented in [2–5,15,22]. Analytical methods for (1) include the Adomian decomposition method (ADM) [23,24,18,17,1,14] and the variational iteration method (VIM) [18,17,20,7]. One disadvantage of these analytical methods is the analytical/symbolic evaluation of the integrations which can be complex even for a computer algebra package.

Another approach that can be applied for solving (1) is to employ the homotopy-perturbation method (HPM), cf. [8,9]. The HPM, in contrast to the traditional perturbation methods, does not require a small parameter in the system and the approximations obtained by the proposed method are uniformly valid not only for small parameters, but also for very large parameters. Odibat and Momani [19] applied HPM to solve quadratic Riccati differential equation of

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fractional order. In [16], Momani and Odibat solved nonlinear fractional partial differential equations by HPM. Wang in [27,26] employed the HPM for solving the classical fractional KdV and KdV-Burgers equations, respectively. Zhang and He [28] obtained approximate solution to the nonlinear Poisson–Boltzmann equation by HPM incorporating the Taylor series expansion.

In this paper, we will apply the homotopy-perturbation method (HPM) to solve the linear and nonlinear fIVPs of the form (1). The modified HPM (mHPM) [21] and the Taylor series-HPM approach [28] shall be adopted.

2. Basic definitions

In this section, we give some definitions and properties of the fractional calculus [22].

Definition 1. A real function $h(t)$, $t>0$, is said to be in the space C_μ, $\mu \in R$, if there exists a real number $p>\mu$, such that $h(t)=t^p h_1(t)$, where $h_1(t)\in C(0,\infty)$, and it is said to be in the space C_μ^n if and only if $h^{(n)}\in C_\mu$, $n\in N$.

Definition 2. The Riemann–Liouville fractional integral operator (J^α) of order $\alpha \geqslant 0$, of a function $h\in C_\mu$, $\mu \geqslant -1$, is defined as

$$J^\alpha h(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}h(\tau)\,\mathrm{d}\tau \quad (\alpha>0),$$
$$J^0 h(t)=h(t), \tag{2}$$

where $\Gamma(z)$ is the well-known Gamma function.

Some of the properties of the operator J^α, which we will need here, are as follows:
For $h\in C_\mu$, $\mu \geqslant -1$, $\alpha,\beta \geqslant 0$ and $\gamma \geqslant -1$:

(1) $J^\alpha J^\beta h(t)=J^{\alpha+\beta}h(t)$,
(2) $J^\alpha J^\beta h(t)=J^\beta J^\alpha h(t)$,
(3) $J^\alpha t^\gamma=(\Gamma(\gamma+1)/\Gamma(\alpha+\gamma+1))t^{\alpha+\gamma}$.

Definition 3. The fractional derivative (D^α) of $h(t)$ in the Caputo's sense is defined as

$$D^\alpha h(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}h^{(n)}(\tau)\,\mathrm{d}\tau, \tag{3}$$

for $n-1<\alpha \leqslant n,\ n\in N,\ t>0,\ h\in C_{-1}^n$.

The following are two basic properties of the Caputo's fractional derivative [6]:

(1) Let $h\in C_{-1}^n$, $n\in N$. Then $D^\alpha h$, $0\leqslant \alpha \leqslant n$ is well defined and $D^\alpha h\in C_{-1}$.
(2) Let $n-1<\alpha\leqslant n$, $n\in N$ and $h\in C_\mu^n$, $\mu\geqslant -1$. Then

$$(J^\alpha D^\alpha)h(t)=h(t)-\sum_{k=0}^{n-1}h^{(k)}(0^+)\frac{t^k}{k!}. \tag{4}$$

3. The homotopy-perturbation method (HPM)

The HPM was first proposed by Chinese mathematician He [8,9]. The essential idea of this method is to introduce a homotopy parameter, say p, which takes the values from 0 to 1. When $p=0$, the system of equations usually reduces to a sufficiently simplified form, which normally admits a rather simple solution. As p gradually increases to 1, the system goes through a sequence of 'deformation', the solution of each of which is 'close' to that at the previous stage of 'deformation'. Eventually at $p=1$, the system takes the original form of the equation and the final stage of 'deformation' gives the desired solution.

