*symmetry*

MDPI

*Article*

# Common Fixed Point Results of Set Valued Maps for $A_\varphi$-Contraction and Generalized $\phi$-Type Weak Contraction

**Murchana Neog [1], Mohammed M. M. Jaradat [2,*] and Pradip Debnath [3]**

[1] Department of Mathematics, North Eastern Regional Institute of Science and Technology, Nirjuli, Arunachal Pradesh 791109, India; murchananeog@gmail.com
[2] Department of Mathematics, Statistics and Physics, Qatar University, Doha P.O. Box 2713, Qatar
[3] Department of Applied Science and Humanities, Assam University, Silchar, Cachar, Assam 788011, India
* Correspondence: mmjst4@qu.edu.qa; Tel.: +974-5579-3995

Received: 9 June 2019; Accepted: 4 July 2019; Published: 8 July 2019

check for updates

**Abstract:** The solutions for many real life problems may be obtained by interpreting the given problem mathematically in the form $f(x) = x$. One such example is that of the famous Borsuk–Ulam theorem in which, using some fixed point argument, it can be guaranteed that at any given time we can find two diametrically opposite places in a planet with same temperature. Thus, the correlation of symmetry is inherent in the study of fixed point theory. In this article, some new results concerning coincidence and a common fixed point for an $A_\varphi$-contraction and a generalized $\phi$-type weak contraction are established. We prove our results for set valued maps without using continuity of the corresponding maps and completeness of the relevant space. Our results generalize and extend several existing results. Some new examples are given to demonstrate the generality and non-triviality of our results.

**Keywords:** common fixed point; $A_\varphi$-contraction; generalized $\phi$-type weak contraction; common property $(EA)$

## 1. Introduction

The Banach fixed point theorem is considered the most versatile work in fixed point theory. The study of similar results in nonlinear contraction maps was initiated by Boyd and Wong [1].

Fixed points for set valued mappings play a fundamental role in nonlinear analysis. Fixed points of multivalued operators are also important for studies in set valued analysis. First, results in this direction were given by Markin [2] and Nadler [3].

Nadler [3] defined the notion of set valued contraction and proved that a set valued contraction has a fixed point if the concerned metric space is complete. Afterwards, many generalizations of Nadler's result were obtained in various directions. In this context, the reference of set valued and multivalued contraction carried out by Assad and Kirk [4] can be estimated. They proved a result for set valued maps defined on a complete metric space by considering another assumption that the space is metrically convex.

There have been enormous developments in the area of existence and uniqueness of fixed point for multi valued and set valued mappings in various directions—see [5–16]. Some references that have been instrumental for the current work are [17–27].

Another important direction of extensions of the Banach Contraction Principle concerns the coincidence points and common fixed point of pair of maps that satisfy contractive type conditions. By introducing weakly commuting maps, Sessa [28] established some results in connection with common fixed points of non commuting generalized contraction maps. Another concept called weak commutativity was generalized by Jungck [29] with the help of compatible maps. Furthermore, by using weak compatibility, he weakened the notion of compatibility. However, Jachymski and his co-author(s) [30–32] have proved that not all generalizations in this respect were meaningful; some of the contractive conditions even turn out to be equivalent.

Aamri and El Moutawakil [33] introduced a property $(EA)$ in connection with self maps that include the class of non-compatible maps. The property $(EA)$ was extended to a hybrid pair of single and multivalued maps by Kamran [34]. After that, Li et al. [35] defined a property called common property $(EA)$ for a hybrid similar pair of maps. By introducing generalized $\phi$-weak contraction, Zhang and Song [36] proved some results that are related to common fixed points for two single valued maps. With the help of common property $(EA)$, in this paper, we newly establish some common fixed point results for $A_\varphi$-contraction and generalized $\phi$-type weak contraction maps. These contractions are new additions to the existing literature for set valued maps filling up the research gap between single valued and set valued contractions in the present context.

After the Introduction part, this paper is divided into four sections: (a) Preliminaries: here we recall the definitions and existing results that are essential for our work; (b) Main Results: in this section, we introduce $A_\varphi$-contraction and generalized $\phi$-type weak contraction for set valued maps and prove our new results; (c) Discussion: here, we discuss the results and how they can be interpreted from the perspective of previous studies and of the working hypotheses. Future research directions have also been highlighted (d) Conclusions: in this section, we discuss how the existing knowledge is enriched as a result of our current work.

## 2. Preliminaries

Below, we list some important definitions and results, which are necessary for our main results. The symbols $\mathbb{Q}$, $\mathbb{Q}'$ and $\mathbb{R}_+$ denote respectively the set of rational numbers, irrationals numbers and non-negative real numbers. In addition, throughout the paper, we use the notation $fx$ instead of $f(x)$ when no confusion arises.

**Definition 1.** *Ref. [37] Consider the set of functions $\Phi = \{\varphi | \varphi : [0, \infty) \to [0, \infty)\}$ that satisfies the assertions given below:*

1. *$u_1 \le u_2$ implies $\varphi(u_1) \le \varphi(u_2)$;*
2. *$\varphi^n(u) \to 0$ as $n \to \infty$ for each $u > 0$;*
3. *$\sum \varphi^n(u)$ converges for each $u > 0$.*

The function $\varphi$ is said to be a comparison function if $(1)$ and $(2)$ are satisfied. A strong comparison function is one for which $(3)$ also holds true.

