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Some fixed point results on G-metric and G_b-metric spaces

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Abstract: The purpose of this paper is to prove some fixed point results using JS-G-contraction on G-metric spaces, also to prove some fixed point results on G_b-complete metric space for a new contraction. Our results extend and improve some results in the literature. Moreover, some examples are presented to illustrate the validity of our results.

Keywords: fixed point, G-metric space, G_b-metric space, JS-G-contraction

MSC: Primary 47H10; Secondary 54H25.

1 Introduction

Mustafa and Sims [1] introduced the notion of G-metric spaces as a generalization of classical metric spaces and obtained some fixed point theorems for mappings satisfying different generalized contractive conditions. Thereafter, the concept of G-metric space has been studied and used to obtain various fixed point theorems by several mathematicians (see ([2–24]).

Definition 1.1. [1] Let X be a non empty and $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following properties

$(G1)$ $G(a, b, c) = 0$ if $a = b = c$,
$(G2)$ $0 < G(a, a, b)$ for all $a, b \in X$ with $a \neq b$,
$(G3)$ $G(a, a, b) \leq G(a, b, c)$ for all $a, b, c \in X$ with $b \neq c$,
$(G4)$ $G(a, b, c) = G(a, c, b) = G(b, c, a) = \cdots$ (symmetry in all three variables),
$(G5)$ $G(a, b, c) \leq G(a, w, w) + G(w, b, c)$ for all $a, b, c, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or, a G-metric on X and the pair (X, G) is called a G-metric space. Throughout this paper we mean by $\mathbb{N}$ the set of all Natural Numbers.

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Definition 1.2. [1] Let (X, G) be a G-metric space, and let (a_n) be a sequence of points of X. Then we say that (a_n) is G-convergent to $a \in X$ if $lim_{n,m\to\infty} G(a, a_n, a_m) = 0$, that is, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(a, a_n, a_m) < \epsilon$ for all, $n, m \geq N$. We call a the limit of the sequence and write $a_n \to x$ or $lim_{n\to\infty} a_n = a$.

Proposition 1.3. *[1] Let (X, G) be a G-metric space. The following statements are equivalent:*

(1) (a_n) is G–convergent to a.
(2) $G(a_n, a_n, a) \to 0$ as $n \to +\infty$.
(3) $G(a_n, a, a) \to 0$ as $n \to +\infty$.
(4) $G(a_n, a_m, a) \to 0$ as $n, m \to +\infty$.

Definition 1.4. [1] Let (X, G) be a G-metric space. A sequence (a_n) is called a G-Cauchy sequence if for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(a_n, a_m, a_l) < \epsilon$ for all $n, m, l \geq N$, that is $G(a_n, a_m, a_l) \to 0$ as $n, m, l \to +\infty$.

Definition 1.5. [1] A G-metric space (X, G) is called G-complete if every G-Cauchy sequence is G-convergent in (X, G).

Corollary 1.6. *[1] Let (X, d) be a metric space, then (X, d) is complete metric space iff (X, G_m) is complete G-metric space where*

$$G_m(a, b, c) = \max\{d(a, b), d(b, c), d(a, c)\}$$

Corollary 1.7. *[1] A G-metric space (X, G) is continuous on its three variables.*

Very recently, Jleli and Samet [25] introduced a new type of contraction which involves the following set of all functions $\psi : (0, \infty) \to (1, \infty)$ satisfying the conditions:

(ψ_1) ψ is nondecreasing;
(ψ_2) for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} \psi(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;
(ψ_3) there exist $r \in (0, 1)$ and $L \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\psi(t)-1}{t^r} = L$.

To be consistent with Jleli and Samet [25], we denote by $\digamma$ the set of all functions $\psi : (0, \infty) \to (1, \infty)$ satisfying the conditions $(\psi_1 - \psi_3)$.

Also, they established the following result as a generalization of Banach Contraction Principle.

Theorem 1.8. *[25, Corollary 2.1] Let (X, d) be a complete metric space and $f : X \to X$ be a mapping. Suppose that there exist $\psi \in \Psi$ and $k \in (0, 1)$ such that*

$$x, y \in X, \quad d(fx, fy) \neq 0 \Rightarrow \psi(d(fx, fy)) \leq [\psi(d(x, y))]^k .$$

Then f has a unique fixed point.

In 2015, Hussain et al. [26] customized the above family of functions and proved a fixed point theorem as a generalization of [25]. They customized the family of functions $\psi : [0, \infty) \to [1, \infty)$ to be as follows:

(ψ_1) ψ is nondecreasing and $\psi(t) = 1$ if and only if $t = 0$;
(ψ_2) for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} \psi(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;
(ψ_3) there exist $r \in (0, 1)$ and $L \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\psi(t)-1}{t^r} = L$;
(ψ_4) $\psi(u + v) \leq \psi(u)\psi(v)$ for all $u, v > 0$.

To be consistent with Hussain et al. [26], we denote by Ψ the set of all functions $\psi : [0, \infty) \to [1, \infty)$ satisfying the conditions $(\psi_1 - \psi_4)$. For more details in this direction, we refer the reader to [27–30].

In this paper, we introduce a new contraction called JS-G-contraction and we prove some fixed point results of such contraction in the setting of G-metric spaces, also we prove some fixed point results on G_b-complete metric space for a new contraction.

2 Fixed Point Results on G- Metric Space

We start this section by introducing the following definition.

Definition 2.1. Let (X, G) be a G-metric space, and let $g : X \to X$ be a self mapping. Then g is said to be a JS-G-contraction whenever there exist a function $\psi \in \Psi$ and positive real numbers r_1, r_2, r_3, r_4 with $0 \leq r_1 + 3r_2 + r_3 + 2r_4 < 1$ such that

$$\begin{aligned} \psi(G(ga, gb, gc)) &\leq [\psi(G(a, b, c))]^{r_1} [\psi(G(a, ga, gc))]^{r_2} [\psi(G(b, gb, gc))]^{r_3} \\ &\quad \times [\psi(G(a, gb, gb) + G(b, ga, ga))]^{r_4}, \end{aligned} \tag{2.1}$$

for all $a, b, c \in X$.

Theorem 2.2. *Let (X, G) be a complete G-metric space and $g : X \to X$ be a JS-G-contraction. Then g has a unique fixed point.*

Proof. Let $a_0 \in X$ be arbitrary. For $a_0 \in X$, we define the sequence $\{a_n\}$ by $a_n = g^n a_0 = ga_{n-1}$. If there exist $n_0 \in \mathbb{N}$ such that $a_{n_0} = a_{n_0+1}$, then a_{n_0} is a fixed point of g, and we have nothing to prove. Thus, we suppose that $a_n \neq a_{n+1}$, i.e., $G(ga_{n-1}, ga_n, ga_n) > 0$ for all $n \in \mathbb{N}$. Now, we will prove that $\lim_{n\to\infty} G(a_n, a_{n+1}, a_{n+1}) = 0$.

Since g is a JS-G-contraction, by using condition (2.1), we get that

$$\begin{aligned} 1 &< \psi(G(a_n, a_{n+1}, a_{n+1})) = \psi(G(ga_{n-1}, ga_n, ga_n)) \\ &\leq [\psi(G(a_{n-1}, a_n, a_n))]^{r_1} [\psi(G(a_{n-1}, ga_{n-1}, ga_n))]^{r_2} [\psi(G(a_n, ga_n, ga_n))]^{r_3} \\ &\quad \times [\psi(G(a_{n-1}, ga_n, ga_n) + G(a_n, ga_{n-1}, ga_{n-1}))]^{r_4} \\ &= [\psi(G(a_{n-1}, a_n, a_n))]^{r_1} [\psi(G(a_{n-1}, a_n, a_{n+1}))]^{r_2} [\psi(G(a_n, a_{n+1}, a_{n+1}))]^{r_3} [\psi(G(a_{n-1}, a_{n+1}, a_{n+1}))]^{r_4}. \end{aligned}$$

Using (G5) and (ψ_4), we get

$$\begin{aligned} \psi(G(a_{n-1}, a_n, a_{n+1})) &\leq \psi(G(a_{n-1}, a_n, a_n) + G(a_n, a_n, a_{n+1})) \\ &\leq \psi(G(a_{n-1}, a_n, a_n) + 2G(a_n, a_{n+1}, a_{n+1})) \\ &\leq \psi(G(a_{n-1}, a_n, a_n))\psi(2G(a_n, a_{n+1}, a_{n+1})) \\ &= \psi(G(a_{n-1}, a_n, a_n))\psi(G(a_n, a_{n+1}, a_{n+1}) + G(a_n, a_{n+1}, a_{n+1})) \\ &\leq \psi(G(a_{n-1}, a_n, a_n))[\psi(G(a_n, a_{n+1}, a_{n+1}))]^2, \end{aligned}$$

and

$$\begin{aligned} \psi(G(a_{n-1}, a_{n+1}, a_{n+1})) &\leq \psi(G(a_{n-1}, a_n, a_n) + G(a_n, a_{n+1}, a_{n+1})) \\ &\leq \psi(G(a_{n-1}, a_n, a_n))\psi(G(a_n, a_{n+1}, a_{n+1})). \end{aligned}$$

Therefore,

$$\begin{aligned} 1 &< \psi(G(a_n, a_{n+1}, a_{n+1})) \\ &\leq [\psi(G(a_{n-1}, a_n, a_n))]^{r_1} \left[\psi(G(a_{n-1}, a_n, a_n))\right]^{r_2} \left[\psi(G(a_n, a_{n+1}, a_{n+1}))\right]^{2r_2} \\ &\quad \times [\psi(G(a_n, a_{n+1}, a_{n+1}))]^{r_3} \left[\psi(G(a_{n-1}, a_n, a_n))\right]^{r_4} \left[\psi(G(a_n, a_{n+1}, a_{n+1}))\right]^{r_4}. \end{aligned}$$

