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COLLEGE OF ARTS AND SCIENCES

# CUMULATIVE EXPOSURE LOGNORMAL MODEL WITH HYBRID CENCORING

BY

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# ABSTRACT

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Title: CUMULATIVE EXPOSURE LOGNORMAL MODEL WITH HYBRID CENSORING

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This research aims to analyze data coming from step stress life testing experiments that are commonly used to make inferences on the reliability of products and machines. Customers expect a reliable product that can still perform its functions for a long period of time. For this reason, factories are pressured to design and make products that can operate for a long enough period of time while performing its functions. Step stress experiments are accelerated experiments for which the stress level increases at a preset time to obtain failure data faster and make the necessary analysis. To analyze step stress data, a model that extrapolates the information obtained from the accelerated tests to normal use conditions needs to be fit to the life test data. In this study, we will use the Cumulative Exposure Model (CEM) to analyze simple step stress lognormal life test data and estimate the model parameter and survival function in the case where hybrid censoring is present in the data. This study uses the maximum likelihood estimation method and the Maximum Likelihood Estimators (MLEs) properties to find the point and interval estimates of the parameters, in addition to finding the point and interval estimates for the survival function. The MLEs are obtained numerically since the ML equations cannot be found explicitly. The approximate confidence interval for estimating the model parameters was constructed based on the asymptotic property of the MLEs. To obtain the approximate confidence interval for estimating the survival function, the delta method is used. The bootstrap-t intervals and percentile intervals were also constructed to estimate the model parameters and survival function. Furthermore, a simulation study has been performed to examine the proposed methods of estimation under different hybrid censoring schemes. The Bias, MSE, coverage probability and average lengths have been calculated to study and compare the performance of the point and interval estimators of the model parameters and survival function. Finally, an illustrative example has been made to view and illustrate how the proposed methods work.

Key words: Maximum Likelihood Estimation (MLE), Cumulative Exposure Model (CEM), simple step stress, hybrid censoring

DEDICATION

*I dedicate this thesis to my friends, family and professor who supported me throughout the journey of writing my thesis.*

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I would like to give my thanks and appreciation to my supervisor professor Ayman Baklizi; he was very patient, kind and supportive throughout the stages of working on my thesis and I believe that I couldn’t have done my thesis if it wasn’t for his continuous support and very helpful feedback.

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# CHAPTER 1: INTRODUCTION

## 1.1 Life Testing

In life testing experiments, a sample of items of interest is placed on a test where the failure times of these items are recorded to analyze the life test data and to estimate the reliability. Reliability in statistics is the probability of the product or unit to do its intended functions under specific conditions for a specified time. The reliability data can be used to test and predict the reliability of some product, measure the time to failure, examine the product's features over the warranty period or predict warranty costs.

## 1.2 Accelerated Life Tests

Due to the competing nature of market today, there is a lot of pressure for manufactures to produce and develop higher technology products in a record time and enhance the productivity, reliability, and overall quality of these products. Therefore, designed experiments are used to improve the quality and reliability of the products. In order to achieve higher reliability for products, it is essential to perform tests on the components and systems. It is difficult to practically test the reliability and performance of highly reliable products that have a long mean lifetime to failure under normal circumstances. Instead, tests running at stress higher than in normal conditions (accelerated tests) are used to obtain information on reliability in a limited time. Such tests use accelerating variables such as use-rate, temperature, voltage, or pressure to obtain more failures in a practical time and therefore statistical models are used to extrapolate the information obtained from accelerated tests to normal use conditions. The results from accelerated tests are then used to assess the product's reliability, detect failure modes, and compare the product from different manufacturers (Meeker W Q, 1998). Overstress testing is one of the most common accelerated life tests which are used to shorten the lifetime of the product (Nelson, 2004). Some of the overstressing tests are stress loading, constant stress, and step stress.

### 1.3 Step Stress Accelerated Life Testing (SSALT)

In overstress testing, the product is run at increasing levels of accelerated stress to shorten the product's life which in return results in having more failure information in shorter time (Nelson, 2004). The failure information can then be analyzed to make estimates regarding the product's reliability. SSALT is an overstress testing in which the product is exposed to higher successive stress levels. If n testing units are put on a SSALT experiment, then the units under the life test are first run on an initial stress $x_0$ and the stress changes to higher stress levels $x_1 < x_2 < \cdots < x_k$ at specified times $\tau_1 < \tau_2 < \cdots < \tau_{k-1}$ , where $\tau_i$ is the preset time to change the stress level from $x_i$ to $x_{i+1}$. Step stress testing is commonly used as it yields more failures in less time and increasing stress levels ensures obtaining failures faster than normal use conditions. Simple step stress accelerated life testing (simple SSALT) is a special case of SSALT where the stress only changes once thereby having only two stress levels: the initial stress level $x_0$ and $x_1$. The units under the simple SSALT are first run under normal stress until time $\tau$ at which time the stress level increases to a stress level $x_1$. Since products usually run on a constant stress, a statistical model must consider the effect of being exposed to successive stress levels and provide estimates under normal use conditions. A number of models were proposed to extrapolate the information observed from accelerated tests to normal use conditions. Such models explain the effect of changing the stress level on the residual lifetime of the experimental testing units.

Four basic models have been proposed to deal with this issue: the tampered random variable model (TRVM), the tampered failure rate model (TFRM), the linear cumulative exposure model (LCEM) and the cumulative exposure model (CEM). In our research we will use the cumulative exposure model to extrapolate the information taken from simple step stress life testing data to normal use conditions. We briefly introduce the TRVM, TFRM, LCEM and show their mathematical formulas and then introduce the CEM and talk about it in more details.

The Tampered Random Variable Model (TRVM)

The TRVM by DeGroot and Goel (1979) is also called the additive accumulative of damages model. The effect of increasing the stress level is expressed mathematically by multiplying the remaining lifetime of the units by a tampering /acceleration factor.

$$F_2(t) = F_1(\gamma t)$$

Where $\gamma$ is the tampering or acceleration factor

The mathematical expression is given below:

$$F_{TRVM}(t) = \begin{cases} F_1(t) & t < \tau \\ F_2(t - \tau + \tau/\gamma) & t > \tau \end{cases}$$

The Tampered Failure Rate Model (TFRM )

The TFRM by Bhattacharrya and Soejoeti (1989) is also known as the proportional hazard model (PHM). The failure rate is given at the initial stress level and the effect of changing the stress is multiplying a factor to the current failure rate. According to this model, the failure rate is dependent only on the present stress and the overall time in which the unit has been exposed to stress. The mathematical expression of this model is shown below:

$$\lambda_{TFRM} = \begin{cases} \lambda_1(t) & t \le \tau \\ \alpha\lambda_1(t) & t > \tau \end{cases}$$

Where $\lambda$ is the failure rate and $\alpha$ is the tampering factor.