**Remark 1.** *Ref. [37] Every strong comparison function is a comparison function.*

**Remark 2.** *Ref. [37] For a comparison function $\varphi : [0, \infty) \to [0, \infty)$, we get $\varphi(u_0) < u_0$, for each $u_0 > 0$, $\varphi(0) = 0$ and $\varphi$ is right continuous at $0$.*

**Definition 2.** *Ref. [38] Suppose $A$ is the collection of functions $\alpha : \mathbb{R}^3_+ \to \mathbb{R}_+$ which satisfies the conditions given below:*

1. *$\alpha$ is continuous on $\mathbb{R}^3_+$ (with respect to the usual metric).*
2. *if $u_0 \le \alpha(u_0, v_0, v_0)$ or $u_0 \le \alpha(v_0, u_0, v_0)$ or $u_0 \le \alpha(v_0, v_0, u_0)$ for all $u_0, v_0$, then $u_0 \le kv_0$ for some $k \in [0, 1)$.*

**Definition 3.** *Ref. [38] Suppose $R$ is a self map on a metric space $(X, d)$. The map $R$ is called an $A$-contraction if*

$$d(Ra_0, Rb_0) \le \alpha(d(a_0, b_0), d(a_0, Ra_0), d(b_0, Rb_0))$$

*for each $a_0, b_0 \in X$ and some $\alpha \in A$.*

The definition of generalized $\phi$-weak contraction that is given below was introduced by Zhang and Song [36].

**Definition 4.** *Two self maps $S, R$ on $(X, d)$ are called generalized $\varphi$-weak contractions if there exists a map $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$ and $\varphi(u) > 0$ for each $u > 0$ satisfying*

$$d(Re_0, Sf_0) \leq N(e_0, f_0) - \varphi(N(e_0, f_0)); \text{ for each } e_0, f_0 \in X,$$

*where*

$$N(e_0, f_0) = \max\{d(e_0, f_0), d(e_0, Re_0), d(f_0, Sf_0), \frac{1}{2}(d(e_0, Sf_0) + d(f_0, Re_0))\}.$$

The following theorem was also established for two single valued generalized $\phi$-weak contractions.

**Theorem 1.** *Ref. [36] Consider the two self maps $S, R$ on $(X, d)$ satisfying*

$$d(Ra_0, Sb_0) \leq N(a_0, b_0) - \phi(N(a_0, b_0)), \text{ for each } a_0, b_0 \in X,$$

*where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a lower semi continuous function with $\phi(0) = 0$ and $\phi(u) > 0$ for all $u > 0$. Then, $R$ and $S$ have a unique common fixed point.*

By $WB(X)$, we denote the set of all nonempty closed and bounded subsets of $X$. We define

$$H(E_0, F_0) = \max\{\sup_{v_0 \in F_0} D(v_0, E_0), \sup_{u_0 \in E_0} D(u_0, F_0)\}, E_0, F_0 \in WB(X),$$

where $D(u_0, F_0) = \inf_{v \in F_0} d(u_0, v)$. Then, $(WB(X), H)$ forms a metric space and $H$ is called Hausdorff metric generated by $d$.

**Definition 5.** *Consider the self map $r$ on $X$ and $R : X \to WB(X)$.*

*1. $z_0 \in X$ is called a fixed point of $r$ if $z_0 = rz_0$ and a fixed point of $R$ if $z_0 \in Rz_0$.*

*2. If $rz_0 \in Rz_0$, then $z_0 \in X$ is called a coincidence point of $r$ and $R$. Let $C(r, R)$ denote the collection of all coincidence points of $r$ and $R$.*

*3. If $z_0 = rz_0 \in Rz_0$, then $z_0 \in X$ is called a common fixed point of $r$ and $R$.*

**Definition 6.** *Ref. [33] Consider the maps $r, s : X \to X$. If there exists a sequence $\{x_k\}$ in $X$ such that $\lim_{k \to \infty} rx_k = \lim_{k \to \infty} sx_k = l \in X$, then the maps are said to have the property $(EA)$.*

**Definition 7.** *Ref. [34] A self map $r$ on $X$ and a map $R : X \to WB(X)$ have the property $(EA)$ if there exists a sequence $\{x_k\}$ in $X$ such that $\lim_{k \to \infty} rx_k = l \in P = \lim_{k \to \infty} Rx_k$ for $l \in X$ and $P \in WB(X)$.*

**Definition 8.** *Ref. [34] Consider the set-valued map $R : X \to WB(X)$. A self map $r$ on $X$ is said to be $R$-weakly commuting at $l \in X$ if $rr(l) \in Rr(l)$.*

## 3. Main Results

In this section, we first introduce some definitions that are essential for establishing our main results.

**Definition 9.** *Let $A_\varphi$ be the collection of all functions $\alpha : \mathbb{R}^3_+ \to \mathbb{R}_+$ satisfying the assertions given below:*

1. *The function $\alpha$ is continuous on $\mathbb{R}^3_+$ (with respect to the usual metric).*
2. *There exists a strong comparison function $\varphi$ such that, for each $u_0, v_0 \in \mathbb{R}_+$ for which $u_0 \leq \alpha(u_0, v_0, v_0)$ or $u_0 \leq \alpha(v_0, u_0, v_0)$ or $u_0 \leq \alpha(v_0, v_0, u_0)$, the inequality $u_0 \leq \varphi(v_0)$ holds.*

**Remark 3.** *By specializing $\varphi(u) = ku$ as $0 < k < 1$ for all $u > 0$, we get the Definition 2.*

**Definition 10.** *(1) Consider four self maps $r, s, R, S$ on the metric space $(X, d)$. The pair of mappings $(r, R)$ and $(s, S)$ have the common property $(EA)$ if two sequences $\{x_k\}$ and $\{y_k\}$ exist in $X$ such that*

$$\lim_{k\to\infty} Sy_k = \lim_{k\to\infty} Rx_k = \lim_{k\to\infty} rx_k = \lim_{k\to\infty} sy_k = l \in X.$$

*(2) Consider the self maps $r, s$ on $X$ and $R, S : X \to WB(X)$. The maps pair $(r, R)$ and $(s, S)$ have the common property $(EA)$ if two sequences $\{x_k\}$ and $\{y_k\}$ exist in $X$ such that*

$$\lim_{k\to\infty} Rx_k = P, \lim_{k\to\infty} Sy_k = Q, \lim_{k\to\infty} rx_k = \lim_{k\to\infty} sy_k = d \in P \cap Q, \text{ for } d \in X \text{ and } P, Q \in WB(X).$$