So, by reordering the product terms of the above inequality, then using the induction, we get that

$$\begin{aligned} 1 < \psi\left(G\left(a_n, a_{n+1}, a_{n+1}\right)\right) &\le \left[\psi\left(G\left(a_{n-1}, a_n, a_n\right)\right)\right]^{\frac{r_1+r_2+r_4}{1-2r_2-r_3-r_4}} \\ &\vdots \\ &\le \left[\psi\left(G\left(a_0, a_1, a_1\right)\right)\right]^{\left(\frac{r_1+r_2+r_4}{1-2r_2-r_3-r_4}\right)^n}. \end{aligned} \tag{2.2}$$

Taking limit as $n \to \infty$, and noting that $\frac{r_1+r_2+r_4}{1-2r_2-r_3-r_4} < 1$, we get

$$\lim_{n\to\infty} \psi\left(G\left(a_n, a_{n+1}, a_{n+1}\right)\right) = 1, \tag{2.3}$$

which implies by (ψ_2) that

$$\lim_{n\to\infty} G\left(a_n, a_{n+1}, a_{n+1}\right) = 0. \tag{2.4}$$

From the condition (ψ_3), there exist $0 < r < 1$ and $L \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\psi\left(G\left(a_{n+1}, a_n, a_n\right)\right) - 1}{\left[G\left(a_n, a_{n+1}, a_{n+1}\right)\right]^r} = L.$$

Suppose that $L < \infty$. In this case, let $B_1 = \frac{L}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left|\frac{\psi\left(G\left(a_n, a_{n+1}, a_{n+1}\right)\right) - 1}{\left[G\left(a_n, a_{n+1}, a_{n+1}\right)\right]^r} - L\right| \le B_1,$$

for all $n > n_0$. This implies that

$$\frac{\psi\left(G\left(a_{n+1}, a_n, a_n\right)\right) - 1}{\left[G\left(a_n, a_{n+1}, a_{n+1}\right)\right]^r} \ge L - B_1 = \frac{L}{2} = B_1,$$

for all $n > n_0$. Then

$$n\left(G\left(a_n, a_{n+1}, a_{n+1}\right)\right)^r \le A_1 n\left[\psi\left(G\left(a_n, a_{n+1}, a_{n+1}\right)\right) - 1\right],$$

where $A_1 = \frac{1}{B_1}$.

Now for $L = \infty$, let $B_2 > 0$ be an arbitrary number. From the definition of the limit there exist $n_1 \in \mathbb{N}$ such that

$$\frac{\psi\left(G\left(a_n, a_{n+1}, a_{n+1}\right)\right) - 1}{\left[G\left(a_n, a_{n+1}, a_{n+1}\right)\right]^r} \ge B_2,$$

for all $n \ge n_1$. Then

$$n\left(G\left(a_n, a_{n+1}, a_{n+1}\right)\right)^r \le A_2 n\left[\psi\left(G\left(a_n, a_{n+1}, a_{n+1}\right)\right) - 1\right],$$

where $A_2 = \frac{1}{B_2}$. Thus, in both cases, there exist $A = \max\{A_1, A_2\} > 0$ and $n_* = \max\{n_0, n_1\} \in \mathbb{N}$ such that

$$n\left(G\left(a_n, a_{n+1}, a_{n+1}\right)\right)^r \le An\left[\psi\left(G\left(a_n, a_{n+1}, a_{n+1}\right)\right) - 1\right] \quad \text{for all } n \ge n_*.$$

Now, using (2.2) we get

$$n\left(G\left(a_n, a_{n+1}, a_{n+1}\right)\right)^r \le A\, n\left[\left[\psi\left(G\left(a_0, a_1, a_1\right)\right)\right]^{\alpha^n} - 1\right],$$

where, $\alpha = \frac{r_1+r_2+r_4}{1-2r_2-r_3-r_4}$. But,

$$\begin{aligned}
\lim_{n\to\infty} n\left[[\psi(G(a_0,a_1,a_1))]^{\alpha^n}-1\right] &= \lim_{n\to\infty}\frac{\left[[\psi(G(a_0,a_1,a_1))]^{\alpha^n}-1\right]}{1/n}\\
&= \lim_{n\to\infty}\frac{\alpha^n\ln(\alpha)\ln(\psi(G(a_0,a_1,a_1)))\left[[\psi(G(a_0,a_1,a_1))]^{\alpha^n}\right]}{-1/n^2}\\
&= \lim_{n\to\infty} -n^2\alpha^n\ln(\alpha)\ln(\psi(G(a_0,a_1,a_1)))\left[[\psi(G(a_0,a_1,a_1))]^{\alpha^n}\right]\\
&= \lim_{n\to\infty}\frac{-n^2\ln(\alpha)\ln(\psi(G(a_0,a_1,a_1)))\left[[\psi(G(a_0,a_1,a_1))]^{\alpha^n}\right]}{\alpha_1^n}\\
&= \lim_{n\to\infty}\frac{-n^2}{\alpha_1^n}\times\lim_{n\to\infty}\ln(\alpha)\ln(\psi(G(a_0,a_1,a_1)))\left[[\psi(G(a_0,a_1,a_1))]^{\alpha^n}\right]\\
&= 0\times\ln(\alpha)\ln(\psi(G(a_0,a_1,a_1)))\\
&= 0\quad(\text{where }\alpha_1=1/\alpha),
\end{aligned}$$

which implies that $\lim_{n\to\infty} n(G(a_n,a_{n+1},a_{n+1}))^r=0$, thus there exists $n_2\in\mathbb{N}$ such that

$$G(a_n,a_{n+1},a_{n+1})\le\frac{1}{n^{1/r}},$$

for all $n>n_2$. Now, for $m>n>n_2$, we have

$$G(a_n,a_m,a_m)\le\sum_{i=n}^{m-1}G(a_i,a_{i+1},a_{i+1})\le\sum_{i=n}^{m-1}\frac{1}{i^{\frac{1}{r}}}\le\sum_{i=1}^{\infty}\frac{1}{i^{\frac{1}{r}}}.$$

Since $0<r<1$, then $\sum_{i=1}^{\infty}\frac{1}{i^{\frac{1}{r}}}$ is convergent and hence $G(a_n,a_m,a_m)\to0$ as $m,n\to\infty$. Thus, we proved that $\{a_n\}$ is a G-Cauchy sequence. Completeness of (X,G) ensures that there exists $a^\star\in X$ such that $a_n\to a^\star$ as $n\to\infty$.

Now we shall show that $a^\star$ is a fixed point of g. Using (G5) we get that

$$\begin{aligned}
G(a^\star,a^\star,ga^\star) &\le G(a^\star,a^\star,a_{n+1})+G(a_{n+1},a_{n+1},ga^\star)\\
&= G(a^\star,a^\star,a_{n+1})+G(ga_n,ga_n,ga^\star)
\end{aligned}\tag{2.5}$$

and

$$G\left(a_n,a_{n+1},ga^\star\right)\le\left(G\left(a_n,a_{n+1},a^\star\right)\right)+\left(G\left(a^\star,a^\star,ga^\star\right)\right).\tag{2.6}$$

Hence, by the properties of ψ we get that

$$\psi(G(a^\star,a^\star,ga^\star))\le\psi(G(a^\star,a^\star,a_{n+1}))\psi(G(ga_n,ga_n,ga^\star))\tag{2.7}$$

$$\psi(G\left(a_n,a_{n+1},ga^\star\right))\le\psi(G\left(a_n,a_{n+1},a^\star\right))\psi(G\left(a^\star,a^\star,ga^\star\right)).\tag{2.8}$$

Thus,

$$\left[\psi(G\left(a_n,a_{n+1},ga^\star\right))\right]^{r_2+r_3}\le\left[\psi(G\left(a_n,a_{n+1},a^\star\right))\right]^{r_2+r_3}\left[\psi(G\left(a^\star,a^\star,ga^\star\right))\right]^{r_2+r_3}.\tag{2.9}$$

However, by using (2.1), (ψ_4) and (2.9) we have

$$\begin{aligned}
\psi\left(G\left(a_{n+1}, a_{n+1}, ga^{\star}\right)\right) &= \psi\left(G\left(ga_n, ga_n, ga^{\star}\right)\right) \\
&\le \left[\psi\left(G\left(a_n, a_n, a^{\star}\right)\right)\right]^{r_1}\left[\psi\left(G\left(a_n, a_{n+1}, ga^{\star}\right)\right)\right]^{r_2} \\
&\quad \times\left[\psi\left(G\left(a_n, a_{n+1}, ga^{\star}\right)\right)\right]^{r_3} \\
&\quad \times\left[\psi\left(G\left(a_n, a_{n+1}, a_{n+1}\right)+G\left(a_n, a_{n+1}, a_{n+1}\right)\right)\right]^{r_4} \\
&= \left[\psi\left(G\left(a_n, a_n, a^{\star}\right)\right)\right]^{r_1}\left[\psi\left(G\left(a_n, a_{n+1}, ga^{\star}\right)\right)\right]^{r_2+r_3} \\
&\quad \times\left[\psi\left(G\left(a_n, a_{n+1}, a_{n+1}\right)\right)\right]^{2r_4} \\
&\le \left[\psi\left(G\left(a_n, a_n, a^{\star}\right)\right)\right]^{r_1}\left[\psi(G\left(a_n, a_{n+1}, a^{\star}\right))\right]^{r_2+r_3} \\
&\quad \left[\psi(G\left(a^{\star}, a^{\star}, ga^{\star}\right))\right]^{r_2+r_3}\left[\psi\left(G\left(a_n, a_{n+1}, a_{n+1}\right)\right)\right]^{2r_4}.
\end{aligned} \tag{2.10}$$