The Linear Cumulative Exposure Model (LCEM)

This model is by Tang et al, 1996 and Tang (2003). In this model the accumulated exposure of the units at each stress is assumed to be linear and it is the ratio of the actual time in which the units has been operating at the current stress level to its lifetime. The units fail when its cumulative exposure reaches 1. The mathematical expression of the LCEM is as follows:

$$F_{LCEM}(t) = \begin{cases} F_1(t) & t \leq \tau \\ F_2\left(t - \tau + \dfrac{t(2,R)}{t(1,R)}\tau\right) & t > \tau \end{cases}$$

Where $t(i,R)$ is the lifetime of a unit T at stress $S_i$ and R is the reliability.

1.4 The Cumulative Exposure Model (CEM)

According to Nelson (1980), this model mainly assumes that the residual lifetime of the testing units is dependent on the cumulative exposure regardless of how this exposure came to be. Increasing the stress results in changing the lifetime distribution from $F_2(t)$ to $F_2(t - \tau + \tau^*)$. The CEM is the most widely used model in analyzing SSALT data. The mathematical expression under simple SSALT:

$$F_{CEM}(t) = \begin{cases} F_1(t), & t < \tau \\ F_2(t - \tau + \tau^*), & t > \tau \end{cases}$$

Where $\tau^*=F_2^{-1}(F_1(t))$ is the corresponding testing time of $\tau$ under the higher stress level.

1.5 Censoring Schemes

In the life tests conducted to assess the reliability of products, there are two possible scenarios: if all the units under the life test fail by the end of the experiment and the exact failure times for all the units are observed, we will have complete data. If however the exact failure times for some units are unknown for any reason, then we are dealing with censoring. According to Nelson (2004), censoring will arise in the situation in which some testing units have been withdrawn from the experiment, as a result their exact failure time cannot be observed or if the experiment ended with some test units not failing, thus their failure time is only known to be beyond the experiment time. It is common for censoring to be present in lifetimes experiments. There are different censoring schemes, however for the sake of our research we will only address hybrid censoring and the two special cases of hybrid censoring.

Type-I censoring scheme: It occurs when the lifetime experiment is set to be terminated at a time prespecified by the researcher or experimenter. In this case, the number of observed failures is random.

Type-II censoring scheme: It occurs in the situation where the experimenter sets the number of failures he wants to observe before terminating the experiment, making the total experiment time a random variable.

Hybrid Censoring Scheme

The hybrid censoring scheme introduced by Epstein (1954) is a mix of both type-I and type-II censoring schemes where the life testing experiment ends when a prefixed number of failures r has occurred or when a prefixed time $t_1$ set by the experimenter has been reached, whichever comes first. If r failures have been reached before the prefixed time $t_1$, the experiment will terminate at the time of failure for the $r^{th}$ unit otherwise the experiment will terminate at time $t_1$. This scheme is also known as type-I hybrid censoring scheme. According to Balakrishnan and Kundu (2013), an advantage of type-I HSC is that the termination time of the experiment is preset by the experimenter. When we are dealing with hybrid censored life testing data, we need to assume that the experiment has at least one observed failure.

## 1.6 The Lognormal Distribution

The lognormal distribution is a commonly used distribution for life test data. It has been used extensively to describe time to breakage due to fatigue crack growth in metals. It is also used as the time to failure distribution of some degradation processes. The lognormal distribution is flexible which makes it suitable for many products. The lognormal hazard has a property that the hazard is zero at time zero then it starts to increase up to a maximum and decreases back to zero with time. Therefore, this distribution is often used to model the population of electronic components that has a decreasing hazard function. The lognormal distribution is called the two-parameter lognormal distribution.

The pdf, CDF, survival function and hazard function of the lognormal distribution are expressed mathematically as follows:

$$f(t;\mu,\sigma) = \frac{1}{\sigma t\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\log t-\mu}{\sigma}\right)^2}, t > 0, -\infty < \mu < \infty, \sigma > 0$$

$$F(t;\mu,\sigma) = \Phi\left(\frac{\log t-\mu}{\sigma}\right),\ t > 0, -\infty < \mu < \infty, \sigma > 0$$

$$S(t) = 1 - \Phi\left(\frac{\log t-\mu}{\sigma}\right),\ t > 0, -\infty < \mu < \infty, \sigma > 0$$

$$h(t) = \frac{\frac{1}{\sigma t\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\log t-\mu}{\sigma}\right)^2}}{1 - \Phi\left(\frac{\log t-\mu}{\sigma}\right)},\ t > 0, -\infty < \mu < \infty, \sigma > 0$$

Where $\mu$ is the mean of the log lifetime data and thereby the log mean of the lifetime data, $\sigma$ is the log standard deviation, and $\Phi$ is the standard normal cumulative distribution function. The mean lifetime for units coming from the lognormal is $e^{\mu+\frac{\sigma^2}{2}}$. The pdf, survival function and hazard function of the lognormal distribution with mean equal to 0 and a standard deviation having values ranging from 1- 4 are graphed and shown in figures 1-3 respectively.

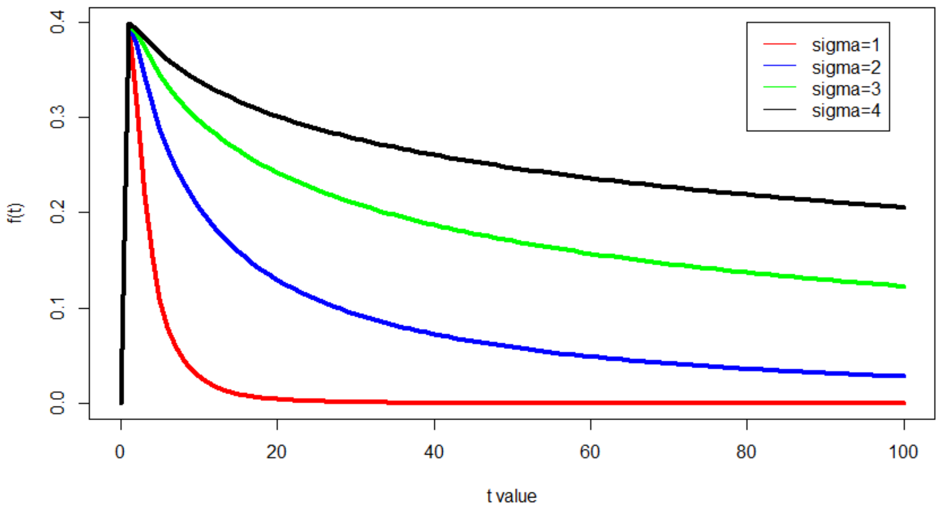


Figure1. The pdf of the lognormal distribution with different scale parameter values