**Example 1.** *Suppose $X = \mathbb{R}$ is endowed with the usual metric. The self maps $r, s$ on $X$ and $R, S : X \to WB(X)$ are defined by $r(x) = \frac{x}{5}$, $s(x) = 2 - \frac{x}{4}$ and $Rx = [0, \frac{x}{5}]$, $Sx = [1, 3 + \frac{x}{2}]$ for each $x \in X$. Consider the sequences $\{x_k\}$ and $\{y_k\}$, where $x_k = 5 + \frac{1}{k}$ and $y_k = 4 + \frac{1}{k}$. Clearly, $\lim_{k\to\infty} Rx_k = [0, 1] = P$, $\lim_{k\to\infty} Sy_k = [1, 5] = Q$, $\lim_{k\to\infty} rx_k = \lim_{k\to\infty} sy_k = 1 \in P \cap Q$. Therefore, $(r, R)$ and $(s, S)$ have the common property $(EA)$.*

**Theorem 2.** *Let $r, s$ be two self maps on a metric space $(X, d)$ and $R, S : X \to WB(X)$ be two set valued maps such that*

*A. $(r, R)$ and $(s, S)$ have the common property $(EA)$;*

*B. There exists some $\alpha \in A_\varphi$ such that, for each $x_0, y_0 \in X$,*

$$H(Rx_0, Sy_0) \leq \alpha(d(rx_0, sy_0), D(rx_0, Rx_0), D(sy_0, Sy_0)). \tag{1}$$

*Let $rX$ and $sX$ be closed subsets of $X$. Then,*

*(a) $r$ and $R$ possess a coincidence point;*

*(b) $s$ and $S$ possess a coincidence point;*

*(c) $r$ and $R$ possess a common fixed point if, for any $l \in C(r, R)$, $r$ is $R$-weakly commuting at $l$ and $rr(l) = r(l)$;*

*(d) $s$ and $S$ possess a common fixed point if, for any $l \in C(s, S)$, $s$ is $S$-weakly commuting at $l$ and $ss(l) = s(l)$;*

*(e) $r, s, R$ and $S$ possess a common fixed point when $(c)$ and $(d)$ hold.*

**Proof.** $(a), (b)$ By virtue of $(A)$, $(r, R)$ and $(s, S)$ have the common property $(EA)$. Thus, two sequences $\{x_k\}$, $\{y_k\}$ exist in $X$ such that

$$\lim_{k\to\infty} Rx_k = P, \lim_{k\to\infty} Sy_k = Q, \text{ for } P, Q \in WB(X),$$

$$\lim_{k\to\infty} rx_k = \lim_{k\to\infty} sy_k = c \in P \cap Q, \text{ for } c \in X.$$

Since $rX$ and $sX$ are closed subsets of $X$, we have $\lim_{k\to\infty} rx_k = rl$ and $\lim_{k\to\infty} sy_k = se$, for some $l, e \in X$. Thus, we have $rl = c$ and $se = c$. We claim that $rl \in Rl$ and $se \in Se$.

Now, from (1), we have

$$H(Rx_k, Se) \leq \alpha(d(rx_k, se), D(rx_k, Rx_k), D(se, Se)).$$

Taking $k \to \infty$, we get

$$H(P, Se) \leq \alpha(0, 0, D(se, Se)).$$

Because $se = c$ belongs to $P$, from the definition of Hausdorff metric, we get

$$\begin{aligned} D(se, Se) &\leq H(P, Se) \\ &\leq \alpha(0, 0, D(se, Se)) \\ \text{i.e., } D(se, Se) &\leq \alpha(0, 0, D(se, Se)). \end{aligned}$$

By (2) of Definition 9, we get

$$D(se, Se) \leq \varphi(0) = 0.$$

Since $Se$ is closed, this implies that $se \in Se$. Again by Equation (1), we have

$$H(Rl, Sy_k) \leq \alpha(d(rl, sy_k), D(rl, Rl), D(sy_k, Sy_k)).$$

Invoking the same procedure as above, we obtain that $rl \in Rl$. Hence, $r$ and $R$ share a coincidence point $l$. In addition, $s$ and $S$ contain a coincidence point $e$.

$(c)$ Since $l \in C(r, R)$, from the condition in $(c)$, we have $rr(l) = r(l)$. This gives $rc = c$. Again, $r$ is $R$—weakly commuting at $l$—thus $rr(l) \in Rr(l)$. This implies $rc \in Rc$. i.e., $c = rc \in Rc$. Hence, $c$ is a common fixed point of $r$ and $R$. Using a similar argument, we can prove $(d)$. In addition, $(c)$ and $(d)$ together give $(e)$. □

If $r = s$ in the above theorem, it gives the following corollary.

**Corollary 1.** *Suppose $r$ is a self map on $(X, d)$ and $R; S : X \to WB(X)$ are two set valued maps such that*

*A. $(r, R)$ and $(r, S)$ have the common property $(EA)$;*

*B. There exist some $\alpha \in A_\varphi$ such that, for each $x_0, y_0 \in X$*

$$H(Rx_0, Sy_0) \leq \alpha(d(rx_0, ry_0), D(rx_0, Rx_0), D(ry_0, Sy_0)).$$

*If $rX$ is a closed subset of $X$, then*

*(a) $r, R$ and $S$ possess a coincidence point;*

*(b) $r, R$ and $S$ possess a common fixed point provided that $r$ is both $R$ weakly commuting and $S$ weakly commuting at $l$ and $rr(l) = r(l)$ for $l \in C(r, R)$.*