Now, substituting (2.10) in (2.7) we get that

$$\begin{aligned}
\psi(G(a^{\star}, a^{\star}, ga^{\star})) &\le \psi(G(a^{\star}, a^{\star}, a_{n+1}))\left[\psi(G\left(a_n, a_n, a^{\star}\right))\right]^{r_1}\left[\psi(G\left(a_n, a_{n+1}, a^{\star}\right))\right]^{r_2+r_3} \\
&\quad \left[\psi(G\left(a^{\star}, a^{\star}, ga^{\star}\right))\right]^{r_2+r_3}\left[\psi\left(G\left(a_n, a_{n+1}, a_{n+1}\right)\right)\right]^{2r_4}.
\end{aligned} \tag{2.11}$$

Hence,

$$\begin{aligned}
1 \le \left[\psi(G(a^{\star}, a^{\star}, ga^{\star}))\right]^{1-r_2-r_3} &\le \psi(G(a^{\star}, a^{\star}, a_{n+1}))\left[\psi\left(G\left(a_n, a_n, a^{\star}\right)\right)\right]^{r_1} \\
&\quad \left[\psi(G\left(a_n, a_{n+1}, a^{\star}\right))\right]^{r_2+r_3}\left[\psi\left(G\left(a_n, a_{n+1}, a_{n+1}\right)\right)\right]^{2r_4}.
\end{aligned} \tag{2.12}$$

By taking the limit as $n \to \infty$ and using (2.4), (ψ_2), Proposition 1.3 and the convergence of a_n to $a^{\star}$ in the above equation we get that

$$\psi(G(a^{\star}, a^{\star}, ga^{\star})) = 1 \tag{2.13}$$

which implies by (ψ_1) that $G(a^{\star}, a^{\star}, ga^{\star}) = 0$ and so $ga^{\star} = a^{\star}$. Thus, $a^{\star}$ is a fixed point of g.

Finally to show the uniqueness, assume that there exist $a' \neq a^{\star}$ such that $a' = ga'$. By (G_2),

$$G(a', a', a^{\star}) = G(ga', ga', ga^{\star}) > 0.$$

Thus, by (2.1) we get

$$\begin{aligned}
\psi(G(a', a', a^{\star})) &= \psi(G(ga', ga', ga^{\star})) \le \left[\psi G(a', a', a^{\star})\right]^{r_1}\left[\psi(G(a', ga', ga^{\star}))\right]^{r_2} \\
&\quad \times\left[\psi(G(a', ga', ga^{\star}))\right]^{r_3}\left[\psi\left(G(a', ga', ga') + G(a', ga', ga')\right)\right]^{r_4}, \\
&= \left[\psi(G(a', a', a^{\star}))\right]^{r_1}\left[\psi(G(a', a', a^{\star}))\right]^{r_2}\left[\psi(G(a', a', a^{\star}))\right]^{r_3} \\
&\quad \times\left[\psi\left(G(a', a', a') + G(a', a', a')\right)\right]^{r_4}, \\
&= \left[\psi(G(a', a', a^{\star}))\right]^{r_1+r_2+r_3},
\end{aligned}$$

which leads to a contradiction because $r_1 + r_2 + r_3 < 1$. Therefore, g has a unique fixed point. □

The following result is a direct consequence of Theorem 2.2 by taking $\psi(t) = e^{\sqrt{t}}$ in (2.1).

Corollary 2.3. *Let (X, G) be a complete G-metric space and $g : X \to X$ be a mapping. Suppose that there exist positive real numbers r_1, r_2, r_3, r_4 with $0 \le r_1 + 3r_2 + r_3 + 2r_4 < 1$ such that*

$$\begin{aligned}\sqrt{G(ga, gb, gc)} \le\; & r_1\sqrt{G(a, b, c)} + r_2\sqrt{G(a, ga, gc)} + r_3\sqrt{G(b, gb, gc)} \\ & + r_4\sqrt{G(a, gb, gb) + G(b, ga, ga)}\end{aligned} \tag{2.14}$$

for all $a, b, c \in X$. Then g has a unique fixed point.

Remark 2.4. Note that condition (2.14) is equivalent to

$$\begin{aligned}G(ga, gb, gc) \le\; & r_1^2 G(a, b, c) + r_2^2 G(a, ga, gc) + r_3^2 G(b, gb, gc) \\ & + r_4^2 [G(a, gb, gb) + G(b, ga, ga)] \\ & + 2r_1r_2\sqrt{G(a, b, c)\, G(a, ga, gc)} + 2r_1r_3\sqrt{G(a, b, c)\, G(b, gb, gc)} \\ & + 2r_1r_4\sqrt{G(a, b, c)[G(a, gb, gb) + G(b, ga, ga)]} \\ & + 2r_2r_3\sqrt{G(a, ga, gc)\, G(b, gb, gc)} \\ & + 2r_2r_4\sqrt{G(a, ga, gc)[G(a, gb, gb) + G(b, ga, ga)]} \\ & + 2r_3r_4\sqrt{G(b, gb, gc)[G(a, gb, gb) + G(b, ga, ga)]}.\end{aligned}$$

Next, in view of Remark 2.4 and by taking $r_2 = r_3 = r_4 = 0$ in Corollary 2.3, we obtain the following corollary.

Corollary 2.5. *Let (X, G) be a complete G-metric space and $g : X \to X$ be a mapping. Suppose that there exist positive real numbers $0 \le r_1 < 1$, such that*

$$G(ga, gb, gc) \le r_1^2 G(a, b, c) \tag{2.15}$$

for all $a, b, c \in X$. Then g has a unique fixed point.

Finally, by taking $\psi(t) = e^{\sqrt[n]{t}}$ in (2.1), we get the following corollary.

Corollary 2.6. *Let (X, G) be a complete G-metric space and $g : X \to X$ be a mapping. Suppose that there exist positive real numbers r_1, r_2, r_3, r_4 with $0 \le r_1 + 3r_2 + r_3 + 2r_4 < 1$, such that*

$$\begin{aligned}\sqrt[n]{G(ga, gb, gc)} \le\; & r_1\sqrt[n]{G(a, b, c)} + r_2\sqrt[n]{G(a, ga, gc)} + r_3\sqrt[n]{G(b, gb, gc)} \\ & + r_4\sqrt[n]{G(a, gb, gb) + G(b, ga, ga)}\end{aligned}$$

for all $a, b, c \in X$. Then g has a unique fixed point.

Remark 2.7. By specifying $r_i = 0$ for some $i \in \{1, 2, 3, 4\}$ in Remark 2.4 and Corollary 2.6 we can get several results.

Example 2.8. Let $X = [0, \infty)$ and the G-metric $G_m(a, b, c) = \max\{|a-b|, |b-c|, |a-c|\}$. Define $g : X \to X$ by $g(x) = \frac{x}{8}$ and $\psi(t) = e^{\sqrt{t}}$. Then clearly all conditions of Theorem 2.2 are satisfied with $r_i = \frac{1}{\sqrt{8}}$; $i = 1, 2, 3, 4$, and $x = 0$ is a unique fixed point of g.

3 Fixed Point Results on G_b-Metric Spaces

In this section, using the concepts of G_b-metric space which was introduced by Aghajani et al. [31] we establish some new fixed point results in this setting.

Definition 3.1. [31] Let X be a nonempty set and $s \ge 1$ be a given real number. Suppose that $G_b : X \times X \times X \to [0, \infty)$ be a function satisfying the following properties

(G_b1) $G_b(u, v, w) = 0$ if $u = v = w$,
(G_b2) $0 < G_b(u, u, v)$ for all $u, v \in X$ with $u \neq v$,
(G_b3) $G_b(u, u, v) \leq G_b(u, v, w)$ for all $u, v, w \in X$ with $v \neq w$,
(G_b4) $G_b(u, v, w) = G_b(p\{u, v, w\})$, where p is a permutation of u, v, w (symmetry),
(G_b5) $G_b(u, v, w) \leq s(G_b(u, c, c) + G_b(c, v, w))$ for all $u, v, w, c \in X$ (rectangle inequality).
Then the function G_b is called a generalized b-metric, or a G_b-metric on X, and the pair (X, G) is called a G_b-metric space.

It is clear that the class of G_b-metric spaces is effectively larger than that of G-metric spaces given in [1]. Indeed, each G-metric space is a G_b-metric space with $s = 1$.

Definition 3.2. [31] A G_b-metric space is said to be symmetric if $G_b(u, v, v) = G_b(v, u, u)$ for all $u, v \in X$.

Proposition 3.3. *[31] Let X be a G_b-metric space. Then for each $u, v, w, c \in X$ it follows that:*

(1) If $G_b(u, v, w) = 0$ then $u = v = w$,
(2) $G_b(u, v, w) \leq s(G_b(u, u, v) + G_b(u, u, w))$,
(3) $G_b(u, v, v) \leq 2sG_b(v, u, u)$,
(4) $G_b(u, v, w) \leq s(G_b(u, c, w) + G_b(c, v, w))$.

Definition 3.4. [31] Let (X, G_b) be a G_b-metric space, and (a_n) be a sequence in X. Then we say that (a_n) is G_b-convergent to $a \in X$ if $lim_{n,m\to\infty} G_b(a, a_n, a_m) = 0$, that is, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G_b(a, a_n, a_m) < \epsilon$, for all, $n, m \geq N$. We call x the limit of the sequence and write $a_n \to a$ or $lim_{n\to\infty} a_n = a$.

Proposition 3.5. *[31] Let (X, G_b) be a G_b-metric space. The following statements are equivalent:*

(1) (a_n) is G_b–convergent to a.
(2) $G_b(a_n, a_n, a) \to 0$ as $n \to +\infty$.
(3) $G_b(a_n, a, a) \to 0$ as $n \to +\infty$.
(4) $G_b(a_n, a_m, a) \to 0$ as $n, m \to +\infty$.

Definition 3.6. [31] Let X be a G_b-metric space. A sequence (a_n) is called a G_b-Cauchy sequence if for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G_b(a_n, a_m, a_l) < \epsilon$ for all $n, m, l \geq N$, that is $G_b(a_n, a_m, a_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 3.7. *[31] Let (X, G_b) be a G_b-metric space. The following statements are equivalent:*

(1) (a_n) is G_b-Cauchy sequence. (2) $G_b(a_n, a_m, a_m) \to 0$ as $n, m \to +\infty$.