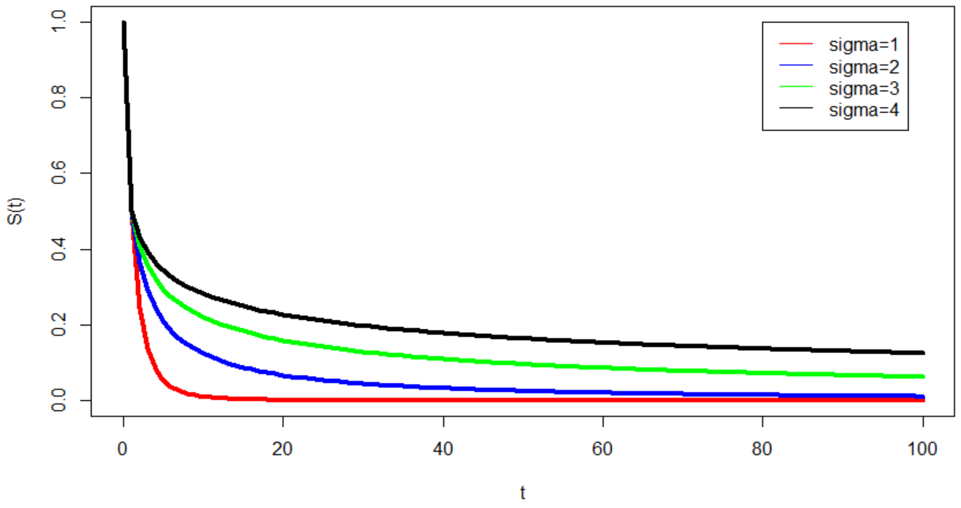


Figure 2. The lognormal survival function with different scale parameter values

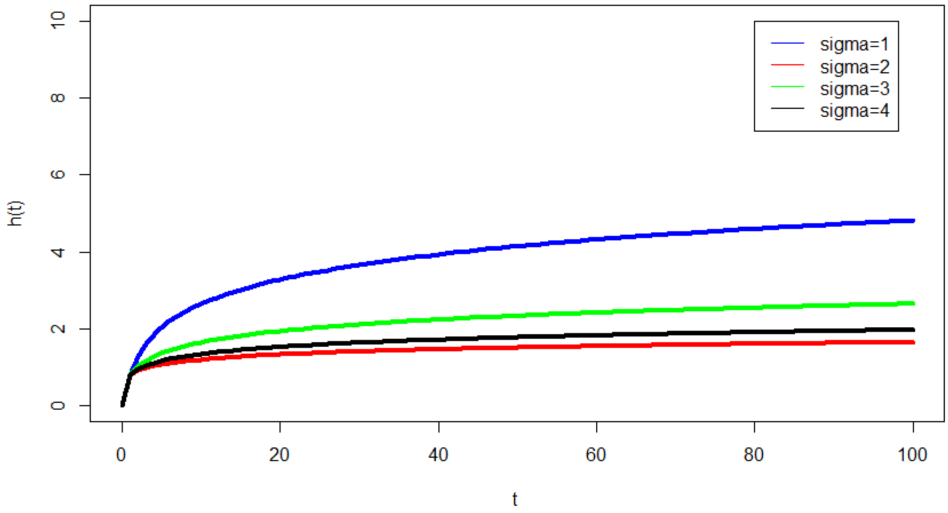


Figure 3. The lognormal hazard function with different scale parameter values

## 1.7 The Statement of the Research Problem

Many products produced nowadays are highly reliable under normal use conditions. Therefore it is nearly impossible to observe failure information for these products in life tests under normal use conditions as few to non-failures will be observed due to the experiment's time restriction. To assess the performance of such highly reliable products that may have a mean time to failure up to 100,000 hours, step stress life tests are used. In Step stress accelerated life tests, the stress level changes after a prespecified period during the life testing, resulting in more test units' failures which guarantees more failure in less time. To extrapolate from the information obtained under overstress to normal use conditions, the cumulative exposure model is used. In lifetime experiments, some test units will still be functioning when the lifetime experiment is over, and thus the lifetime of these units are only known to be greater than the experiment time (censoring). In this study we will estimate the parameters of the lognormal simple step stress data with hybrid censoring

## 1.8 Literature Review

We are mainly interested in reviewing research papers in which the cumulative exposure model has been fit to lognormally distributed step stress data to estimate the model parameters. We first review a research paper introducing four main models that has been used to model step stress data and methods for choosing the best model to fit the step stress data.

Comparison between the four step stress models

Xu and Fei (2012) presented and compared between four basic models for fitting step stress life test data TRVM, CEM, TFRM and LCEM. It was noticed that the difference between these four models lies in the expression of the equivalent operating time of the stress changing time. The equivalent operating stress time is equal for the CEM, LCEM and TRVM if the tampering coefficient $\gamma$ in the TRVM is constant (in other words if it is independent from the stress changing time $\tau$). The authors then proceeded with comparing the TFRM with the TFRM. It was concluded that both models are identical only when the lifetime distribution is exponential. The best model was selected according to a selection criterion that a higher log likelihood value means the best model. The authors performed a simulation study to examine the performance of the CEM and TFRM when the distribution is Weibull. Finally, they discussed the limitations of  the CEM and the LCEM.

- Step stress data having other distributions

1- Estimating the model parameters

Several research papers addressed the problem of finding good estimates for step stress data modeled with the CEM assuming that the life data comes from exponential and Weibull distributions.

Balakrishnan and Kundu (2007) considered a simple step stress data with exponentially distributed lifetimes with the presence of type I censoring. They obtained the MLEs and their conditional exact distributions. The authors derived the conditional moment generating functions (CMGF) and obtained the exact conditional distributions of the MLEs of the model. The authors also constructed an approximate and adjusted percentile confidence interval. Finally, they ran a simulation study to evaluate the point estimates of the model parameters and compare between the different confidence intervals for the MLEs.

Balakrishnan et al.(2007) considered a simple step stress model for exponentially distributed lifetimes with type II censoring. The authors obtained the MLEs and derived their exact conditional distributions by finding the inverse of the CMGF of the MLEs. The authors then obtained the exact, approximate, and bootstrap confidence intervals and ran a Monte Carlo simulation study to assess the performance of the of the point and interval estimates of the model parameters. Balakrishnan, Xie and Kundu (2007) considered a CEM for simple step stress exponentially distributed life test data with hybrid censoring. They obtained the MLEs and their exact distributions by deriving the CMGF. The authors then obtained the exact, approximate, and bootstrap confidence intervals and ran a simulation study to compare the performance of the point and interval estimates of the model parameters.