**Corollary 2.** *Suppose $r, s, R$ and $S$ are four self maps on $(X, d)$ such that*

*A. $(r, R)$ and $(s, S)$ have the common property $(EA)$;*

*B. There exist some $\alpha \in A_\varphi$ such that, for each $x_0, y_0 \in X$*

$$d(Rx_0, Sy_0) \leq \alpha(d(rx_0, sy_0), d(rx_0, Rx_0), d(sy_0, Sy_0)).$$

*If $rX$ and $sX$ are closed subsets of $X$, then*

*(a) $r$ and $R$ possess a coincidence point;*

*(b) $s$ and $S$ possess a coincidence point;*

*(c) $r$ and $R$ possess a common fixed point if $r$ is a $R$-weakly commuting map at $l$ and $rr(l) = r(l)$ for $l \in C(r, R)$;*

*(d) $s$ and $S$ possess a common fixed point when $s$ is a $S$-weakly commuting map at $l$ and $ss(l) = s(l)$ for $l \in C(s, S)$;*

*(e) $r, s, R$ and $S$ possess a common fixed point when both $(c)$ and $(d)$ hold.*

Dropping the assumption that "both pairs $(r, R)$ and $(s, S)$ are weakly commuting" in the above result, we prove the next result.

**Theorem 3.** *Suppose that $r, s$ are two self maps on $(X, d)$ and $R, S : X \to WB(X)$ are two set valued maps such that*

1. *$(r, R)$ and $(s, S)$ have the common property $(EA)$;*
2. *there exist some $\alpha \in A_\varphi$ such that, for each $x_0, y_0 \in X$,*

$$H(Rx_0, Sy_0) \leq \alpha(d(rx_0, sy_0), D(rx_0, Rx_0), D(sy_0, Sy_0)),$$

3. *$rr(l_0) = r(l_0)$ for $l_0 \in C(r, R)$ and $ss(l_0) = s(l_0)$ for $l_0 \in C(s, S)$.*

*If $rX$ and $sX$ are closed subsets of $X$, then $r, s, R$ and $S$ possess a common fixed point.*

**Proof.** By virtue of (1), $(r, R)$ and $(s, S)$ fulfill the common property $(EA)$. Thus, two sequences $\{x_k\}$, $\{y_k\}$ exist in $X$ such that

$$\lim_{k\to\infty} Rx_k = P, \lim_{k\to\infty} Sy_k = Q, \text{ for } P, Q \in WB(X)$$

$$\lim_{k\to\infty} rx_k = \lim_{k\to\infty} sy_k = c_0 \in P \cap Q, \text{ for } c_0 \in X.$$

Since $rX$ and $sX$ are closed subsets of $X$, we have $\lim_{k\to\infty} rx_k = rl_0$ and $\lim_{k\to\infty} sy_k = se$, for some $l_0, e \in X$. Thus, we have $rl_0 = c_0$ and $se = c_0$. Using the similar process as in Theorem 2 and from assumption (2), we have $rl_0 \in Rl_0$ and $se \in Se$.

As $l_0 \in C(r, R)$, thus, from (3), we have $rr(l_0) = r(l_0)$. Thus, $c_0 = r(l_0) = rr(l_0) = rc_0$. In addition, $rr(l_0) \in Rl_0$. This implies $rc_0 \in Rl_0$. Now, using the definition of the Pompeiu–Hausdorff metric, we get

$$\begin{aligned} D(rc_0, Rc_0) &\leq H(Rl_0, Rc_0) \\ &\leq \alpha(d(rl_0, rc_0), D(rl_0, Rl_0) D(rc_0, Rc_0)) \\ &= \alpha(d(c_0, c_0), D(c_0, Rl_0), D(rc_0, Rc_0)) \\ &= \alpha(0, 0, D(rc_0, Rc_0)), \end{aligned}$$

which implies by $(ii)$ of Definition 9 that

$$D(rc_0, Rc_0) \leq \varphi(0) = 0.$$

Since $Rc_0$ is closed, this implies that $rc_0 \in Rc_0$. Hence, $c_0 = rc_0 \in Rc_0$.

Similarly, from $ss(e) = s(e)$, for $e \in C(s, S)$, we obtain $c_0 = sc_0 \in Sc_0$. Hence, $c_0$ is a common fixed point of $r, s, R$ and $S$. □

**Corollary 3.** *Suppose $r$ is a self map on $(X, d)$ and $R, S : X \to WB(X)$ are two set valued maps such that*

1. *$(r, R)$ and $(s, S)$ have the common property $(EA)$;*
2. *there exist some $\alpha \in A_\varphi$ such that for each $x_0, y_0 \in X$*

$$H(Rx_0, Sy_0) \leq \alpha(d(rx_0, ry_0), D(rx_0, Rx_0), D(ry_0, Sy_0)),$$

3. *$rr(l_0) = r(l_0)$ for $l_0 \in C(r, R)$.*

*If $rX$ is a closed subset of $X$, then $r, s, R$ and $S$ possess a common fixed point.*

**Corollary 4.** *Suppose $r, s, R$ and $S$ are four self maps on $(X, d)$ such that*

1. *$(r, R)$ and $(s, S)$ have the common property $(EA)$;*

2. *there exist some* $\alpha \in A_{\varphi}$ *such that for each* $x_0, y_0 \in X$

$$d(Rx_0, Sy_0) \leq \alpha(d(rx_0, sy_0), d(rx_0, Rx_0), d(sy_0, Sy_0)),$$

3. $rr(l_0) = r(l_0)$ *for* $l_0 \in C(r, R)$ *and* $ss(l_0) = s(l_0)$ *for* $l_0 \in C(s, S)$.

*If* $rX$ *and* $sX$ *are closed subsets of* $X$*, then* $r, s, R$ *and* $S$ *possess a common fixed point.*

Below, we give an example to show that the common property $(EA)$ is necessary for the existence of a common fixed point.