The fIVPs (1) is first written in the operator form:

$$D^{\alpha}y(t)+Ly(t)+Ny(t)=g(t), \tag{5}$$

$$y^{(k)}(0)=c_k, \quad k=0,1,\ldots,n-1, \tag{6}$$

where c_k is the initial conditions, L is the linear operator which might include other fractional derivative operators $D^{\beta}(\beta<\alpha)$, and N is the nonlinear operator in the space C^n_{-1}, while the function g, the source function, is assumed to be in C_{-1} if α is an integer, and in C^1_{-1} if α is not an integer. The solution $y(t)$ is to be determined in C^n_{-1}.

We shall next present the solution approaches based on the standard HPM [8,9] and the modified HPM (mHPM) of [21] separately.

3.1. Standard HPM

In view of HPM, we construct the following homotopy:

$$(1-p)D^{\alpha}y+p[D^{\alpha}y+Ly(t)+Ny(t)-g(t)]=0, \tag{7}$$

or

$$D^{\alpha}y+p[Ly(t)+Ny(t)-g(t)]=0, \tag{8}$$

where $p\in[0,1]$ is an embedding parameter. If $p=0$, Eqs. (7) and (8) become

$$D^{\alpha}y=0, \tag{9}$$

and when $p=1$, both (7) and (8) turn out to be the original fDE (5).

Using the parameter p, we expand the solution in the following form:

$$y(t)=y_0(t)+py_1(t)+p^2y_2(t)+p^3y_3(t)+\cdots. \tag{10}$$

The convergence of the above series is discussed in [10] and the asymptotic behavior of the series is illustrated in [11,12].

Setting $p=1$ results in the solution of Eq. (5)

$$y(t)=y_0(t)+y_1(t)+y_2(t)+y_3(t)+\cdots. \tag{11}$$

For the nonlinear term $Ny(t)$ in (5), let us set $Ny(t)=h(y)$.

Substituting (10) in (8) and collecting the terms with the same powers of p, we obtain

$$p^0: D^{\alpha}y_0=0, \tag{12}$$

$$p^1: D^{\alpha}y_1=-Ly_0(t)-h_1(y_0)+g(t), \tag{13}$$

$$p^2: D^{\alpha}y_2=-Ly_1(t)-h_2(y_0,y_1), \tag{14}$$

$$p^3: D^{\alpha}y_3=-Ly_2(t)-h_3(y_0,y_1,y_2), \tag{15}$$

and so on, where the functions $h_1,h_2,h_3,\ldots,$ satisfy the following equation:

$$h(y_0+py_1+p^2y_2+\cdots)=h_1(y_0)+ph_2(y_0,y_1)+p^2h_3(y_0,y_1,y_2)+\cdots. \tag{16}$$

Applying the operator J^{α}, the inverse operator of D^{α}, which is defined by (2) on both sides of the above linear equations, with considering the initial conditions by using (4), the first few terms of the HPM solution can be given by

$$y_0=\sum_{k=0}^{n-1}y^{(k)}(0)\frac{t^k}{k!}=\sum_{k=0}^{n-1}c_k\frac{t^k}{k!},$$

$$y_1 = -J^\alpha[Ly_0(t)] - J^\alpha[h_1(y_0)] + J^\alpha[g(t)],$$

$$y_2 = -J^\alpha[Ly_1(t)] - J^\alpha[h_2(y_0, y_1)],$$

$$y_3 = -J^\alpha[Ly_2(t)] - J^\alpha[h_3(y_0, y_1, y_2)].$$

3.2. *Modified HPM*

In the standard HPM as described above, finding the fractional integrations can be complex. One way to avoid this difficulty is to employ the mHPM of Odibat [21] by taking the Taylor series of the source term $g(t)$, i.e.