Ismail (2014) considered a TRVM for step stress partially accelerated life test (SSPALT) with type-I progressive hybrid censoring where the distribution of the life test data is Weibull. In partially accelerated life testing experiments, some of the test units are put under higher stress levels than normal. In progressively type-I hybrid censoring scheme, all the units are first run under normal stress that changes to a higher stress level at a predetermined time at which a predetermined number of the unfailed units are removed randomly from the life test experiment. The experiment ends when a predetermined number of failures has occurred or when a predetermined time has been reached whichever occurs first. The author used the Newton Raphson method to find the MLEs and obtained the approximate confidence interval according to the asymptotic property of the MLEs. Finally, the author ran a Monte Carlo simulation to evaluate the performance of the point and interval MLEs

2- Finding the optimal design plan

Several research papers considered planning a step stress experiment where the cumulative exposure model has been fit to the data.

Samanta et al. (2019) considered a step stress exponentially distributed life testing model with two stress levels. The authors proposed a type II hybrid stress changing time to obtain more information at the initial stress level. In type II hybrid stress changing time, the stress level changes to a higher stress level when a prespecified number of failures occur or at a preset time, whichever occurs later. The experiment then continues until all the units have failed. The authors obtained the conditional pdf of the MLEs and the approximate confidence intervals by pivoting the CDF of the MLEs provided that the CDF of the MLEs is a monotonically decreasing function. The authors also obtained a bias adjusted bootstrap confidence interval because the exact distributions are very complex to obtain. Finally, the authors defined an optimality criterion to obtain the best possible choices of the predetermined number of failures and the stress changing time.

- Research done on the lognormal distribution

There are several research papers that considered a lognormally distributed step stress life test data.

Balakrishnan, Zhang and Xie (2009), considered a lognormally distributed simple step stress model with type I censoring. The authors used three numerical methods to obtain the MLEs of the model parameters: Newton-Raphson(NR) , Davidon-Fletcher-Powell (DPF) and Broyden-Fletcher-Goldfarb-Shannon (BFGS) method. The authors then performed a numerical simulation study to assess the performance of the three algorithms above. The simulation results showed that the BFGS method gave the most satisfying results and thus this numerical method has been used throughout the research to obtain the point and interval MLEs. The authors also obtained the approximate, Bootstrap-t intervals, percentile intervals and adjusted percentile (BCa) intervals. Finally, they ran a simulation study to compare and assess the properties of the point and interval estimators. Dube et al.(2011), considered the case where the life test data are following the lognormal distribution with hybrid censoring. They obtained the MLEs numerically using the EM algorithm. The authors then performed a simulation study to compare the performances of the MLEs obtained from the EM algorithm and the approximate MLEs obtained by using the Taylor series expansion for the nonlinear term in the likelihood equations. The authors also discussed finding the optimal censoring scheme.

Lin and Chou (2012) considered k step stress life test data and proposed a two-stage global optimization strategy where a modified simulated annealing algorithm (MSA) is integrated with a Newton Raphson (NR) algorithm for obtaining the MLEs. The MSA algorithm was used to adjust the tolerance level for convergence, number of iterations and the parameter values. The authors ran a numerical simulation to compare different algorithms for obtaining the MLE and found that the NR method requires the least number of iterations. The authors then derived the approximate, likelihood ratio and parametric bootstrap intervals. Finally, they performed a Monte Carlo simulation study to assess the performance of the point intervals of the model parameters and compare the performance of the different confidence intervals.

Hakamipour (2017) considered a plan for step stress lognormal life test data with two stress factors having two stress levels where a linear relationship exists between the mean of the log lifetime and the stress levels. The author proposed minimizing the asymptotic variance of the estimate of the reliability under normal use conditions and maximizing the determinant of the fisher information matrix as two optimality criteria for obtaining the optimal test design. Finally, the author did a sensitivity analysis to explore whether changing the initial parameters affects the optimal values of the time to increase the stress levels.

## 1.9 The Research Objectives:

The main objective of this research is to estimate the parameters of the lognormal simple step stress model with hybrid censoring and to estimate the reliability under normal conditions.

To reach our main objective, the following specific objectives must be achieved.

1- Obtaining the MLEs of the model parameters and studying their properties.

Since the likelihood equations cannot be found in a closed form, numerical methods are used to obtain the MLEs of the step stress lognormal model with hybrid censoring. To study the properties of the MLEs, a simulation study is performed where the MLEs of 2000 replications are obtained and their Bias and MSE are calculated.

2- Deriving and studying the performance of the asymptotic, bootstrap-t confidence intervals and percentile confidence intervals.

The approximate confidence interval is constructed depending on the asymptotic property that the MLEs approach normal distribution as the sample size increases.

To construct the bootstrap-t and percentile confidence intervals, 1000 bootstrap samples are simulated for each of the 2000 replications to obtain the interval estimates of the bootstrap intervals for estimating the model parameters.

To study the performance of the interval estimates, the coverage probabilities and average interval lengths are calculated. The coverage probability indicates how many times the true parameter value was included in 2000 confidence intervals. The average interval length is the sum of the lengths of 2000 intervals (the interval length is the difference between the upper and lower bounds of the confidence interval) divided by 2000.

3- Assessing the performance of the point and interval estimates of the survival function.

To assess the performance of the point and interval estimates of the survival function, the Bias, mean square error (MSE), coverage probability and average interval lengths are calculated.

4- Analyzing the results obtained for different hybrid censoring schemes.

The results are analyzed by checking whether the Bias and MSE decrease with increasing the sample size and comparing the performance of the coverage probability for the three confidence intervals by checking which of the confidence intervals have coverage probability close to the nominal level for different hybrid censoring schemes.

5- Presenting a detailed example to illustrate the application of the methods discussed in this thesis.

Since a real step stress data has not been found, a generated sample is used instead. In the illustrative example, a step stress lognormal sample with hybrid censoring is generated and the point and interval estimates of the model parameters are obtained.

# CHAPTER 2: MAXIMUM LIKELIHOOD INFERENCE

## 2.1 Overview of the Maximum Likelihood Inference

One of the most commonly used methods to estimate the model parameters is the maximum likelihood estimation method (ML methods) . This method applies to most data types and can be used with censored data in addition to providing good point and interval estimates of the model parameters. Functions of the model parameters to be estimated can also be estimated with the ML methods according to the invariance property of the MLE. To find the MLE of the reliability function at specific times, we evaluate the reliability function at the MLEs of the model parameters.