**Example 2.** *Suppose* $X = [0, 1]$*. Define the self maps* $r, s$ *on* $X$ *and* $R, S : X \to WB(X)$ *as*

$$rx = \frac{x+1}{6}\,, Rx = [0, \frac{x}{11}],$$

$$sx = \frac{1}{2} \text{ and } Sx = [0, \frac{x}{12}].$$

*In addition, define the metric* $d$ *as*

$$d(x_0, y_0) = \begin{cases} 0, & \text{if } x_0 = y_0, \\ \max\{x_0^2, y_0^2\}, & \text{if } x_0, y_0 \in [0, 1/3], \\ \max\{x_0, y_0\}, & \text{otherwise.} \end{cases}$$

*In addition, so the Hausdorff metric is defined as*

$$H(P, Q) = \begin{cases} \max_{x_0 \in P, y_0 \in Q}\{x_0^2, y_0^2\}, & \text{if } P, Q \subseteq [0, \frac{1}{3}], \\ \max_{x_0 \in P, y_0 \in Q}\{x_0, y_0\}, & \text{if } P \text{ or } Q \text{ (or both) } \not\subseteq [0, \frac{1}{3}], \\ 0, & \text{if } P = Q. \end{cases}$$

*One can see that, for any sequences* $x_k, y_k \in X$*, if* $\lim_{k\to\infty} rx_k = \lim_{k\to\infty} sy_k = \lim_{k\to\infty} \frac{1}{2} = \frac{1}{2}$*, then* $\lim_{k\to\infty} d(rx_k, \frac{1}{2}) = 0$*. However,* $d(rx_k, \frac{1}{2}) = \max\{rx_k, \frac{1}{2}\} = \frac{1}{2}$*, which implies that* $\lim_{k\to\infty} d(rx_k, \frac{1}{2}) = \frac{1}{2}$*. Thus,* $(r, R)$ *and* $(s, S)$ *do not have the common property (EA).*

*Now, for each* $x_0, y_0 \in [0, 1]$*, we have*

$$H(Rx_0, Sy_0) \leq \max\{\frac{x_0}{11}, \frac{y_0}{12}\}\,, d(rx_0, sy_0) = \frac{1}{2},$$

$$D(rx_0, Rx_0) = \frac{1+x_0}{6} \text{ and } D(sy_0, Sy_0) = \frac{1}{2},$$

*which yields that*

$$\max\{\frac{x_0}{11}, \frac{y_0}{12}\} \leq \frac{1+x_0}{6} \leq \alpha(\frac{1}{2}, \frac{1+x_0}{6}, \frac{1}{2}),$$

*where* $\alpha(x_0, y_0, z_0) = \max\{\frac{2x_0}{3}, \frac{2y_0}{3}, \frac{2z_0}{3}\}$ *and* $\varphi(t) = \frac{2t}{3}$*. Thus,*

$$H(Rx_0, Sy_0) \leq \alpha(d(rx_0, sy_0), D(rx_0, Rx_0), D(sy_0, Sy_0)).$$

*Hence, condition* $(2)$ *of Theorem 3 is satisfied. However, the mappings* $r, s, R$ *and* $S$ *do not have a common fixed point because the pairs of mappings* $(r, R)$ *and* $(s, S)$ *do not have the common property* $(EA)$*.*

**Theorem 4.** *Suppose* $r, s$ *are two self maps on* $(X, d)$ *and* $R, S : X \to WB(X)$ *are two set valued maps such that*

1. *$(r,R)$ and $(s,S)$ have the common property $(EA)$;*
2. *for each $x_0, y_0 \in X$ with $x_0 \neq y_0$,*

$$H(Rx_0, Sy_0) \leq M(x_0, y_0) - \phi(M(x_0, y_0)), \tag{2}$$

*where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function with $0 < \phi(u) < u$, for each $u \in (0, \infty)$ and $\varphi(0) = 0$, and $M(x_0, y_0) = \max\{d(rx_0, sy_0), D(rx_0, Rx_0), D(sy_0, Sy_0), \frac{1}{2}(D(rx_0, Sy_0) + D(sy_0, Rx_0))\}$.*

3. *$rr(l_0) = r(l_0)$ for $l_0 \in C(r,R)$ and $ss(l_0) = s(l_0)$ for $l_0 \in C(s,S)$.*

*If $rX$ and $sX$ are closed subsets of $X$, then $r, s, R$ and $S$ possess a common fixed point.*

**Proof.** By virtue of $(1)$, $(r,R)$ and $(s,S)$ have the common property $(EA)$. Thus, two sequences $\{x_k\}$, $\{y_k\}$ exist in $X$ such that for $c_0 \in X, P_0, Q_0 \in WB(X)$

$$\lim_{k\to\infty} Rx_k = P_0\ ,\ \lim_{k\to\infty} Sy_k = Q_0,$$

$$\lim_{k\to\infty} rx_k = \lim_{k\to\infty} sy_k = c_0 \in P_0 \cap Q_0.$$

Since $rX$ and $sX$ are closed subsets of $X$, we have $\lim_{k\to\infty} rx_k = rl_0$ and $\lim_{k\to\infty} sy_k = se_0$, for some $l_0, e_0 \in X$. Thus, we have $rl_0 = c_0$ and $se_0 = c_0$. Now, we claim that $rl_0 \in Rl_0$ and $se_0 \in Se_0$.

From inequality (2), we have

$$\begin{aligned} H(Rx_k, Se_0) &\leq M(x_k, e_0) - \phi(M(x_k, e_0)) \\ &= \max\{d(rx_k, se_0), D(rx_k, Rx_k), D(se_0, Se_0), \frac{1}{2}(D(x_k, Se_0) + D(se_0, Rx_k))\} \\ &- \phi[\max\{d(rx_k, se_0), D(rx_k, Rx_k), D(se_0, Se_0), \frac{1}{2}(D(rx_k, Se_0) + D(se_0, Rx_k))\}]. \end{aligned}$$

Taking the upper limits as $k \to \infty$, we get

$$\begin{aligned} H(P_0, Se_0) &\leq \max\{d(c_0, se_0), D(c_0, P_0), D(se_0, Se_0), \frac{1}{2}(D(c_0, Se_0) + D(se_0, P_0))\} \\ &- \phi[\max\{d(c_0, se_0), D(c_0, P_0), D(se_0, Se_0), \frac{1}{2}(D(c_0, Se_0) + D(se_0, P_0))\}]. \end{aligned} \tag{3}$$

Since $se = c \in P$, inequality (3) implies

$$H(P_0, Se_0) \leq D(se_0, Se_0) - \phi(D(se_0, Se_0)).$$

Now, using the definition of Hausdorff metric, we get

$$\begin{aligned} D(se_0, Se_0) &\leq H(P_0, Se_0) \\ &\leq D(se_0, Se_0) - \phi(D(se_0, Se_0)), \end{aligned}$$

which implies that $\phi(D(se_0, Se_0)) = 0$, by the property of the function $\phi$ and obtain $D(se_0, Se_0) = 0$. Since $Se_0$ is closed, this implies that $se_0 \in Se_0$, hence $e_0 \in C(s,S)$.