Definition 3.8. [31] A G_b-metric space X is called G_b-complete if every G_b-Cauchy sequence is G_b-convergent in X.

Lemma 3.9. *Let X be a G_b-metric space with $s \geq 1$. If a sequence $(a_n) \subseteq X$ is G_b-convergent, then it has a unique limit point.*

Very recently, Ahmad et al. [27] studied JS-contraction and considered a new set of real functions, say Ω. They replaced condition (ψ_3) by another condition called (Θ_3).

Applying this condition we can have a new range of functions. Thus, consistent with Ahmad et al. [27] we denote by Ω the set of all functions $\theta : [0, \infty) \to [1, \infty)$ satisfying the following conditions:

(Θ_1): θ is nondecreasing and $\Theta(t) = 1$ if and only if $t = 0$;
(Θ_2): for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;
(Θ_3): θ is continuous.

Example 3.10. [27] Let $\theta_1(t) = e^{\sqrt{t}}$, $\theta_2(t) = e^{\sqrt{te^t}}$, $\theta_3(t) = e^t$, $\theta_4(t) = \cosh t$ and $\theta_5(t) = 1 + \ln(1 + t)$ for all $t > 0$. Then $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \in \Omega$.

Remark 3.11. [27] Note that the conditions (ψ_3) and (Θ_3) are independent of each other. Indeed, for $p \geq 1$, $\theta(t) = e^{t^p}$ satisfies the conditions (ψ_1) and (ψ_2) but it does not satisfy (ψ_3), while it satisfies the condition (Θ_3). Therefore $\Omega \not\subseteq \Psi$. Again, for $a > 1$, $m \in (0, \frac{1}{a})$, $\theta(t) = 1 + t^m(1 + [t])$, where $[t]$ denotes the integral part of t, satisfies the conditions (ψ_1) and (ψ_2) but it does not satisfy (Θ_3), while it satisfies the condition (ψ_3) for any $r \in (\frac{1}{a}, 1)$. Therefore $\Psi \not\subseteq \Omega$. Also, if we take $\theta(t) = e^{\sqrt{t}}$, then $\theta \in \Psi$ and $\theta \in \Omega$. Therefore $\Psi \cap \Omega \neq \emptyset$.

Definition 3.12. [4] Let $g : X \to X$ and $\alpha : X \times X \times X \to [0, \infty)$. Then g is called α-admissible if for all $u, v, w \in X$ with $\alpha(u, v, w) \geq 1$ implies $\alpha(gu, gv, gw) \geq 1$.

Definition 3.13. Let $g : X \to X$ and $\alpha : X \times X \times X \to [0, \infty)$. Then g is called rectangular-α-admissible if
1. g is α-admissible,
2. $\alpha(u, c, c) \geq 1$ and $\alpha(c, v, w) \geq 1$ implies that $\alpha(u, v, w) \geq 1$

where $u, v, w, c \in X$.

Lemma 3.14. *Let g ba a rectangular α-admissible mapping. Suppose that there exist $a_0 \in X$ such that $\alpha(a_0, ga_0, ga_0) \geq 1$. Define the sequence $a_n = g^n a_0$. Then*

$$\alpha(a_m, a_n, a_n) \geq 1, \text{ for all } m, n \in N \text{ with } m < n$$

Proof. Let $a_n = g^n a_0$ and assume that $n = m + k$ for some integer $k \geq 1$. Since $\alpha(a_0, ga_0, ga_0) \geq 1$ and g is α-admissible, then

$$\alpha(a_1, a_2, a_2) = \alpha(a_1, ga_1, ga_1) = \alpha(ga_0, g^2a_0, g^2a_0) \geq 1.$$

Continuing this process we get that $\alpha(a_m, a_{m+1}, a_{m+1}) \geq 1$. Similarly we have

$$\alpha(a_{m+1}, a_{m+2}, a_{m+2}) \geq 1.$$

Hence, by rectangular α–admissible we have $\alpha(a_m, a_{m+2}, a_{m+2}) \geq 1$, now repeating the same process we get that $\alpha(a_m, a_n, a_n) = \alpha(a_m, a_{m+k}, a_{m+k}) \geq 1$. □

Now, we are ready to state our main theorem in this section.

Theorem 3.15. *Let (X, G_b) be a G_b-complete metric space with $s > 1$. Let $\alpha : X \times X \times X \to (0, \infty)$ and g be a rectangular α-admissible mapping. Suppose that there exist $\theta \in \Omega$ and $r \in (0, 1)$ such that*

$$\frac{1}{3s^2} G_b(u, gu, gu) \leq G_b(u, v, w) \Rightarrow \alpha(u, v, w)\, \theta\left(s^2 G_b(gu, gv, gw)\right) \leq [\theta(M(u, v, w))]^r \tag{3.1}$$

for all $u, v, w \in X$ with at least two of gu, gv and gw being not equal, where

$$M(u, v, w) = \max\left\{\begin{array}{c} G_b(u, v, w),\ \frac{G_b(u,gu,gu)G_b(u,gv,gw)+G_b(v,gv,gw)G_b(v,gu,gu)}{1+s[G_b(u,gu,gw)+G_b(v,gv,gw)]}, \\ \\ \frac{G_b(u,gu,gu)G_b(u,gv,gw)+G_b(v,gv,gw)G_b(v,gu,gu)}{1+G_b(u,gv,gw)+G_b(v,gu,gw)} \end{array}\right\}.$$

Also, suppose that the following assertions hold:
(i) There exists $a_0 \in X$ such that $\alpha(a_0, ga_0, ga_0) \geq 1$.
(ii) For any convergence sequence $\{a_n\}$ to a with $\alpha(a_n, a_{n+1}, a_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, we have $\alpha(a_n, a, a) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.
Then g has a fixed point.
(iii) Moreover, if for all $u, v \in Fix(g)$ implies $\alpha(u, v, v) \geq 1$, then the fixed point is unique where $Fix(g) = \{u : gu = u\}$.

Proof. Let $a_0 \in X$ be such that $\alpha(a_0, ga_0, ga_0) \geq 1$. Define a sequence $\{a_n\}$ by $a_n = g^n a_0$ for all $n \in \mathbb{N}$. Since g is an α-admissible mapping and $\alpha(a_0, a_1, a_1) = \alpha(a_0, ga_0, ga_0) \geq 1$, we deduce that $\alpha(a_1, a_2, a_2) = \alpha(ga_0, ga_1, ga_1) \geq 1$. Continuing this process, we get that $\alpha(a_n, a_{n+1}, a_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Without loss of generality, we assume that $a_n \neq a_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. We shall proceed in proving the theorem using the following two steps.

Step 1: We shall show that $\lim_{n\to\infty} G_b(a_{n+1}, a_n, a_n) = 0$.

Now,

$$\begin{aligned}
M(a_{n-1}, a_n, a_n) &= \max\left\{\begin{array}{c} G_b(a_{n-1}, a_n, a_n), \\ \frac{G_b(a_{n-1}, ga_{n-1}, ga_{n-1})G_b(a_{n-1}, ga_n, ga_n)+G_b(a_n, ga_n, ga_n)G_b(a_n, ga_{n-1}, ga_{n-1})}{1+s[G_b(a_{n-1}, ga_{n-1}, ga_n)+G_b(a_n, ga_n, ga_n)]}, \\ \frac{G_b(a_{n-1}, ga_{n-1}, ga_{n-1})G_b(a_{n-1}, ga_n, ga_n)+G_b(a_n, ga_n, ga_n)G_b(a_n, ga_{n-1}, ga_{n-1})}{1+G_b(a_{n-1}, ga_n, ga_n)+G_b(a_n, ga_{n-1}, ga_n)} \end{array}\right\} \\
&= \max\left\{\begin{array}{c} G_b(a_{n-1}, a_n, a_n), \\ \frac{G_b(a_{n-1}, a_n, a_n)G_b(a_{n-1}, a_{n+1}, a_{n+1})+G_b(a_n, a_{n+1}, a_{n+1})G_b(a_n, a_n, a_n)}{1+s[G_b(a_{n-1}, a_n, a_{n+1})+G_b(a_n, a_{n+1}, a_{n+1})]}, \\ \frac{G_b(a_{n-1}, a_n, a_n)G_b(a_{n-1}, a_{n+1}, a_{n+1})+G_b(a_n, a_{n+1}, a_{n+1})G_b(a_n, a_n, a_n)}{1+G_b(a_{n-1}, a_{n+1}, a_{n+1})+G_b(a_n, a_n, a_{n+1})} \end{array}\right\} \\
&= \max\left\{\begin{array}{c} G_b(a_{n-1}, a_n, a_n), \\ G_b(a_{n-1}, a_n, a_n)\frac{G_b(a_{n-1}, a_{n+1}, a_{n+1})}{1+s[G_b(a_{n-1}, a_n, a_{n+1})+G_b(a_n, a_{n+1}, a_{n+1})]}, \\ G_b(a_{n-1}, a_n, a_n)\frac{G_b(a_{n-1}, a_{n+1}, a_{n+1})}{1+G_b(a_{n-1}, a_{n+1}, a_{n+1})+G_b(a_n, a_n, a_{n+1})} \end{array}\right\}
\end{aligned} \tag{3.2}$$

But, from (G_b3), we have $G_b(a_{n-1}, a_{n+1}, a_{n+1}) \leq G_b(a_{n-1}, a_n, a_{n+1})$, and so

$$\frac{G_b(a_{n-1}, a_{n+1}, a_{n+1})}{1+s[G_b(a_{n-1}, a_n, a_{n+1})+G_b(a_n, a_{n+1}, a_{n+1})]} \leq 1$$

also

$$\frac{G_b(a_{n-1}, a_{n+1}, a_{n+1})}{1+G_b(a_{n-1}, a_{n+1}, a_{n+1})+G_b(a_n, a_n, a_{n+1})} \leq 1.$$

Therefore, $M(a_{n-1}, a_n, a_n) = G_b(a_{n-1}, a_n, a_n)$.