The likelihood function is expressed mathematically as the function of the model parameters as shown below:

$$L(\theta) = L(\theta; data) = C\prod_{i=1}^{n} L_i(\theta; data_i)$$

Where C is a constant that does not contain the model parameters or the data and thus can be removed when finding the roots of the equations. The maximum likelihood estimator MLE is the unique value that maximizes the likelihood function. For the mathematical ease, the log likelihood function is more commonly used instead of the likelihood function.

## 2.2 Likelihood for Estimating the Model Parameters

In this section, we will construct the likelihood function for the step stress life test data with the presence of hybrid censoring. There are two cases that we have to consider when constructing the likelihood function. The case in which the preset experiment time is reached before having r failures (type I censoring) and the case where r units fail before the predetermined time of the experiment is reached ( type II censoring).

We first introduce some notations that we will be using throughout this research.

Notations

$t_1$= The preset experiment time

r= The predetermined number of failures

$\tau$= The stress changing time

$x_0$= The initial stress level

$x_1$= The higher stress level

$N_1$= The number of failures at the initial stress level

$N_2$= The number of failures at the higher stress level

m= The total number of failures in the experiment, m=$N_1$+$N_2$

The MLE $\hat{\mu}_1$ doesn't exist when the number of failures $N_1$ in the initial level $x_0$ is zero and the MLE $\hat{\mu}_2$ doesn't exist when the number of failures $N_2$ in the second stress level $x_1$ is zero, therefore, we will condition on that $N_1 > 0$ $and$ $N_2 > 0$.

The simple step stress data follow a lognormal distribution. We fit a CEM to the step stress lognormal distributed life test data to extrapolate the data to normal use conditions and relate the distribution to the stress levels.

The pdf and CDF of the lognormal simple step stress model are shown in eq 1 and 2 respectively:

$$g(t) = \begin{cases} g_1(t) = \frac{1}{\sigma t\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{log t-\mu_1}{\sigma}\right)^2}, 0 \le t < \tau \\ g_2(t) = \frac{1}{\sigma[t-\alpha(\mu)]\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{log[t-\alpha(\mu)]-\mu_2}{\sigma}\right)^2}, \tau \le t < \infty \end{cases} \quad (1)$$

$$G(t) = \begin{cases} G_1(t) = \Phi\left(\frac{log t-\mu_1}{\sigma}\right), 0 \le\ t < \tau \\ G_2(t) = \Phi\left(\frac{\log\,[t-\alpha(\mu)]-\mu_2}{\sigma}\right), \tau \le\ t < \infty \end{cases} \quad (2)$$

$$Where\ \ \alpha(\mu) = \tau(1 - e^{\mu_2 - \mu_1}) \tag{3}$$

$g_1(t)$ is the distribution of the lifetime of the units under normal conditions $x_0$ and $g_2(t)$ is the distribution at the higher stress level $x_1$.

The graphical representation of the CDF of the lognormal simple step stress model where $\tau$=30 and model parameters $\mu_1 = log(200)$, $\mu_2 = log(5)$, $\sigma = 3$ is shown in figure 4:

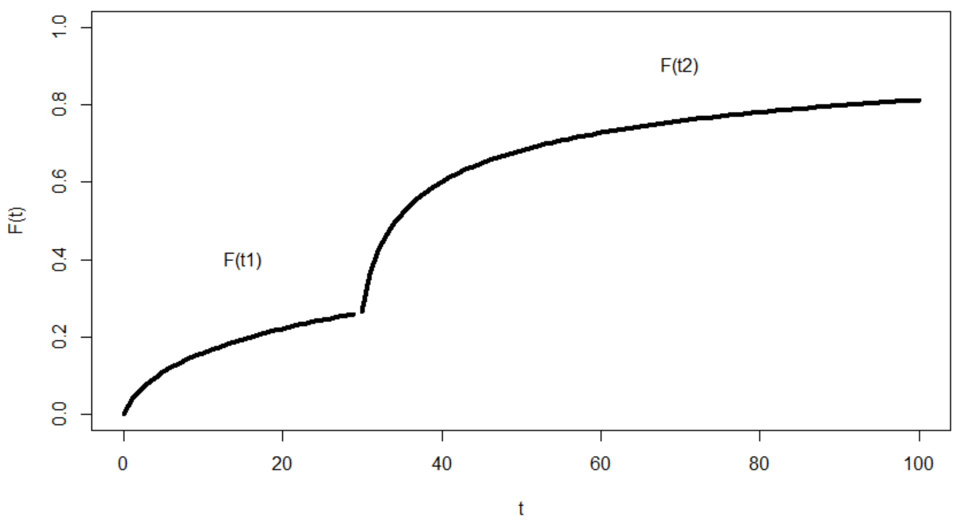


Figure 4. Cumulative exposure lognormal model

Likelihood construction

Case 1: Let $T^* = min(t_1, t_{(r)}) = t_1$, where $N_1 + N_2 < r$

The observed time to failure of the n units in this case will be in the form shown in equations 4:

$$t_{1:n} < t_{2:n} < \cdots < t_{N_1:n} < \tau < t_{N_1+1:n} < \cdots < t_{m:n} < t_1 \tag{4}$$

The likelihood function is constructed as shown in equation 5:

$$L(\theta|t) = \left\{\prod_{k=1}^{N_1} g_1(t_{k:n})\right\} * \left\{\prod_{k=N_1+1}^{m} g_2(t_{k:n})\right\} * (1 - G_2(t_1\ ))^{n-m} \quad (5)$$

Substituting the pdf and CDF of the step stress model, we get equation 6:

$$L(\theta|t) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^m \frac{1}{\prod_{k=1}^{N_1} t_k} \frac{1}{\prod_{k=N_1}^{m}[t_{k:n} - \alpha(\mu)]} *$$

$$e^{\frac{-1}{2}\sum_{k=1}^{N_1}\left(\frac{log t_k - \mu_1}{\sigma}\right)^2} e^{-\frac{1}{2}\sum_{k=N_1+1}^{m}\left(\frac{log[t_k - \alpha(\mu)] - \mu_2}{\sigma}\right)^2} * \left\{1 - \Phi\left(\frac{\log\,[t_1 - \alpha(\mu)] - \mu_2}{\sigma}\right)^{n-m}\right\} \quad (6)$$