Again by $(2)$, we have

$$H(Rl_0, Sy_k) \leq M(l_0, y_k) - \phi(M(l_0, y_k)).$$

Employing a similar procedure as above, we obtain $rl_0 \in Rl_0$. As $l_0 \in C(r,R)$, so from assumption $(3)$, we have $rr(l_0) = rl_0$. Thus, $c_0 = rl_0 = rr(l_0) = rc_0$. In addition, $rr(l_0) \in Rl_0$. This implies $rc_0 \in Rl_0$.

Now, by the definition of Hausdorff metric, we get

$$\begin{aligned} D(rc_0, Rc_0) &\leq H(Rl_0, Rc_0) \\ &\leq M(l_0, c_0) - \phi(M(l_0, c_0)), \end{aligned} \tag{4}$$

where

$$\begin{aligned} M(l_0, c_0) &= \max\left\{d(rl_0, rc_0), D(rl_0, Rl_0), D(rc_0, Rc_0), \frac{1}{2}\Big(D(rl_0, Rc_0) + D(rc_0, Rl_0)\Big)\right\} \\ &= D(rc_0, Rc_0). \end{aligned}$$

Thus, inequality (4) implies

$$D(rc_0, Rc_0) \leq D(rc_0, Rc_0) - \phi(D(rc_0, Rc_0)),$$

which implies $\phi(D(rc_0, Rc_0)) \leq 0$. This gives $\phi(D(rc_0, Rc_0)) = 0$; by the property of the function $\phi$, we get $D(rc_0, Rc_0) = 0$. Hence, $rc_0 \in Rc_0$. Thus, $c_0 = rc_0 \in Rc_0$.

Similarly, from $ss(e_0) = s(e_0)$, for $e_0 \in C(s, S)$, we obtain $c_0 = sc_0 \in Sc_0$. Hence, $c_0$ is a common fixed point of $r, s, R$ and $S$. □

**Corollary 5.** *Consider a self map $r$ on $(X, d)$ and two set valued maps $R, S : X \to WB(X)$ such that*

1. *$(r, R)$ and $(s, S)$ have the common property $(EA)$;*
2. *for each $a_0, b_0 \in X$ with $a_0 \neq b_0$,*

$$H(Ra_0, Sb_0) \leq J(a_0, b_0) - \phi(J(a_0, b_0)),$$

*where $\phi : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $0 < \phi(u) < u$, for all $u \in (0, \infty)$ and $\phi(0) = 0$,*

$$J(a_0, b_0) = \max\left\{d(ra_0, rb_0), D(ra_0, Ra_0), D(rb_0, Sb_0), \frac{1}{2}\Big(D(ra_0, Sb_0) + D(rb_0, Ra_0)\Big)\right\}.$$

3. *$rr(l_0) = r(l_0)$ for $l_0 \in C(r, R)$.*

*Then, $r, R$ and $S$ possess a common fixed point provided $rX$ a closed subsets of $X$.*

An example is given below to demonstrate our results.

**Example 3.** *Suppose $X = [0, 1]$ is endowed with the usual metric. Define the self maps $r, s$ on $X$ and $R, S : X \to WB(X)$ as*

$$rx = \begin{cases} x^2 + \frac{x}{2}, & x \in [0, \frac{1}{4}] \\ \frac{2x+1}{4}, & x \in (\frac{1}{4}, 1], \end{cases}$$

$$sx = \frac{[x] + 1}{2}, 0 \leq x \leq 1,$$

$$Rx = \begin{cases} [0, 2x], & x \in [0, \frac{1}{4}] \\ [0, \frac{2x+3}{8}], & x \in (\frac{1}{4}, 1], \end{cases}$$

*and*

$$Sx = \begin{cases} [0, \frac{[x]+1}{4}], & x \in [0, \frac{1}{4}] \\ [0, \frac{1}{2}], & x \in (\frac{1}{4}, 1]. \end{cases}$$

*$(r, R)$ and $(s, S)$ have the common property $(EA)$ for the sequences $x_k = \frac{k+2}{2k}$, $y_k = \frac{1}{k}, k = 1, 2, 3 \ldots$*

*For each* $x, y \in [0, \frac{1}{4}]$ *with* $x \neq y$,

$$H(Rx, Sy) = \frac{1}{2}\left|4x - \frac{[y]+1}{2}\right|,$$

$d(rx, sy) = \left|x^2 + \frac{x}{2} - \frac{[y]+1}{2}\right|$ , $D(rx, Rx) = \left|x^2 - \frac{3x}{2}\right|$ , $D(sy, Sy) = \left|\frac{[y]+1}{4}\right|$,

$$\text{and } \frac{1}{2}\Big[D(rx, Sy) + D(sy, Rx)\Big] = \frac{1}{2}\left[\left|x^2 + \frac{x}{2} - \frac{[y]+1}{4}\right| + \left|\frac{[y]+1}{2} - 2x\right|\right].$$