$$g(t) = \sum_{n=0}^{\infty} g_n(t). \tag{17}$$

Following Odibat [21], we construct the following homotopy:

$$(1-p)D^\alpha y + p[D^\alpha y + Ly(t) + Ny(t)] = \sum_{n=0}^{\infty} p^n g_n(t), \tag{18}$$

or

$$D^\alpha y + p\,[Ly(t) + Ny(t)] = \sum_{n=0}^{\infty} p^n g_n(t). \tag{19}$$

If we set $g_1(t) = g(t)$, $g_n(t) = 0$ for $n = 0$ or $n \geqslant 2$, then the homotopy (18) or (19) reduces to the homotopy (7) or (8), respectively.

The form of homotopy (19) allows us to obtain the individual terms $y_0, y_1, \ldots$ in (10). Substituting (10) in (19) and collecting the terms with the same powers of p, we obtain

$$p^0 : D^\alpha y_0 = g_0(t), \tag{20}$$

$$p^1 : D^\alpha y_1 = g_1(t) - Ly_0(t) - h_1(y_0), \tag{21}$$

$$p^2 : D^\alpha y_2 = g_2(t) - Ly_1(t) - h_2(y_0, y_1), \tag{22}$$

$$p^3 : D^\alpha y_3 = g_3(t) - Ly_2(t) - h_3(y_0, y_1, y_2), \tag{23}$$

and so on, where the functions $h_1, h_2, h_3, \ldots,$ satisfy (16). Again, by applying the operator J^α on both sides of the above linear equations, the first few terms of the HPM solution can be given by

$$y_0 = \sum_{k=0}^{n-1} y^{(k)}(0)\frac{t^k}{k!} + J^\alpha[g_0(t)] = \sum_{k=0}^{n-1} c_k \frac{t^k}{k!} + J^\alpha[g_0(t)],$$

$$y_1 = J^\alpha[g_1(t)] - J^\alpha[Ly_0(t)] - J^\alpha[h_1(y_0)],$$

$$y_2 = J^\alpha[g_2(t)] - J^\alpha[Ly_1(t)] - J^\alpha[h_2(y_0, y_1)],$$

$$y_3 = J^\alpha[g_3(t)] - J^\alpha[Ly_2(t)] - J^\alpha[h_3(y_0, y_1, y_2)].$$

4. Test examples

In this section, we shall illustrate the applicability of HPM to linear and nonlinear fIVPs.

4.1. Problem 1

First, we consider the following linear fIVP:

$$D^{\alpha} y = -y, \quad 0 < \alpha \leqslant 2, \tag{24}$$

$$y(0) = 1, \quad y'(0) = 0. \tag{25}$$

The second initial condition is for $\alpha > 1$ only.

According to Eq. (7), we can construct the following homotopy:

$$D^{\alpha} y + p(y) = 0. \tag{26}$$

Substituting (10) into (26), and collecting terms of the same power of p, yields the following linear equations:

$$p^0 : D^{\alpha} y_0 = 0, \tag{27}$$

$$p^1 : D^{\alpha} y_1 = -y_0, \tag{28}$$

$$p^2 : D^{\alpha} y_2 = -y_1, \tag{29}$$

$$p^3 : D^{\alpha} y_3 = -y_2, \tag{30}$$

$$\vdots$$

Applying the operator J^{α}, the inverse operator of D_t^{α}, on both sides of the linear equations (27)–(30) and using the initial condition (25), we obtain

$$y_0 = y(0) = 1,$$

$$y_1 = -J^{\alpha}[y_0] = -\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$y_2 = -J^{\alpha}[y_1] = -J^{\alpha}\left[-\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right] = \frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$y_3 = -J^{\alpha}[y_2] = -J^{\alpha}\left[\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right] = -\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$\vdots$$

Hence the solution is

$$y = y_0 + y_1 + y_2 + y_3 + \cdots = \sum_{k=0}^{\infty} \frac{(-t^{\alpha})^k}{\Gamma(\alpha k+1)}, \tag{31}$$

which is the exact solution [4].