Since $\alpha(a_n, a_{n+1}, a_{n+1}) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$ and $\frac{1}{3s^2} G_b(a_{n-1}, ga_{n-1}, ga_{n-1}) \leq G_b(a_{n-1}, a_n, a_n)$, as a result by (3.1) we have

$$\begin{aligned}
\theta(G_b(a_n, a_{n+1}, a_{n+1})) &= \theta(G_b(ga_{n-1}, ga_n, ga_n)), \\
&\leq \alpha(a_{n-1}, a_n, a_n)\,\theta\left(s^2 G_b(ga_{n-1}, ga_n, ga_n)\right), \\
&\leq [\theta(M(a_{n-1}, a_n, a_n))]^r, \\
&= [\theta(G_b(a_{n-1}, a_n, a_n))]^r \\
&< \theta(G_b(a_{n-1}, a_n, a_n)).
\end{aligned} \tag{3.3}$$

Therefore, we have

$$1 < \theta(G_b(a_n, a_{n+1}, a_{n+1})) \leq [\theta(G_b(a_{n-1}, a_n, a_n))]^r \leq \cdots \leq [\theta(G_b(a_0, a_1, a_1))]^{r^n}.$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \theta(G_b(a_n, a_{n+1}, a_{n+1})) = 1.$$

This gives us, by (θ_2),

$$\lim_{n\to\infty} G_b(a_n, a_{n+1}, a_{n+1}) = 0. \tag{3.4}$$

But $G_b(a_{n+1}, a_n, a_n) \leq 2sG_b(a_n, a_{n+1}, a_{n+1})$, therefore

$$\lim_{n\to\infty} G_b(a_{n+1}, a_n, a_n) = 0. \tag{3.5}$$

Step 2: We shall prove that the sequence $\{a_n\}$ is a G_b–Cauchy sequence. Suppose on the contrary that $\{a_n\}$ is not a G_b–Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{a_{m_i}\}$ and $\{a_{n_i}\}$ of $\{a_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \text{ and } G_b(a_{m_i}, a_{n_i}, a_{n_i}) \geq \varepsilon. \tag{3.6}$$

This means that

$$G_b\left(a_{m_i}, a_{n_i-1}, a_{n_i-1}\right) < \varepsilon. \tag{3.7}$$

By using (3.6) and (G_b5), we get

$$\varepsilon \leq G_b\left(a_{m_i}, a_{n_i}, a_{n_i}\right) \leq sG_b\left(a_{m_i}, a_{m_i+1}, a_{m_i+1}\right) + sG_b\left(a_{m_i+1}, a_{n_i}, a_{n_i}\right).$$

Taking the upper limit as $i \to \infty$ and using (3.5) we get

$$\frac{\varepsilon}{s} \leq \lim_{i\to\infty} \sup G_b\left(a_{m_i+1}, a_{n_i}, a_{n_i}\right). \tag{3.8}$$

Notice that from (3.3) and (θ_1), we get

$$G_b\left(a_n, a_{n+1}, a_{n+1}\right) \leq G_b\left(a_{n-1}, a_n, a_n\right) \text{ for all } n \in \mathbb{N}, \tag{3.9}$$

Suppose that there exists $i_0 \in \mathbb{N}$ such that

$$\frac{1}{3s^2} G_b\left(a_{m_{i_0}}, ga_{m_{i_0}}, ga_{m_{i_0}}\right) > G_b\left(a_{m_{i_0}}, a_{n_{i_0}-1}, a_{n_{i_0}-1}\right)$$

and

$$\frac{1}{3s^2} G_b\left(a_{m_{i_0}+1}, ga_{m_{i_0}+1}, ga_{m_{i_0}+1}\right) > G_b\left(a_{m_{i_0}+1}, a_{n_{i_0}-1}, a_{n_{i_0}-1}\right).$$

Then from (G_b5), (3.9) we have

$$\begin{aligned} G_b\left(a_{m_{i_0}}, a_{m_{i_0}+1}, a_{m_{i_0}+1}\right) &\leq s\left[G_b\left(a_{m_{i_0}}, a_{n_{i_0}-1}, a_{n_{i_0}-1}\right) + G_b\left(a_{n_{i_0}-1}, a_{m_{i_0}+1}, a_{m_{i_0}+1}\right) \right] \\ &\leq s\left[G_b\left(a_{m_{i_0}}, a_{n_{i_0}-1}, a_{n_{i_0}-1}\right) + 2sG_b\left(a_{m_{i_0}+1}, a_{n_{i_0}-1}, a_{n_{i_0}-1}\right) \right] \\ &\leq s\left[\tfrac{1}{3s^2} G_b\left(a_{m_{i_0}}, ga_{m_{i_0}}, ga_{m_{i_0}}\right) + \tfrac{2s}{3s^2} G_b\left(a_{m_{i_0}+1}, ga_{m_{i_0}+1}, ga_{m_{i_0}+1}\right) \right] \\ &= \left[\frac{1}{3s} G_b\left(a_{m_{i_0}}, a_{m_{i_0}+1}, a_{m_{i_0}+1}\right) + \frac{2}{3} G_b\left(a_{m_{i_0}+1}, a_{m_{i_0}+2}, a_{m_{i_0}+2}\right)\right] \\ &\leq \left(\frac{1}{3s} + \frac{2}{3}\right) G_b\left(a_{m_{i_0}}, a_{m_{i_0}+1}, a_{m_{i_0}+1}\right) \\ &< G_b\left(a_{m_{i_0}}, a_{m_{i_0}+1}, a_{m_{i_0}+1}\right), (\text{ since } s > 1), \end{aligned} \tag{3.10}$$

which is a contradiction. Hence, either

$$\frac{1}{3s^2} G_b\left(a_{m_i}, ga_{m_i}, ga_{m_i}\right) \leq G_b\left(a_{m_i}, a_{n_i-1}, a_{n_i-1}\right)$$

or

$$\frac{1}{3s^2} G_b\left(a_{m_i+1}, ga_{m_i+1}, ga_{m_i+1}\right) \leq G_b\left(a_{m_i+1}, a_{n_i-1}, a_{n_i-1}\right),$$

holds for all $i \in \mathbb{N}$. First suppose that

$$\frac{1}{3s^2} G_b\left(a_{m_i}, ga_{m_i}, ga_{m_i}\right) \leq G_b\left(a_{m_i}, a_{n_i-1}, a_{n_i-1}\right). \tag{3.11}$$

From the definition of $M(u, v, w)$ and using (3.5) and (3.7) we have

$$\begin{aligned} &\lim_{i\to\infty} \sup M\left(a_{m_i}, a_{n_i-1}, a_{n_i-1}\right) \\ &= \lim_{i\to\infty} \sup \max \left\{ \begin{array}{c} G_b\left(a_{m_i}, a_{n_i-1}, a_{n_i-1}\right), \\ \frac{G_b\left(a_{m_i}, ga_{m_i}, ga_{m_i}\right) G_b\left(a_{m_i}, ga_{n_i-1}, ga_{n_i-1}\right) + G_b\left(a_{n_i-1}, ga_{n_i-1}, ga_{n_i-1}\right) G_b\left(a_{n_i-1}, ga_{m_i}, ga_{m_i}\right)}{1 + s\left[G_b\left(a_{m_i}, ga_{m_i}, ga_{n_i-1}\right) + G_b\left(a_{n_i-1}, ga_{n_i-1}, ga_{n_i-1}\right)\right]}, \\ \frac{G_b\left(a_{m_i}, ga_{m_i}, ga_{m_i}\right) G_b\left(a_{m_i}, ga_{n_i-1}, ga_{n_i-1}\right) + G_b\left(a_{n_i-1}, ga_{n_i-1}, ga_{n_i-1}\right) G_b\left(a_{n_i-1}, ga_{m_i}, ga_{m_i}\right)}{1 + \left[G_b\left(a_{m_i}, ga_{n_i-1}, ga_{n_i-1}\right) + G_b\left(a_{n_i-1}, ga_{m_i}, ga_{n_i-1}\right)\right]}, \end{array} \right\} \\ &= \lim_{i\to\infty} \sup \max \left\{ \begin{array}{c} G_b\left(a_{m_i}, a_{n_i-1}, a_{n_i-1}\right), \\ \frac{G_b\left(a_{m_i}, a_{m_i+1}, a_{m_i+1}\right) G_b\left(a_{m_i}, a_{n_i}, a_{n_i}\right) + G_b\left(a_{n_i-1}, a_{n_i}, a_{n_i}\right) G_b\left(a_{n_i-1}, a_{m_i+1}, a_{m_i+1}\right)}{1 + s\left[G_b\left(a_{m_i}, a_{m_i+1}, a_{n_i}\right) + G_b\left(a_{n_i-1}, a_{n_i}, a_{n_i}\right)\right]}, \\ \frac{G_b\left(a_{m_i}, a_{m_i+1}, a_{m_i+1}\right) G_b\left(a_{m_i}, a_{n_i}, a_{n_i}\right) + G_b\left(a_{n_i-1}, a_{n_i}, a_{n_i}\right) G_b\left(a_{n_i-1}, a_{m_i+1}, a_{m_i+1}\right)}{1 + \left[G_b\left(a_{m_i}, a_{n_i}, a_{n_i}\right) + G_b\left(a_{n_i-1}, a_{m_i+1}, a_{n_i}\right)\right]}, \end{array} \right\} \leq \varepsilon. \end{aligned}$$