*For each* $x, y \notin [0, \frac{1}{4}]$ *with* $x \neq y$

$$H(Rx, S(y)) = \frac{1}{2}\left|\frac{x}{2} - \frac{1}{4}\right|,$$

$d(rx, sy) = \left|\frac{2x+1}{4} - \frac{[y]+1}{2}\right|$ , $D(rx, Rx) = \frac{1}{2}\left|\frac{x}{2} - \frac{1}{4}\right|$ ,$D(sy, Sy) = \left|\frac{[y]+1}{2} - \frac{1}{2}\right| = \frac{[y]}{2}$,

$$\text{and } \frac{1}{2}\Big[D(rx, Sy) + D(s(y), Rx)\Big] = \frac{1}{2}\left[\left|\frac{x}{2} - \frac{1}{4}\right| + \left|\frac{[y]+1}{2} - \frac{2x+3}{8}\right|\right].$$

*For each* $x \in [0, \frac{1}{4}]$, $y \notin [0, \frac{1}{4}]$

$$H(Rx, Sy) = \frac{1}{2}\left|4x - 1\right|,$$

$d(rx, sy) = \left|x^2 + \frac{x}{2} - \frac{[y]+1}{2}\right|$ , $D(rx, Rx) = \left|x^2 - \frac{3x}{2}\right|$ , $D(sy, Sy) = \left|\frac{[y]+1}{2} - \frac{1}{2}\right| = \frac{[y]}{2}$,

$$\text{and } \frac{1}{2}\Big[D(rx, Sy) + D(sy, Rx)\Big] = \frac{1}{2}\left[\left|x^2 + \frac{x}{2} - \frac{1}{2}\right| + \left|\frac{[y]+1}{2} - 2x\right|\right].$$

*For each* $x \notin [0, \frac{1}{4}]$, $y \in [0, \frac{1}{4}]$,

$$H(R(x, Sy)) = \frac{1}{2}\left|\frac{2x+3}{4} - \frac{[y]+1}{2}\right|,$$

$d(rx, sy) = \left|\frac{2x+1}{4} - \frac{[y]+1}{2}\right|$ , $D(rx, Rx) = \frac{1}{2}\left|\frac{2x-1}{4}\right|$ , $D(sy, Sy) = \left|\frac{[y]+1}{2} - \frac{[y]+1}{4}\right| = \frac{[y]+1}{4}$,

$$\text{and } \frac{1}{2}\Big[D(rx, Sy) + D(sy, Rx)\Big] = \frac{1}{2}\left[\left|\frac{2x+1}{4} - \frac{[y]+1}{4}\right| + \left|\frac{[y]+1}{2} - \frac{2x+3}{8}\right|\right].$$

*In all the cases, assumption* (2) *holds for the function* $\phi(t) = \frac{t}{7}$. *In addition,* $rr(\frac{1}{2}) = r(\frac{1}{2})$ *for* $\frac{1}{2} \in C(r, R)$ *and* $ss(\frac{1}{2}) = s(\frac{1}{2})$ *for* $\frac{1}{2} \in C(s, S)$. *Hence, all conditions of Theorem 4 are satisfied. Thus, we conclude that* $r, s, R$ *and* $S$ *possess a common fixed point* $\frac{1}{2}$.

## 4. Discussion

Common fixed point results for single valued maps have been mainly used to solve nonlinear integral equations. However, in recent times, the study of fixed point for set valued maps have gone beyond mere generalization of the single valued case. Such studies have also been applied to prove the existence of equilibria in the context of game theory. Similar generalizations of such contractions for the mappings of the type $R : WB(X) \to WB(X)$ would be a special topic for future study. Another direction of future work would be to apply our results in the solution of set valued fractional differential equations.

## 5. Conclusions

We proved some interesting results dealing with common fixed point for $A_\varphi$-contraction and generalized $\phi$-type weak contraction without using the continuity of any map. Our results are unified and extended forms of some existing results in literature. The proofs also give us schemes regarding how to find the desired common fixed point of such maps.

**Author Contributions:** Conceptualization, M.N., M.M.M.J. and P.D.; methodology, M.N., M.M.M.J. and P.D.; investigation, M.N., M.M.M.J. and P.D.; writing—original draft preparation, M.N. and P.D.; writing—review and editing, M.M.M.J.

**Funding:** This research received no external funding.

**Acknowledgments:** The authors express their heartfelt gratitude to the anonymous referees and the editors for their extremely valuable comments, suggestions and ideas which have improved this paper a great deal.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References