4.2. Problem 2

Now consider the following nonlinear fIVP:

$$D^{\alpha} y = \frac{40320}{\Gamma(9-\alpha)} t^{8-\alpha} - 3\frac{\Gamma(5+\alpha/2)}{\Gamma(5-\alpha/2)} t^{4-\alpha/2} + \frac{9}{4}\Gamma(\alpha+1) + \left(\frac{3}{2} t^{\alpha/2} - t^4\right)^3 - y^{3/2}, \quad 0 < \alpha \leqslant 2, \tag{32}$$

$$y(0) = 0, \quad y'(0) = 0. \tag{33}$$

The second initial condition is for $\alpha > 1$ only.

Following Zhang and He [28], the nonlinear term $y^{3/2}$ in (32) is expanded using the Taylor series as follows:

$$y^{3/2} \approx 1 + \frac{3}{2}(y-1) + \frac{3}{8}(y-1)^2 = -\frac{1}{8} + \frac{3}{4}y + \frac{3}{8}y^2. \tag{34}$$

According to (19), we can construct the following homotopy:

$$D^{\alpha}y + p\left[-\frac{1}{8} + \frac{3}{4}y + \frac{3}{8}y^2\right] = \sum_{n=0}^{\infty} p^n g_n(t), \tag{35}$$

where we take $g_n(t)$ to be given by

$$g_0(t) = \frac{40320}{\Gamma(9-\alpha)}t^{8-\alpha} - 3\frac{\Gamma(5+\alpha/2)}{\Gamma(5-\alpha/2)}t^{4-\alpha/2} + \frac{9}{4}\Gamma(\alpha+1), \tag{36}$$

$$g_1(t) = (\frac{3}{2}t^{\alpha/2} - t^4)^3, \tag{37}$$

$$g_n(t) = 0, \quad n \geqslant 2. \tag{38}$$

Substituting (10) into (35) and equating the terms with the same power of p, we obtain

$$p^0 : D^{\alpha}y_0 = g_0(t), \tag{39}$$

$$p^1 : D^{\alpha}y_1 = g_1(t) - \left[-\frac{1}{8} + \frac{3}{4}y_0 - \frac{3}{8}y_0^2\right], \tag{40}$$

$$p^2 : D^{\alpha}y_2 = -\left[\frac{3}{4}y_1 - \frac{3}{8}(2y_0y_1)\right], \tag{41}$$

$$p^3 : D^{\alpha}y_3 = -\left[\frac{3}{4}y_2 - \frac{3}{8}(y_1^2 + 2y_0y_2)\right], \tag{42}$$

$$\vdots$$

In (34) we have taken the first three terms of the Taylor expansion series of the nonlinear term $y^{3/2}$ in order to show that the computation of $y_n, n \geqslant 2$, depends heavily on y_0 and y_1, but if we use the whole terms of the Taylor expansion series, i.e.

$$y^{3/2} = \omega(y) = \sum_{k=0}^{\infty} \omega^{(k)}(y)(y-1)^k, \tag{43}$$

then the first two linear equations can be given by

$$p^0 : D^{\alpha}y_0 = g_0(t),$$

$$p^1 : D^{\alpha}y_1 = g_1(t) - \sum_{k=0}^{\infty} \omega^{(k)}(y_0)(y_0-1)^k = g_1(t) - y_0^{3/2}.$$

Applying the operator J^α, which is the inverse operator of D^α, and using (4), we obtain

$$\begin{aligned} y_0 &= y(0) + y'(0)t + J^\alpha[g_0] \\ &= J^\alpha\left[\frac{40320}{\Gamma(9-\alpha)}t^{8-\alpha} - 3\frac{\Gamma(5+\alpha/2)}{\Gamma(5-\alpha/2)}t^{4-\alpha/2} + \frac{9}{4}\Gamma(\alpha+1)\right] \\ &= \left(\frac{3}{2}t^{\alpha/2} - t^4\right)^2, \\ y_1 &= J^\alpha[g_1] - J^\alpha[y_0^{3/2}] \\ &= J^\alpha\left[(\frac{3}{2}t^{\alpha/2} - t^4)^3\right] - J^\alpha\left[(\frac{3}{2}t^{\alpha/2} - t^4)^3\right] = 0. \end{aligned}$$