Note that, $m_i \neq n_i - 1$, as otherwise $G_b(a_{m_i}, a_{n_i-1}, a_{n_i-1}) = 0$ and so, by (3.11)

$$G_b(a_{m_i}, a_{m_i+1}, a_{m_i+1}) = G_b(a_{m_i}, ga_{m_i}, ga_{m_i}) = 0$$

which contradicts our assumption that $a_n \neq a_{n+1}$ for all $n \in N$. Hence, $\alpha(a_{m_i}, a_{n_i-1}, a_{n_i-1}) \geq 1$. Based on the assumption (3.11), (θ_1), $\alpha(a_{m_i}, a_{n_i-1}, a_{n_i-1}) \geq 1$, (3.8), (3.1) and the above inequality, we obtain that

$$\begin{aligned} \theta\left(s^2.\frac{\varepsilon}{s}\right) &\leq \alpha(a_{m_i}, a_{n_i-1}, a_{n_i-1})\,\theta\left(s^2.\lim_{i\to\infty}\sup G_b(a_{m_i+1}, a_{n_i}, a_{n_i})\right) \\ &= \alpha(a_{m_i}, a_{n_i-1}, a_{n_i-1})\,\theta\left(s^2.\lim_{i\to\infty}\sup G_b(ga_{m_i}, ga_{n_i-1}, ga_{n_i-1})\right) \\ &\leq \left[\theta\left(\lim_{i\to\infty}\sup M(a_{m_i}, a_{n_i-1}, a_{n_i-1})\right)\right]^r \leq [\theta(\varepsilon)]^r, \end{aligned}$$

which implies that $\theta(s\varepsilon) \leq [\theta(\varepsilon)]^r$, a contradiction. Now suppose that

$$\frac{1}{3s^2}G_b(a_{m_i+1}, ga_{m_i+1}, ga_{m_i+1}) \leq G_b(a_{m_i+1}, a_{n_i-1}, a_{n_i-1}) \tag{3.12}$$

holds for all $i \in \mathbb{N}$. Further, from (3.6) and using (G_b5), we get

$$\begin{aligned} \varepsilon \leq G_b(a_{m_i}, a_{n_i}, a_{n_i}) &\leq sG_b(a_{m_i}, a_{m_i+2}, a_{m_i+2}) + sG_b(a_{m_i+2}, a_{n_i}, a_{n_i}). \\ &\leq s^2G_b(a_{m_i}, a_{m_i+1}, a_{m_i+1}) + s^2G_b(a_{m_i+1}, a_{m_i+2}, a_{m_i+2}) \\ &+ sG_b(a_{m_i+2}, a_{n_i}, a_{n_i}). \end{aligned}$$

Taking the upper limit as $i \to \infty$, and using (3.5) we get

$$\frac{\varepsilon}{s} \leq \lim_{i\to\infty}\sup G_b(a_{m_i+2}, a_{n_i}, a_{n_i}). \tag{3.13}$$

Also, from (G_b5), we get

$$G_b(a_{m_i+1}, a_{n_i-1}, a_{n_i-1}) \leq sG_b(a_{m_i+1}, a_{n_i}, a_{n_i}) + sG_b(a_{n_i}, a_{n_i-1}, a_{n_i-1}).$$

Taking the upper limit as $i \to \infty$, and using (3.5) and (3.7) we get

$$\lim_{i\to\infty}\sup G_b(a_{m_i+1}, a_{n_i-1}, a_{n_i-1}) \leq s\varepsilon. \tag{3.14}$$

From the definition of $M(u, v, w)$ and using (3.5) and (3.14), we have
$\lim_{i\to\infty}\sup M(a_{m_i+1}, a_{n_i-1}, a_{n_i-1}) =$

$$\lim_{i\to\infty}\sup\max\left\{\begin{array}{c} G_b(a_{m_i+1}, a_{n_i-1}, a_{n_i-1}), \\ \frac{G_b(a_{m_i+1}, a_{m_i+2}, a_{m_i+2})G_b(a_{m_i+1}, a_{n_i}, a_{n_i}) + G_b(a_{n_i-1}, a_{n_i}, a_{n_i})G_b(a_{n_i-1}, a_{m_i+2}, a_{m_i+2})}{1+s[G_b(a_{m_i+1}, a_{m_i+2}, a_{n_i}) + G_b(a_{n_i-1}, a_{n_i}, a_{n_i})]}, \\ \frac{G_b(a_{m_i+1}, a_{m_i+2}, a_{m_i+2})G_b(a_{m_i+1}, a_{n_i}, a_{n_i}) + G_b(a_{n_i-1}, a_{n_i}, a_{n_i})G_b(a_{n_i-1}, a_{m_i+2}, a_{m_i+2})}{1+[G_b(a_{m_i+1}, a_{n_i}, a_{n_i}) + G_b(a_{n_i-1}, a_{m_i+2}, a_{n_i})]}, \end{array}\right\} \leq s\varepsilon.$$

Note that, $m_i + 1 \neq n_i - 1$, as otherwise

$$G_b(a_{m_i+1}, a_{n_i-1}, a_{n_i-1}) = 0$$

and so, by (3.12) $G_b(a_{m_i+1}, a_{m_i+2}, a_{m_i+2}) = G_b(a_{m_i+1}, ga_{m_i+1}, ga_{m_i+1}) = 0$, which contradicts our assumption that $a_n \neq a_{n+1}$ for all $n \in N$. Hence, $\alpha(a_{m_i+1}, a_{n_i-1}, a_{n_i-1}) \geq 1$.

Based on the assumption (3.12), (θ_1), $\alpha(a_{m_i+1}, a_{n_i-1}, a_{n_i-1}) \geq 1$, (3.13), (3.1) and the above inequality we obtain that

$$\begin{aligned} \theta\left(s^2.\frac{\varepsilon}{s}\right) &\leq \alpha(a_{m_i+1}, a_{n_i-1}, a_{n_i-1})\,\theta\left(s^2.\lim_{i\to\infty}\sup G_b(a_{m_i+2}, a_{n_i}, a_{n_i})\right) \\ &= \alpha(a_{m_i+1}, a_{n_i-1}, a_{n_i-1})\,\theta\left(s^2.\lim_{i\to\infty}\sup G_b(ga_{m_i+1}, ga_{n_i-1}, ga_{n_i-1})\right) \\ &\leq \left[\theta\left(\lim_{i\to\infty}\sup M(a_{m_i+1}, a_{n_i-1}, a_{n_i-1})\right)\right]^r \leq [\theta(s\varepsilon)]^r, \end{aligned}$$

a contradiction. Therefore, in all cases $\{a_n\}$ is a G_b-Cauchy sequence, thus by G_b-completeness of X yields that $\{a_n\}$ is G_b-convergent to a point $a^\star \in X$. By an argument similar to that in (3.10), we get either

$$\frac{1}{3s^2} G_b\left(a_n, ga_n, ga_n\right) \leq G_b\left(a_n, x^\star, a^\star\right)$$

or

$$\frac{1}{3s^2} G_b\left(a_{n+1}, ga_{n+1}, ga_{n+1}\right) \leq G_b\left(a_{n+1}, a^\star, a^\star\right)$$

holds for all $n \in \mathbb{N}$. First, suppose that

$$\frac{1}{3s^2} G_b\left(a_n, ga_n, ga_n\right) \leq G_b\left(a_n, a^\star, a^\star\right).$$

Now,

$$M\left(a_n, a^\star, a^\star\right) = \max\left\{\begin{array}{c} G_b\left(a_n, a^\star, a^\star\right), \frac{G_b(a_n,ga_n,ga_n)G_b\left(a_n,ga^\star,ga^\star\right)+G_b\left(a^\star,ga^\star,ga^\star\right)G_b\left(a^\star,ga_n,ga_n\right)}{1+s[G_b(a_n,ga_n,ga^\star)+G_b(a^\star,ga^\star,ga^\star)]}, \\ \frac{G_b(a_n,ga_n,ga_n)G_b\left(a_n,ga^\star,ga^\star\right)+G_b\left(a^\star,ga^\star,ga^\star\right)G_b\left(a^\star,ga_n,ga_n\right)}{1+[G_b(a_n,ga^\star,ga^\star)+G_b(a^\star,ga_n,ga^\star)]} \end{array}\right\}$$

So, $\lim_{n\to\infty} M\left(a_n, a^\star, a^\star\right) = 0$. Hence from (3.1) and assertion (ii) of the theorem, we have

$$\begin{aligned} 1 \leq \theta\left(G_b\left(ga_n, ga^\star, ga^\star\right)\right) &\leq \theta\left(s^2 G_b\left(ga_n, ga^\star, ga^\star\right)\right) \\ &\leq \alpha(a_n, a^\star, a^\star)\theta\left(s^2 G_b\left(ga_n, ga^\star, ga^\star\right)\right) \\ &\leq \left[\theta\left(M\left(a_n, a^\star, a^\star\right)\right)\right]^r \end{aligned}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, in the above inequality we get that

$$\lim_{n\to\infty} \theta\left(G_b\left(ga_n, ga^\star, ga^\star\right)\right) = 1.$$

This implies by (Θ_1) that

$$\lim_{n\to\infty} G_b\left(ga_n, ga^\star, ga^\star\right) = 0.$$

Hence, $ga^\star = \lim_{n\to\infty} ga_n = \lim_{n\to\infty} a_{n+1} = a^\star$. Thus, we deduce that $ga^\star = a^\star$.

Now if

$$\frac{1}{3s^2} G_b\left(a_{n+1}, ga_{n+1}, ga_{n+1}\right) \leq G_b\left(a_{n+1}, a^\star, a^\star\right),$$

holds, then by repeating the same process as above we can get $ga^\star = a^\star$. Therefore, we proved that $a^\star$ is a fixed point of g.