1. Boyd, D.W.; Wong, J.S. On nonlinear contractions. *Proc. Am. Math. Soc.* **1969**, *20*, 458–464. [CrossRef]
2. Markin, J.T. A fixed point theorem for set- valued mappings. *Bull. Am. Math. Soc.* **1968**, *74*, 639–640. [CrossRef]
3. Nadler, S.B. Multi-valued contraction mappings. *Pac. J. Math.* **1969**, *30*, 475–488. [CrossRef]
4. Assad, N.A.; Kirk, W.A. Fixed point theorems for set-valued mappings of contractive type. *Pac. J. Math.* **1972**, *43*, 553–561. [CrossRef]
5. Berinde, M.; Berinde, V. On a general class of multi-valued weakly Picard mappings. *J. Math. Anal. Appl.* **2007**, *326*, 772–782. [CrossRef]
6. Chaipunya, P.; Mongkolkeha, C.; Sintunavarat, W.; Kumam, P. Fixed point theorems for multivalued mappings in modular metric spaces. *Abstr. Appl. Anal.* **2012**, *2012*, 14. [CrossRef]
7. Ćirić, L. Fixed point theorems for multivalued contractions in complete metric spaces. *J. Math. Anal. Appl.* **2008**, *348*, 499–507. [CrossRef]
8. Daffer, P.Z.; Kaneko, H. Fixed points of generalized contractive multi-valued mappings. *J. Math. Anal. Appl.* **1995**, *192*, 655–666. [CrossRef]
9. Dag, H.; Minak, G.; Altun, I. Some fixed point results for multivalued F-contractions on quasi metric spaces. *RACSAM* **2017**, *111*, 177. [CrossRef]
10. Debnath, P.; Choudhury, B.S.; Neog, M. Fixed set of set valued mappings with set valued domain in terms of start set on a metric space with a graph. *Fixed Point Theory Appl.* **2017**, *5*. [CrossRef]
11. Feng, Y.; Liu, S. Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings. *J. Math. Anal. Appl.* **2006**, *317*, 103–112. [CrossRef]
12. Kamran, T. Mizoguchi-Takahashi's type fixed point theorem. *Comput. Math. Appl.* **2009**, *57*, 507–511. [CrossRef]
13. Klim, D.; Wardowski, D. Fixed point theorems for set valued contractions in complete metric spaces. *J. Math. Anal. Appl.* **2007**, *334*, 132–139. [CrossRef]
14. Lin, L.J.; Wang, S.U. Common fixed point theorems for a finite family of discontinuous and noncommutative maps. *Fixed Point Theory Appl.* **2011**, *211*, 847170. [CrossRef]
15. Mizoguchi, N.; Takahashi, W. Fixed point theorems formultivaluedmappings on completemetric spaces. *J. Math. Anal. Appl.* **1989**, *141*, 177–188. [CrossRef]
16. Neog, M.; Debnath, P.; Radenović, S. New extension of some common fixed point theorems in complete metric spaces. *Fixed Point Theory* **2019**, *20*, 567–580.
17. Abdou, A.A.N. Common fixed point results for multi-valued mappings with some examples. *J. Nonlinear Sci. Appl.* **2016**, *9*, 787–798. [CrossRef]
18. Agarwal, R.P.; El-Gebeily, M.A.; O'Regan, D. Generalized contractions in partially ordered metric spaces. *Appl. Anal.* **2008**, *87*, 109–116. [CrossRef]
19. Agarwal, R.P.; O'Regan, D. Fixed point theory for generalized contractions on spaces with two metrics. *J. Math. Anal. Appl.* **2000**, *248*, 402–414. [CrossRef]

20. Agarwal, R.P.; Sintunavarat, W.; Kumam, P. Coupled coincidence point and common coupled fixed point theorems lacking the mixed monotone property. *Fixed Point Theory Appl.* **2013**, *22*, 1–20. [CrossRef]
21. Debnath, P.; Neog, M.; Radenović, S. New extension of some fixed point results in complete metric spaces. *Tbilisi Math. J.* **2018**, *10*, 201–210. [CrossRef]
22. Ding, H.S.; Kadelburg, Z.; Nashine, H.K. Common fixed point theorems for weakly increasing mappings on ordered orbitally complete metric spaces. *Fixed Point Theory Appl.* **2012**, *85*. [CrossRef]
23. Huang, Q.; Zhu, L.; Chen, X.; Zhang, C. On stable perturbations of the generalized Drazin inverses of closed linear operators in Banach spaces. *Abstr. Appl. Anal.* **2012**, *2012*, 12. [CrossRef]
24. Khan, M.S.; Cho, Y.J.; Park, W.T.; Mumtaz, T. Coincidence and common fixed points of hybrid contractions. *Fixed Point Theory Appl.* **1993**, *55*, 369–385. [CrossRef]
25. Liu, Z.; Zhang, X.; Ume, J.S.; Kang, S.M. Common fixed point theorems for four mappings satisfying *psi*-weakly contractive conditions. *Fixed Point Theory Appl.* **2015**, *20*. [CrossRef]
26. Mustafa, Z.; Jaradat, M.M.M.; Ansari, A.H.; Popović, B.Z.; Jaradat, H.M. C-class functions with new approach on coincidence point results for generalized $(\psi, \varphi)$-weakly contractions in ordered b-metric spaces. *SpringerPlus* **2016**, *5*, 802. [CrossRef] [PubMed]
27. Sintunavarat, W.; Kumam, P. Coincidence and common fixed points for hybrid strict contractions without the weakly commuting condition. *Appl. Math. Lett.* **2009**, *22*, 1877–1881. [CrossRef]
28. Sessa, S. On a weak commutativity condition of mappings in fixed point consideration. *Publ. Inst. Math.* **1982**, *32*, 149–153.
29. Jungck, G.; Rhoades, B.E. Fixed point for set valued functions without continuity. *Indian J. Pure Appl. Math.* **1998**, *29*, 227–238.
30. Jachymski, J. Equivalence of some contractivity properties over metrical structures. *Proc. Am. Math. Soc.* **1997**, *125*, 2327–2335. [CrossRef]
31. Jachymski, J.; Jozwik, I. Nonlinear contractive conditions: A comparison and related problems, fixed point theory and its applications. *Pol. Acad. Sci.* **2007**, *77*, 123–146.
32. Jachymski, J. Equivalent conditions for generalized contractions on (ordered) metric spaces. *Nonlinear Anal.* **2011**, *74*, 768–774. [CrossRef]
33. Aamri, M.; Moutawakil, D.E. Some new common fixed point theorems under strict contractive conditions. *J. Math. Anal. Appl.* **2002**, *270*, 181–188. [CrossRef]
34. Kamran, T. Coincidence and fixed points for hybrid strict contractions. *J. Math. Anal. Appl.* **2004**, *299*, 235–241. [CrossRef]
35. Liu, Y.; Wu, J.; Li, Z. Common fixed points of single valued and multivalued maps. *Int. J. Math. Math. Sci.* **2005**, *19*, 3045–3055. [CrossRef]
36. Zhang, Q.; Song, Y. Fixed point theory for generalized $\varphi$-weak contractions. *Appl. Math. Lett.* **2009**, *22*, 75–78. [CrossRef]
37. Rus, I.A.; Petrusel, A.; Petrusel, G. *Fixed Point Theory*; Cluj University Press: Cluj-Napoca, Romania, 2008.
38. Akram, M.; Zafar, A.A.; Siddiqui, A.A. A general class of contractions: A-contractions. *Novi Sad J. Math.* **2008**, *38*, 25–33.


© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).