According to (39)–(42), it is clear that $y_m = 0$, $m \geqslant 2$. Hence, the exact solution [4],

$$y = \left(\frac{3}{2}t^{\alpha/2} - t^4\right)^2, \tag{44}$$

is reached.

4.3. Problem 3

Let us consider the following nonlinear fIVP:

$$D^\alpha y = y^2 + 1, \quad p-1 < \alpha \leqslant p, \quad p \in N, \quad 0 < t < 1, \tag{45}$$

$$y^{(k)}(0) = 0, \quad k = 0, \ldots, p-1. \tag{46}$$

The exact solution of this initial value problem for $\alpha = 1$, the ODE case, is $y = \tan t$.

Using Eq. (19) and setting $g_0(t) = 1$ and $g_n(t) = 0$, $n \geqslant 1$, we obtain the following homotopy:

$$D^\alpha y - p(y^2) = 1. \tag{47}$$

Substituting (10) into (47) and equating the terms with the identical powers of p, we obtain the following linear equations:

$$p^0 : D^\alpha y_0 = 1, \tag{48}$$

$$p^1 : D^\alpha y_1 = y_0^2, \tag{49}$$

$$p^2 : D^\alpha y_2 = 2y_0 y_1, \tag{50}$$

$$p^3 : D^\alpha y_3 = y_1^2 + 2y_0 y_2, \tag{51}$$

$$p^4 : D^\alpha y_4 = 2y_0 y_3 + 2y_1 y_2, \tag{52}$$

$$\vdots$$

Applying the operator J^α on both sides of (48)–(52) and using the initial conditions (46) we obtain $y_0, y_1, \ldots, y_4$. The 10-term approximate solution can be given by

$$\phi_{10} = \sum_{k=0}^{9} C_k t^{(2k+1)\alpha}, \tag{53}$$

where

$$C_0 = \frac{1}{\Gamma(\alpha+1)}, \quad C_1 = \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}C_0^2, \quad C_2 = \frac{\Gamma(4\alpha+1)}{\Gamma(5\alpha+1)}(2C_0C_1),$$

Table 1
Approximate solution of (45)–(46) for some values of α using ϕ_{10}

t	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1.0$	$\alpha = 1.5$	$\alpha = 2.5$	$\alpha = 3.5$
0.0	0.00000	0.00000	0.00000	0.00000	0.0000000	0.000000000
0.1	0.39198	0.19705	0.10033	0.02379	0.0009515	0.000027187
0.2	0.62411	0.34318	0.20271	0.06733	0.0053827	0.000307582
0.3	0.89704	0.48835	0.30934	0.12390	0.0148330	0.001271394
0.4	1.31180	0.64571	0.42279	0.19136	0.0304499	0.003479892
0.5	2.12064	0.82828	0.54630	0.26886	0.0531966	0.007598903
0.6	4.01314	1.05548	0.68414	0.35624	0.0839245	0.014384195
0.7	8.75972	1.36308	0.84229	0.45395	0.1234118	0.024671763
0.8	20.53092	1.82633	1.02964	0.56301	0.1723911	0.039370780
0.9	48.37395	2.61830	1.26014	0.68506	0.2315740	0.059458246
1.0	110.54022	4.15450	1.55724	0.82251	0.3016763	0.085974877

$$C_3 = \frac{\Gamma(6\alpha+1)}{\Gamma(7\alpha+1)}(2C_0C_2 + C_1^2), \quad C_4 = \frac{\Gamma(8\alpha+1)}{\Gamma(9\alpha+1)}(2C_0C_3 + 2C_1C_2),$$