Taking the log of the likelihood of the function:

$$l(\theta|t) = -m log\left(\sqrt{2\pi}\sigma\right) - \sum_{k=1}^{N_1}\log(t_{k:n}) - \sum_{k=N_1+1}^{m}\log[t_{k:n} - \alpha(\mu)] -$$

$$\frac{1}{2}\sum_{k=1}^{N_1}\left(\frac{log t_k - \mu_1}{\sigma}\right)^2 - \frac{1}{2}\sum_{k=N_1+1}^{m}\left(\frac{\log[t_{k:n} - \alpha(\mu)] - \mu_2}{\sigma}\right)^2 + (n-m) log\left(1 - \right. \quad (7)$$

$$\left.\Phi\left(\frac{\log[t_1 - \alpha(\mu)] - \mu_2}{\sigma}\right)\right)$$

The first derivative of the log likelihood function with respect to $\mu_1, \mu_2$ and $\sigma$ respectively is shown in equations 8 to 10:

$$\frac{\partial}{\partial\mu_1} l(\theta|t) = \sum_{k=N_1+1}^{m} \frac{\tau e^{\mu_2 - \mu_1}}{[t_k - \alpha(\mu)]} + \sum_{k=1}^{N_1}\left(\frac{log(t_k) - \mu_1}{\sigma^2}\right) +$$

$$\sum_{k=N_1+1}^{m}\left(\frac{log[t_k - \alpha(\mu)] - \mu_2}{\sigma}\right)\frac{(\tau e^{\mu_2 - \mu_1})}{\sigma[t_k - \alpha(\mu)]} - (n-m) * \frac{\varphi\left(\frac{1}{\sigma} log[t_1 - \alpha(\mu)] - \mu_2\right)}{1 - \Phi\left[\frac{1}{\sigma} log[t_1 - \alpha(\mu)] - \mu_2\right]} * \quad (8)$$

$$\frac{(\tau e^{\mu_2 - \mu_1})}{\sigma[t_1 - \alpha(\mu)]}$$

$$\frac{\partial}{\partial\mu_2} l(\theta|t) = \sum_{k=N_1+1}^{m} \frac{\tau e^{\mu_2 - \mu_1}}{t_k - \alpha(\mu)} + \sum_{k=N_1+1}^{m}\left(\frac{log[t_k - \alpha(\mu)]}{\sigma}\right) * \left(\frac{\tau e^{\mu_2 - \mu_1}}{\sigma[t_k - \alpha(\mu)]} - \frac{1}{\sigma}\right) +$$

$$(n-m)\ \frac{\varphi\left(\frac{1}{\sigma} log[t_1 - \alpha(\mu)] - \mu_2\right)}{1 - \Phi\left(\frac{log[t_1 - \alpha(\mu)] - \mu_2}{\sigma}\right)} * \left(\frac{\tau e^{\mu_2 - \mu_1}}{\sigma[t_1 - \alpha(\mu)]} - \frac{1}{\sigma}\right) \quad (9)$$

$$\frac{\partial}{\partial\sigma} l(\theta|t) = \frac{-m}{\sigma} + \sum_{k=1}^{N_1}\left(\frac{(log(t_k)-\mu_1)^2}{\sigma^3}\right) + \sum_{k=N_1+1}^{m}\left(\frac{(log[t_k-\alpha(\mu)]-\mu_2)^2}{\sigma^3}\right) + (n - m)\frac{\varphi\left(\frac{1}{\sigma}log[t_1-\alpha(\mu)]-\mu_2\right)}{1-\Phi\left[\frac{1}{\sigma}log[t_1-\alpha(\mu)]-\mu_2\right]} * \frac{log[t_1-\alpha(\mu)]-\mu_2}{\sigma^2} \quad (10)$$

Case 2: $T^* = min\left(t_1, t_{(r)}\right) = t_{(r)}$, where $N_1 + N_2 = r$

The observed time to failure of the n units in case 2 will be in the form shown in equation 11:

$$t_{1:n} < t_{2:n} < \cdots < t_{N1:n} < \tau < t_{N_1+1:n} < \cdots < t_{(r)} \quad (11)$$

The likelihood function is constructed as shown in equation 12:

$$L(\theta|t) = \left\{\prod_{k=1}^{N_1} g_1(t_{k:n})\right\} * \left\{\prod_{k=N_1+1}^{r} g_2(t_{k:n})\right\} * (1 - G_2(t_r\ ))^{n-r} \quad (12)$$

Substituting the pdf and CDF of the step stress model:

$$L(\theta|t) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^r \frac{1}{\prod_{k=1}^{N_1} t_k} \frac{1}{\prod_{k=N_1}^{r}[t_{k:n}-\alpha(\mu)]} * e^{\frac{-1}{2}\sum_{k=1}^{N_1}\left(\frac{logt_k-\mu_1}{\sigma}\right)^2} e^{-\frac{1}{2}\sum_{k=N_1+1}^{r}\left(\frac{log[t_k-\alpha(\mu)]-\mu_2}{\sigma}\right)^2} * \left\{1 - \Phi\left(\frac{\log[t_r-\alpha(\mu)]-\mu_2}{\sigma}\right)^{n-r}\right\} \quad (13)$$

Taking the log of the likelihood of the function:

$$l(\theta|t) = -rlog\left(\sqrt{2\pi}\sigma\right) - \sum_{k=1}^{N_1}\log(t_{k:n}) - \sum_{k=N_1+1}^{r}\log[t_{k:n} - \alpha(\mu)] - \frac{1}{2}\sum_{k=1}^{N_1}\left(\frac{logt_k-\mu_1}{\sigma}\right)^2 - \frac{1}{2}\sum_{k=N_1+1}^{r}\left(\frac{\log[t_{k:n}-\alpha(\mu)]-\mu_2}{\sigma}\right)^2 + (n-r)log\left(1 - \Phi\left(\frac{\log[t_r-\alpha(\mu)]-\mu_2}{\sigma}\right)\right) \quad (14)$$