Now to prove uniqueness, suppose there exist $u, v \in Fix(g)$ with $u \neq v$, that is $u = gu$ and $v = gv$. Therefore by (iii), $\alpha(u, v, v) \geq 1$ and so, by (3.1) and (G_{b2}) we have

$$0 = \frac{1}{3s^2} G(u, gu, gu) \leq G(u, v, v)$$

and

$$\begin{aligned} \theta(G_b(u, v, v)) &\leq \alpha(u, v, v)\theta(s^2 G_b(gu, gv, gv)) \\ &\leq \left[\theta(M(u, v, v))\right]^r \\ &= \left[\theta(G_b(u, v, v))\right]^r \\ &< \theta(G_b(u, v, v)). \end{aligned}$$

Thus the contradiction implies that the fixed point is unique. □

Theorem 3.16. *Let (X, G_b) be a G_b-complete metric space with $s > 1$. Let $\alpha : X \times X \times X \to (0, \infty)$ and g be a rectangular α-admissible mapping. Suppose that there exist $\theta \in \Omega$ and $r \in (0, 1)$ such that*

$$\frac{1}{3s^2} G_b(u, gu, gu) \le G_b(u, v, w) \Rightarrow \alpha(u, v, w)\, \theta\left(s^2 G_b(gu, gv, gw)\right) \le [\theta(M(u, v, w))]^r \tag{3.15}$$

for all $x, y, z \in X$ with at least two of gx, gy and gz being not equal, where

$$M(u, v, w) = \max\left\{G_b(u, v, w), \frac{G_b(u, gu, gu)\, G_b(v, gv, gw)}{1 + G_b(u, v, w)}, \frac{G_b(u, gu, gu)\, G_b(v, gv, gw)}{1 + G_b(gu, gv, gw)}\right\}.$$

Also, suppose that the following assertions hold:
(i) There exists $a_0 \in X$ such that $\alpha(a_0, ga_0, ga_0) \ge 1$.
(ii) For any convergent sequence $\{a_n\}$ to a with $\alpha(a_n, a_{n+1}, a_{n+1}) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$, we have $\alpha(a_n, a, a) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$.
Then g has a fixed point.
(iii) Moreover, if for all $u, v \in Fix(g)$ implies $\alpha(u, v, v) \ge 1$, then the fixed point is unique where $Fix(g) = \{u; gu = u\}$.

Proof. Let $a_0 \in X$ be such that $\alpha(a_0, ga_0, ga_0) \ge 1$. Define a sequence $\{a_n\}$ by $a_n = g^n a_0$ for all $n \in \mathbb{N}$. Since g is an α-admissible mapping and $\alpha(a_0, a_1, a_1) = \alpha(a_0, ga_0, ga_0) \ge 1$, we deduce that $\alpha(a_1, a_2, a_2) = \alpha(ga_0, ga_1, ga_1) \ge 1$. Continuing this process, we get that $\alpha(a_n, a_{n+1}, a_{n+1}) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$. Without loss of generality, assume that $a_n \ne a_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. We shall show that $\lim_{n\to\infty} G_b(a_{n+1}, a_n, a_n) = 0$. Now,

$$\begin{aligned} M(a_{n-1}, a_n, a_n) &= \max\left\{\begin{array}{c} G_b(a_{n-1}, a_n, a_n), \frac{G_b(a_{n-1}, ga_{n-1}, ga_{n-1}) G_b(a_n, ga_n, ga_n)}{1+G_b(a_{n-1}, a_n, a_n)}, \\ \frac{G_b(a_{n-1}, ga_{n-1}, ga_{n-1}) G_b(a_n, ga_n, ga_n)}{1+G_b(ga_{n-1}, ga_n, ga_n)} \end{array}\right\} \\ &= \max\left\{\begin{array}{c} G_b(a_{n-1}, a_n, a_n), \frac{G_b(a_{n-1}, a_n, a_n) G_b(a_n, a_{n+1}, a_{n+1})}{1+G_b(a_{n-1}, a_n, a_n)}, \\ \frac{G_b(a_{n-1}, a_n, a_n) G_b(a_n, a_{n+1}, a_{n+1})}{1+G_b(a_n, a_{n+1}, a_{n+1})} \end{array}\right\} \end{aligned} \tag{3.16}$$

Since, $\frac{G_b(a_{n-1}, a_n, a_n)}{1+G_b(a_{n-1}, a_n, a_n)} < 1$ and $\frac{G_b(a_n, a_{n+1}, a_{n+1})}{1+G_b(a_n, a_{n+1}, a_{n+1})} < 1$. Therefore,

$$M(a_{n-1}, a_n, a_n) = \max\{G_b(a_{n-1}, a_n, a_n), G_b(a_n, a_{n+1}, a_{n+1})\}.$$

If $\max\{G_b(a_{n-1}, a_n, a_n), G_b(a_n, a_{n+1}, a_{n+1})\} = G_b(a_n, a_{n+1}, a_{n+1})$, then since $\alpha(a_{n-1}, a_n, a_n) \ge 1$ for each $n \in \mathbb{N}$, $\frac{1}{3s^2} G_b(a_{n-1}, ga_{n-1}, ga_{n-1}) \le G_b(a_{n-1}, a_n, a_n)$ and so by (3.15) we have

$$\begin{aligned} \theta(G_b(a_n, a_{n+1}, a_{n+1})) &= \theta(G_b(ga_{n-1}, ga_n, ga_n)), \\ &\le \alpha(a_{n-1}, a_n, a_n)\, \theta\left(s^2 G_b(ga_{n-1}, ga_n, ga_n)\right), \\ &\le [\theta(M(a_{n-1}, a_n, a_n))]^r, \\ &= [\theta(G_b(a_n, a_{n+1}, a_{n+1}))]^r \\ &< \theta(G_b(a_n, a_{n+1}, a_{n+1})) \end{aligned} \tag{3.17}$$

which is a contradiction since $r \in (0, 1)$. Thus, $M(a_{n-1}, a_n, a_n) = G_b(a_{n-1}, a_n, a_n)$.

The rest of the proof is the same as the proof of Theorem 3.15. □

Analogously, we can prove the following theorem.

Theorem 3.17. *Let (X, G_b) be a complete G_b– metric space with $s > 1$. Let $\alpha : X \times X \times X \to (0, \infty)$ and g be a rectangular α-admissible mapping. Suppose that there exist $\theta \in \Omega$ and $r \in (0, 1)$ such that*

$$\frac{1}{3s^2} G_b(u, gu, gu) \le G_b(u, v, w) \Rightarrow \alpha(u, v, w)\, \theta\left(s^2 G_b(gu, gv, gw)\right) \le [\theta(M(u, v, w))]^r$$

for all $u, v, w \in X$ with at least two of gu, gv and gw are not equal, where

$$M(u,v,w) = \max\left\{ \begin{array}{c} G_b(u,v,w), \frac{G_b(u,gu,gu)G_b(v,gv,gw)}{1+s[G_b(u,v,w)+G_b(v,gu,gu)+G_b(u,gv,gv)]}, \\ \frac{G_b(u,gv,gv)G_b(u,v,w)}{1+sG_b(u,gu,gu)+s^2[G_b(v,gv,gv)+G_b(v,gu,gu)]} \end{array} \right\}.$$

Also, suppose that the following assertions hold:
(i) There exists $a_0 \in X$ such that $\alpha(a_0, ga_0, ga_0) \geq 1$;
(ii) For any convergent sequence $\{a_n\}$ to a with $\alpha(a_n, a_{n+1}, a_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, we have $\alpha(a_n, a, a) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.
Then g has a fixed point.
(iii) Moreover, if for all $u, v \in Fix(g)$ implies $\alpha(u,v,v) \geq 1$, then the fixed point is unique where $Fix(g) = \{u; gu = u\}$.

Now, we give an example to support Theorem 3.1

Example 3.18. Let $X = [0,\infty)$ and $G_b : X \times X \times X \to R$ be a G_b-metric space defined by $G_b(u,v,w) = (|u-v| + |v-w| + |u-w|)^2$. Clearly (X, G_b) is a complete G_b-metric space with $s = 2$. Also let $r = \frac{3}{5}$ and define $g : X \to X$, $\alpha : X \times X \times X \to R$ and $\theta : [0,\infty) \to [1,\infty)$ by

$$g(x) = \begin{cases} \frac{x}{5}, & \text{if } x \in [0,1] \\ x^2, & \text{otherwise,} \end{cases}$$

$$\alpha(u,v,w) = \begin{cases} 1, & \text{if } u,v,w \in [0,1] \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\theta(t) = e^t.$$

Assume that $\frac{1}{12}G_b(u,gu,gu) \leq G_b(u,v,w)$. If one of $u, v, w \notin [0,1]$, then $\alpha(u,v,w) = 0$ and so, the conclusion of (3.1) is satisfied. If $u, v, w \in [0,1]$, then $gu, gv, gw \in [0,1]$ and $\alpha(u,v,w) \geq 1$ with $gu \neq gv \neq gw$. Hence,

$$\begin{aligned} \alpha(u,v,w)\theta(4G_b(gu,gv,gw)) &= e^{4(\frac{1}{5}(|u-v|+|v-w|+|u-w|))^2} \\ &= e^{\frac{4}{25}(|u-v|+|v-w|+|u-w|)^2} \\ &\leq e^{(3/5)(|u-v|+|v-w|+|u-w|)^2} \\ &= \left(e^{(|u-v|+|v-w|+|u-w|)^2}\right)^{\frac{3}{5}} \\ &= \left(e^{G_b(u,v,w)}\right)^{\frac{3}{5}} \\ &= \left(\theta(G_b(u,v,w))\right)^{\frac{3}{5}}. \end{aligned}$$

Thus all conditions of Theorem 3.15 are satisfied and $x = 0$ is the unique fixed point of g.