$$C_5 = \frac{\Gamma(10\alpha+1)}{\Gamma(11\alpha+1)}(2C_0C_4 + 2C_1C_3 + C_2^2),$$

$$C_6 = \frac{\Gamma(12\alpha+1)}{\Gamma(13\alpha+1)}(2C_0C_5 + 2C_1C_4 + 2C_2C_3),$$

$$C_7 = \frac{\Gamma(14\alpha+1)}{\Gamma(15\alpha+1)}(2C_0C_6 + 2C_1C_5 + 2C_2C_4 + C_3^2),$$

$$C_8 = \frac{\Gamma(16\alpha+1)}{\Gamma(17\alpha+1)}(2C_0C_7 + 2C_1C_6 + 2C_2C_5 + 2C_3C_4),$$

$$C_9 = \frac{\Gamma(18\alpha+1)}{\Gamma(19\alpha+1)}(2C_0C_8 + 2C_1C_7 + 2C_2C_6 + 2C_3C_5 + C_4^2).$$

Table 1 shows the 10-term approximate solutions for (45)–(46) for different values of α.

4.4. Problem 4

Finally, we consider the nonlinear fIVP:

$$D^\alpha y = \frac{9}{4}\sqrt{y} + y, \quad 1 < \alpha \leqslant 2, \quad t \geqslant 0, \tag{54}$$

$$y(0) = 1, \quad y'(0) = 2. \tag{55}$$

The exact solution of the initial value problem (54)–(55) for $\alpha = 2$, i.e. the ODE case, is

$$y = \frac{9}{4}\left[\frac{3}{2}\exp(0.5t) + \frac{1}{6}\exp(-0.5t) - 1\right]^2. \tag{56}$$

If we expand the nonlinear term $\sqrt{y}$ in (54) using the Taylor series, we obtain

$$\sqrt{y} \approx 1 + \frac{1}{2}(y-1) - \frac{1}{8}(y-1)^2 + \frac{1}{16}(y-1)^3. \tag{57}$$

Hence, we can approximate (54) as follows:

$$D^\alpha y = \frac{45}{64} + \frac{199}{64}y - \frac{45}{64}y^2 + \frac{9}{64}y^3. \tag{58}$$

In view of (8), we can construct the following homotopy:

$$D^\alpha y - p\left(\frac{45}{64} + \frac{199}{64}y - \frac{45}{64}y^2 + \frac{9}{64}y^3\right) = 0. \tag{59}$$

Substituting (10) into (59) and equating the terms with the identical powers of p, yields the following linear equations:

$$p^0 : D^\alpha y_0 = 0, \tag{60}$$
$$p^1 : D^\alpha y_1 = \frac{45}{64} + \frac{199}{64}y_0 - \frac{45}{64}y_0^2 + \frac{9}{64}y_0^3, \tag{61}$$
$$p^2 : D^\alpha y_2 = \frac{199}{64}y_1 - \frac{45}{32}y_0 y_1 + \frac{27}{64}y_0^2 y_1, \tag{62}$$
$$p^3 : D^\alpha y_3 = \frac{199}{64}y_2 - \frac{45}{64}(2y_0 y_2 + y_1^2) + \frac{27}{64}(y_0 y_1^2 + y_0^2 y_2), \tag{63}$$
$$p^4 : D^\alpha y_4 = \frac{199}{64}y_3 - \frac{45}{32}(y_0 y_3 + y_1 y_2) + \frac{9}{64}(6y_0 y_1 y_2 + 3y_0^2 y_3 + y_1^3), \tag{64}$$
$$\vdots$$