The first derivative of the log likelihood function with respect to $\mu_1, \mu_2$ and $\sigma$ respectively are as shown in equations 15 to 17:

$$\frac{\partial}{\partial\mu_1}l(\theta|t) = \sum_{k=N_1+1}^{r}\frac{\tau e^{\mu_2-\mu_1}}{[t_k-\alpha(\mu)]} + \sum_{k=1}^{N_1}\left(\frac{log(t_k)-\mu_1}{\sigma^2}\right) + \sum_{k=N_1+1}^{r}\left(\frac{log[t_k-\alpha(\mu)]-\mu_2}{\sigma}\right)\frac{(\tau e^{\mu_2-\mu_1})}{\sigma[t_k-\alpha(\mu)]} - (n-r) * \frac{\varphi\left(\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right)}{1-\Phi\left[\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right]} * \frac{(\tau e^{\mu_2-\mu_1})}{\sigma[t_r-\alpha(\mu)]} \quad (15)$$

$$\frac{\partial}{\partial\mu_2}l(\theta|t) = \sum_{k=N_1+1}^{r}\frac{\tau e^{\mu_2-\mu_1}}{t_k-\alpha(\mu)} + \sum_{k=N_1+1}^{r}\left(\frac{log[t_k-\alpha(\mu)]}{\sigma}\right) * \left(\frac{\tau e^{\mu_2-\mu_1}}{\sigma[t_k-\alpha(\mu)]} - \frac{1}{\sigma}\right) + (n-r)\ \frac{\varphi\left(\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right)}{1-\Phi\left(\frac{log[t_r-\alpha(\mu)]-\mu_2}{\sigma}\right)} * \left(\frac{\tau e^{\mu_2-\mu_1}}{\sigma[t_r-\alpha(\mu)]} - \frac{1}{\sigma}\right) \quad (16)$$

$$\frac{\partial}{\partial\sigma}l(\theta|t) = \frac{-r}{\sigma} + \sum_{k=1}^{N_1}\left(\frac{(log(t_k)-\mu_1)^2}{\sigma^3}\right) + \sum_{k=N_1+1}^{r}\left(\frac{(log[t_k-\alpha(\mu)]-\mu_2)^2}{\sigma^3}\right) + (n-r)\frac{\varphi\left(\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right)}{1-\Phi\left[\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right]} * \frac{log[t_r-\alpha(\mu)]-\mu_2}{\sigma^2} \quad (17)$$

## 2.3 Likelihood for Estimating the Survival Function

The Survival function is given by $S(t) = 1 - \Phi\left(\frac{log(t)-\mu_1}{\sigma}\right)$. To find the MLE of the survival function, we substitute the values of the MLEs $\hat{\mu}$ and $\hat{\sigma}$ in the survival function $\hat{S}(t) = 1 - \Phi\left(\frac{log(t)-\hat{\mu}_1}{\hat{\sigma}}\right)$ based on the invariance property.

To calculate the variance of a function of the MLEs, we can use the delta method (see Casella G, Berger, R.L, 2002 )

$$\sqrt{n}\left[g\left(\hat{\theta}_1, \ldots, \hat{\theta}_s\right) - g\left(\theta_1, \ldots, \theta_p\right)\right] \to N\left(0, \sum\sum\sigma_{ij}\frac{\partial g(\theta)}{\partial\theta_i} * \frac{\partial g(\theta)}{\partial\theta_j}\right)$$

Where $\sigma_{ij} = cov\left(X_i, X_j\right)$

The variance of the survival function is calculated as follows:

$$G = \left(\frac{\partial S(t)}{\partial\mu_1}\ \ \frac{\partial S(t)}{\partial\mu_2}\frac{\partial S(t)}{\partial\sigma}\right)$$

Where,

$$\frac{\partial S(t)}{\partial \mu_1} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{log(t)-\mu_1}{\sigma}\right)^2} \tag{18}$$

$$\frac{\partial S(t)}{\partial \mu_2} = 0 \tag{19}$$

$$\frac{\partial S(t)}{\partial \sigma} = \frac{\mu_1 - log(t)}{\sigma^2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{log(t)-\mu_1}{\sigma}\right)^2} \tag{20}$$

$$\widehat{var}\left(\hat{S}(t)\right) \approx [G^t I^{-1}(\mu_1, \mu_2, \sigma) G]_{(\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma})}$$

## 2.4 Asymptotic Confidence Intervals for the Model Parameters

The MLE has an asymptotic property in which the MLE has approximate distributions where $\sqrt{n}(\hat{\theta} - \theta) \to N(0, \sigma^2)$(see Casella G, Berger, R.L, 2002 )

We can obtain approximate confidence intervals for the MLEs by using the asymptotic property. The number of failures in the sample should be large in order to have a good normality approximation (based on the large sample theory).

Finding the approximate distributions for the MLEs requires the knowledge of the information matrix. And since the inverse of the information matrix is usually difficult to analytically derive, the inverse of a numerical approximation of the information matrix is usually used instead.

The information matrix is given by:

$$I = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

Where $c_{ij} = -E\left[\frac{\partial^2 l(\theta|t)}{\partial\theta_i \partial\theta_j}\right], \quad i, j = 1,2,3$

The second partial derivatives of the log likelihood function for case I (type I censoring) are as follows:

$$\frac{\partial^2 l(\theta|t)}{\partial \mu^2{}_1} = -\sum_{k=N_1+1}^{m} \tau e^{\mu_2-\mu_1} - \frac{N_1}{\sigma^2} + \sum_{k=N_1+1}^{m}\left[\left(\frac{\tau e^{\mu_2-\mu_1}}{(\sigma[t_k-\alpha(\mu)])}\right)^2 + \left(\frac{log[t_k-\alpha(\mu)]}{\sigma}\right) * \frac{(-\sigma\tau e^{\mu_2-\mu_1}[t_k-\alpha(\mu)])+\sigma(\tau e^{\mu_2-\mu_1})^2}{(\sigma[t_k-\alpha(\mu)])^2}\right] - (n - m) * \left[\frac{1-\Phi\left(\frac{\log[t_1-\alpha(\mu)]-\mu_2}{\sigma}\right)*\varphi\left(\frac{1}{\sigma}log[t_1-\alpha(\mu)]-\mu_2\right)*\frac{1-\tau e^{\mu_2-\mu_1}}{\sigma\,[t_1-\alpha(\mu)]}*\frac{\left(\tau e^{\mu_2-\mu_1}\right)}{\sigma[t_1-\alpha(\mu)]}}{\left(1-\Phi\left(\frac{\log[t_1-\alpha(\mu)]-\mu_2}{\sigma}\right)\right)^2}\right] + \left[\frac{\varphi\left(\frac{1}{\sigma}log[t_1-\alpha(\mu)]-\mu_2\right)}{1-\Phi\left[\frac{1}{\sigma}log[t_1-\alpha(\mu)]-\mu_2\right]} * \frac{(-\sigma[t_1-\alpha(\mu)]*\tau e^{\mu_2-\mu_1})+\left(\sigma(\tau e^{\mu_2-\mu_1})^2\right)}{(\sigma[t_1-\alpha(\mu)])^2}\right] \quad (21)$$