Corollary 3.19. *Let (X, G_b) be a complete G_b- metric space with $s > 1$. Let $\alpha : X \times X \times X \to (0,\infty)$ and g be a rectangular α-admissible mapping. Suppose that there exist $\theta \in \Omega$ and $r, \delta, \beta, \gamma \in (0,1)$ with $\delta + \beta + \gamma < 1$ such that*

$$\begin{aligned} &\frac{1}{3s^2}G_b(u,gu,gu) \leq G_b(u,v,w) \Rightarrow \alpha(u,v,w)\,\theta\left(s^2 G_b(gu,gv,gw)\right) \\ \leq\; &\left[\theta\left(\delta G_b(u,v,w) + \beta\frac{G_b(u,gu,gu)\,G_b(v,gv,gw)}{1+G_b(u,v,w)} + \gamma\frac{G_b(u,gu,gu)\,G_b(v,gv,gw)}{1+G_b(gu,gv,gw)}\right)\right]^r \end{aligned}$$

for all $u, v, w \in X$ with at least two of gu, gv and gw being not equal. Also, suppose that the following assertions hold:

(i) There exists $a_0 \in X$ such that $\alpha(a_0, ga_0, ga_0) \geq 1$;
(ii) For any convergent sequence $\{a_n\}$ to a with $\alpha(a_n, a_{n+1}, a_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$, we have $\alpha(a_n, a, a) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.
Then g has a fixed point.
(iii) Moreover, if for all $u, v \in Fix(g)$ implies $\alpha(u, v, v) \geq 1$, then the fixed point is unique where $Fix(g) = \{u; gu = u\}$.

Corollary 3.20. *Let (X, G_b) be a complete G_b-metric space with $s > 1$. Let $\alpha : X \times X \times X \to (0, \infty)$ and g be a rectangular α-admissible mapping. Suppose that there exist $\theta \in \Omega$ and $r, \delta, \beta, \gamma \in (0, 1)$ with $\delta + \beta + \gamma < 1$ such that*

$$\begin{aligned} \frac{1}{3s^2} G_b(u, gu, gu) &\leq G_b(u, v, w) \Rightarrow \alpha(u, v, w)\, \theta\left(s^2 G_b(gu, gv, gw)\right) \\ &\leq \left[\theta\left(\begin{array}{c} \delta G_b(u, v, w) + \beta \frac{G_b(u,gu,gu)G_b(u,gv,gw)+G_b(v,gv,gw)G_b(v,gu,gu)}{1+s[G_b(u,gu,gw)+G_b(v,gv,gw)]} \\ +\gamma \frac{G_b(u,gu,gu)G_b(u,gv,gw)+G_b(v,gv,gw)G_b(v,gu,gu)}{1+G_b(u,gv,gw)+G_b(v,gu,gw)} \end{array}\right)\right]^r \end{aligned}$$

for all $u, v, w \in X$ with at least two of gu, gv and gw being not equal. Also, suppose that the following assertions hold:
(i) There exists $a_0 \in X$ such that $\alpha(a_0, ga_0, ga_0) \geq 1$;
(ii) for any convergent sequence $\{a_n\}$ to a with $\alpha(a_n, a_{n+1}, a_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$, we have $\alpha(a_n, a, a) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.
Then g has a fixed point.
(iii) Moreover, if for all $u, v \in Fix(g)$ implies $\alpha(u, v, v) \geq 1$, then the fixed point is unique where $Fix(g) = \{u; gu = u\}$.

Corollary 3.21. *Let (X, G_b) be a complete G_b- metric space with $s > 1$. Let $\alpha : X \times X \times X \to (0, \infty)$ and g be a rectangular α-admissible mapping. Suppose that there exist $\theta \in \Omega$ and $r, \delta, \beta, \gamma \in (0, 1)$ with $\delta + \beta + \gamma < 1$ such that*

$$\begin{aligned} \frac{1}{3s^2} G_b(u, gu, gu) &\leq G_b(u, v, w) \Rightarrow \alpha(u, v, w)\, \theta\left(s^2 G_b(gu, gv, gw)\right) \\ &\leq \left[\theta\left(\begin{array}{c} \delta G_b(u, v, w) + \beta \frac{G_b(u,gu,gu)G_b(v,gv,gw)}{1+s[G_b(u,v,w)+G_b(u,gv,gv)+G_b(v,gu,gu)]} \\ +\gamma \frac{G_b(u,gv,gv)G_b(u,v,w)}{1+sG_b(u,gu,gu)+s^2[G_b(v,gu,gu)+G_b(v,gv,gv)]} \end{array}\right)\right]^r \end{aligned}$$

for all $u, v, w \in X$ with at least two of gu, gv and gw being not equal. Also, suppose that the following assertions hold:
(i) There exists $a_0 \in X$ such that $\alpha(a_0, ga_0, ga_0) \geq 1$;
(ii) For any convergent sequence $\{a_n\}$ to a with $\alpha(a_n, a_{n+1}, a_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, we have $\alpha(a_n, a, a) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.
Then g has a fixed point.
(iii) Moreover, if for all $u, v \in Fix(g)$ implies $\alpha(u, v, v) \geq 1$, then the fixed point is unique where $Fix(g) = \{u; gu = u\}$.

Taking $\theta(t) = e^t$ for all $t > 0$, in the above corollaries we get the following new results.

Corollary 3.22. *Let (X, G_b) be a complete G_b- metric space with $s > 1$. Let $\alpha : X \times X \times X \to (0, \infty)$ and g be a rectangular α-admissible mapping. Suppose that there exist $\theta \in \Omega$ and $r, \delta, \beta, \gamma \in (0, 1)$ with $\delta + \beta + \gamma < 1$ such that*

$$\begin{aligned} \frac{1}{3s^2} G_b(u, gu, gu) &\leq G_b(u, v, w) \Rightarrow \ln \alpha(u, v, w) + s^2 G_b(gu, gv, gw) \\ &\leq r\left[\delta G_b(u, v, w) + \beta \frac{G_b(u, gu, gu)\, G_b(v, gv, gw)}{1 + G_b(u, v, w)} + \gamma \frac{G_b(u, gu, gu)\, G_b(v, gv, gw)}{1 + G_b(gu, gv, gw)}\right] \end{aligned}$$

for all $u, v, w \in X$ with at least two of gu, gv and gw being not equal. Also, suppose that the following assertions hold:

(i) *There exists* $a_0 \in X$ *such that* $\alpha(a_0, ga_0, ga_0) \geq 1$;
(ii) *For any Convergent sequence* $\{a_n\}$ *to* a *with* $\alpha(a_n, a_{n+1}, a_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$, *we have* $\alpha(a_n, a, a) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$.
Then g has a fixed point.
(iii) *Moreover, if for all* $u, v \in Fix(g)$ *implies* $\alpha(u, v, v) \geq 1$, *then the fixed point is unique where* $Fix(g) = \{u; gu = u\}$.

Corollary 3.23. *Let* (X, G_b) *be a complete* G_b*- metric space (with parameter* $s > 1$). *Let* $\alpha : X \times X \times X \to (0, \infty)$ *and* g *be a rectangular* α*-admissible mapping. Suppose that there exist* $\theta \in \Omega$, *and* $r, \delta, \beta, \gamma \in (0, 1)$ *with* $\delta + \beta + \gamma < 1$ *such that*

$$\begin{aligned} \frac{1}{3s^2} G_b(u, gu, gu) &\leq G_b(u, v, w) \Rightarrow \ln \alpha(u, v, w) + s^2 G_b(gv, gv, gw) \\ &\leq r \begin{bmatrix} \delta G_b(u, v, w) + \beta \frac{G_b(u,gu,gu)G_b(u,gv,gw)+G_b(v,gv,gw)G_b(v,gu,gu)}{1+s[G_b(u,gu,gw)+G_b(v,gv,gw)]} \\ +\gamma \frac{G_b(u,gu,gu)G_b(u,gv,gw)+G_b(v,gv,gw)G_b(v,gu,gu)}{1+G_b(u,gv,gw)+G_b(v,gu,gw)} \end{bmatrix} \end{aligned}$$

for all $u, v, w \in X$ *with at least two of* gu, gv *and* gw *being not equal. Also, suppose that the following assertions hold:*
(i) *There exists* $a_0 \in X$ *such that* $\alpha(a_0, ga_0, ga_0) \geq 1$;
(ii) *For any Convergent sequence* $\{a_n\}$ *to* a *with* $\alpha(a_n, a_{n+1}, a_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$, *we have* $\alpha(a_n, a, a) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$.
Then g has a fixed point.
(iii) *Moreover, if for all* $u, v \in Fix(g)$ *implies* $\alpha(u, v, v) \geq 1$, *then the fixed point is unique where* $Fix(g) = \{u; gu = u\}$.

Corollary 3.24. *Let* (X, G_b) *be a complete* G_b*-complete metric space with* $s > 1$. *Let* $\alpha : X \times X \times X \to (0, \infty)$ *and* g *be a rectangular* α*-admissible mapping. Suppose that there exist* $\theta \in \Omega$, *and* $r, \delta, \beta, \gamma \in (0, 1)$ *with* $\delta+\beta+\gamma < 1$ *such that*

$$\begin{aligned} \frac{1}{3s^2} G_b(u, gu, gu) &\leq G_b(u, v, w) \Rightarrow \ln \alpha(u, v, w) + s^2 G_b(gu, gv, gw) \\ &\leq r \begin{bmatrix} \delta G_b(u, v, w) + \beta \frac{G_b(u,gu,gu)G_b(v,gv,gw)}{1+s[G_b(u,v,w)+G_b(u,gv,gv)+G_b(v,gu,gu)]} \\ +\gamma \frac{G_b(u,gv,gv)G_b(u,v,w)}{1+sG_b(u,gu,gu)+s^2[G_b(v,gu,gu)+G_b(v,gv,gv)]} \end{bmatrix} \end{aligned}$$

for all $u, v, w \in X$ *with at least two of* gu, gv *and* gw *being not equal. Also, suppose that the following assertions hold:*
(i) *There exists* $a_0 \in X$ *such that* $\alpha(a_0, ga_0, ga_0) \geq 1$;
(ii) *For any convergent sequence* $\{a_n\}$ *to* a *with* $\alpha(a_n, a_{n+1}, a_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$, *such that* $a_n \to x$ *as* $n \to \infty$, *we have* $\alpha(a_n, a, a) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$.
Then g has a fixed point.
(iii) *Moreover, if for all* $u, v \in Fix(g)$ *implies* $\alpha(u, v, v) \geq 1$, *then the fixed point is unique where* $Fix(g) = \{u; gu = u\}$.

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