Here we choose y_0 to be the simplest term of the initial conditions (55), i.e. the first condition, and the remaining initial conditions will be added to y_1, [24]. Hence, applying the fractional integration operator J^α, and according to (4), the first four terms of the approximate solution can be given by

$$\begin{aligned}
y_0 &= y(0) = 1,\\
y_1 &= y'(0)t + J^\alpha\left[\frac{45}{64} + \frac{199}{64}y_0 - \frac{45}{64}y_0^2 + \frac{9}{64}y_0^3\right] = 2t + C_1 t^\alpha,\\
y_2 &= J^\alpha\left[\frac{199}{64}y_1 - \frac{45}{32}y_0 y_1 + \frac{27}{64}y_0^2 y_1\right] = C_2 t^{\alpha+1} + C_3 t^{2\alpha},\\
y_3 &= J^\alpha\left[\frac{199}{64}y_2 - \frac{45}{64}(2y_0 y_2 + y_1^2) + \frac{27}{64}(y_0 y_1^2 + y_0^2 y_2)\right]\\
&= C_4 t^{\alpha+2} + C_5 t^{2\alpha+1} + C_6 t^{3\alpha},\\
y_4 &= J^\alpha\left[\frac{199}{64}y_3 - \frac{45}{32}(y_0 y_3 + y_1 y_2) + \frac{9}{64}(6y_0 y_1 y_2 + 3y_0^2 y_3 + y_1^3)\right]\\
&= C_7 t^{\alpha+3} + C_8 t^{2\alpha+2} + C_9 t^{3\alpha+1} + C_{10} t^{4\alpha},
\end{aligned}$$

where

$$\begin{aligned}
C_1 &= \frac{13}{4\Gamma(\alpha+1)}, \quad C_2 = \frac{17}{4\Gamma(\alpha+2)}, \quad C_3 = \frac{221}{32\Gamma(2\alpha+1)},\\
C_4 &= \frac{-9}{4\Gamma(\alpha+3)}, \quad C_5 = \frac{\Gamma(\alpha+2)}{8\Gamma(2\alpha+2)}(17C_2 - 9C_1),\\
C_6 &= \frac{\Gamma(2\alpha+1)}{32\Gamma(3\alpha+1)}(68C_3 - 9C_1^2), \quad C_7 = \frac{27}{4\Gamma(\alpha+4)},\\
C_8 &= \frac{\Gamma(\alpha+3)}{16\Gamma(2\alpha+3)}(34C_4 - 18C_2 + 27C_1),\\
C_9 &= \frac{\Gamma(2\alpha+2)}{32\Gamma(3\alpha+2)}(68C_5 - 36C_3 - 18C_1C_2 + 27C_1^2),\\
C_{10} &= \frac{\Gamma(3\alpha+1)}{64\Gamma(4\alpha+1)}(136C_6 - 36C_1C_3 + 9C_1^3).
\end{aligned}$$

In Table 2 we present the 5-term approximate solution of (54)–(55). We note that the approximate solution obtained by HPM is the same approximate solution obtained by ADM [24]. The accuracy of the HPM approximate solution is remarkably good in view of the crude approximation taken in (57).

Table 2
Approximate solution of (54)–(55) for some values of α using ϕ_5

t	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2.0$
0.0	1.00000000	1.00000000	1.00000000	1.00000000
0.1	1.37720809	1.28245453	1.23780724	1.21697814
0.2	1.86543117	1.65028248	1.53420555	1.47098992
0.3	2.45171393	2.09444288	1.88844768	1.76704394
0.4	3.14134795	2.61939160	2.30514239	2.11071067
0.5	3.94282826	3.23340806	2.79109495	2.50817057
0.6	4.86628633	3.94743003	3.35490799	2.96626225
0.7	5.92309716	4.77490977	4.00703217	3.49253054
0.8	7.12576345	5.73195406	4.75997836	4.09527467
0.9	8.48788677	6.83756778	5.62861104	4.78359634
1.0	10.0241699	8.11394079	6.63049227	5.56744792

5. Conclusions

In this work, the HPM was applied to derive exact and approximate analytical solutions of both linear and nonlinear fIVPs. The nonlinear terms involving radical powers were expanded by Taylor series. The reliability of HPM and the reduction in computations give HPM a wider applicability. It was also demonstrated that HPM is more efficient than the ADM.

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