$$\frac{\partial^2 l(\theta|t)}{\partial \mu^2{}_2} = \sum_{k=N_1+1}^{m} \frac{[t_k-\alpha(\mu)]*\tau e^{\mu_2-\mu_1}-\left((\tau e^{\mu_2-\mu_1})^2\right)}{[t_k-\alpha(\mu)]^2} + \sum_{k=N_1+1}^{m}\left[\frac{\tau e^{\mu_2-\mu_1}}{(\sigma[t_k-\alpha(\mu)])} * \frac{1}{\sigma}\left[\frac{\tau e^{\mu_2-\mu_1}}{([t_k-\alpha(\mu)])} - 1\right]\right] + \left(\frac{log[t_k-\alpha(\mu)]}{\sigma}\right) * \frac{1}{\sigma}\frac{t_k-\alpha(\mu)*\tau e^{\mu_2-\mu_1}-\left((\tau e^{\mu_2-\mu_1}-1)*\tau e^{\mu_2-\mu_1}\right)}{[t_k-\alpha(\mu)]^2} + (n-m) * \left[\frac{1-\Phi\left(\frac{\log[t_1-\alpha(\mu)]-\mu_2}{\sigma}\right)*\varphi\left(\frac{1}{\sigma}log[t_1-\alpha(\mu)]-\mu_2\right)*\frac{1}{\sigma}\left(\frac{\tau e^{\mu_2-\mu_1}}{[t_1-\alpha(\mu)]}-1\right)*\frac{\left(\tau e^{\mu_2-\mu_1}\right)}{\sigma[t_1-\alpha(\mu)]}-1+\left(\left(\varphi\left(\frac{1}{\sigma}log[t_1-\alpha(\mu)]-\mu_2\right)\right)^2*\frac{1}{\sigma}\left[\frac{\tau e^{\mu_2-\mu_1}}{(\sigma[t_1-\alpha(\mu)])}-1\right]\right)}{\left(1-\Phi\left(\frac{\log[t_1-\alpha(\mu)]-\mu_2}{\sigma}\right)\right)^2}\right] + \left[\frac{\varphi\left(\frac{1}{\sigma}log[t_1-\alpha(\mu)]-\mu_2\right)}{1-\Phi\left[\frac{1}{\sigma}log[t_1-\alpha(\mu)]-\mu_2\right]} * \frac{1}{\sigma}\frac{([t_1-\alpha(\mu)]*\tau e^{\mu_2-\mu_1})-\left((\tau e^{\mu_2-\mu_1}-1)\tau e^{\mu_2-\mu_1}\right)}{([t_1-\alpha(\mu)])^2}\right] \quad (22)$$

$$\frac{\partial^2 l(\theta|t)}{\partial\sigma^2} = \frac{m}{\sigma^2} - \sum_{k=1}^{N_1} \frac{(log(t_k)-\mu_1)^2}{\sigma^4} - \sum_{k=N_1+1}^{m} \frac{(log[t_k-\alpha(\mu)]-\mu_2)^2}{\sigma^4} + (n-m) * \frac{log[t_1-\alpha(\mu)]-\mu_2}{\sigma^2} * \tag{23}$$

$$\frac{-\left(1-\Phi\left(\frac{\log[t_1-\alpha(\mu)]-\mu_2}{\sigma}\right)*\varphi\left(\frac{1}{\sigma}log[t_1-\alpha(\mu)]-\mu_2\right)*\left(\frac{1}{\sigma^2}log[t_1-\alpha(\mu)]-\mu_2\right)\right)}{\left(1-\Phi\left(\frac{\log[t_1-\alpha(\mu)]-\mu_2}{\sigma}\right)\right)^2} - \frac{\left(\frac{\log[t_1-\alpha(\mu)]-\mu_2}{\sigma}\right)*-\varphi\left(\frac{1}{\sigma}log[t_1-\alpha(\mu)]-\mu_2\right)*-\frac{1}{\sigma^2}log[t_1-\alpha(\mu)]-\mu_2}{\left(1-\Phi\left(\frac{\log[t_1-\alpha(\mu)]-\mu_2}{\sigma}\right)\right)^2} +$$

$$\frac{\varphi\left(\frac{1}{\sigma}log[t_1-\alpha(\mu)]-\mu_2\right)}{1-\Phi\left(\frac{\log[t_1-\alpha(\mu)]-\mu_2}{\sigma}\right)} * \frac{-log[t_1-\alpha(\mu)]-\mu_2}{\sigma^3}$$

$$\frac{\partial^2 l(\theta|t)}{\partial\mu_1\partial\mu_2} = \sum_{K=N_1+1}^{m} \frac{([t_k-\alpha(\mu)]*\tau e^{\mu_2-\mu_1})-((\tau e^{\mu_2-\mu_1})^2)}{([t_k-\alpha(\mu)])^2} + \sum_{k=N_1+1}^{m}\left(\left(\frac{1}{\sigma}\left[\frac{\tau e^{\mu_2-\mu_1}}{([t_k-\alpha(\mu)])}-1\right]\right) * \frac{\tau e^{\mu_2-\mu_1}}{([t_k-\alpha(\mu)])}\right) + \left(\left(\frac{log[t_k-\alpha(\mu)]-\mu_2}{\sigma}\right) * \right. \tag{24}$$

$$\left. \frac{1}{\sigma}\frac{\left((t_k-\alpha(\mu))*\tau e^{\mu_2-\mu_1}\right)-((\tau e^{\mu_2-\mu_1})^2)}{[t_k-\alpha(\mu)]^2}\right) - (n-m) *$$

$$\left[\frac{1-\Phi\left(\frac{\log[t_1-\alpha(\mu)]-\mu_2}{\sigma}\right)*\varphi\left(\frac{1}{\sigma}log[t_1-\alpha(\mu)]-\mu_2\right)*\frac{1}{\sigma}\left(\frac{\tau e^{\mu_2-\mu_1}}{[t_1-\alpha(\mu)]}-1\right)+\left(\left(\varphi\left(\frac{1}{\sigma}log[t_1-\alpha(\mu)]-\mu_2\right)\right)^2*\frac{1}{\sigma}\left[\frac{\tau e^{\mu_2-\mu_1}}{(\sigma[t_1-\alpha(\mu)])}-1\right]\right)}{\left(1-\Phi\left(\frac{\log[t_1-\alpha(\mu)]-\mu_2}{\sigma}\right)\right)^2}\right]$$

$$\frac{\partial^2 l(\theta|t)}{\partial\sigma\partial\mu_1} = \sum_{k=1}^{N_1} \frac{-2}{\sigma^3}\left(log(t_k-\mu_1)\right) * - \sum_{k=N_1+1}^{m} \frac{2log[t_k-\alpha(\mu)]}{\sigma^3} * \left[\frac{\tau e^{\mu_2-\mu_1}}{([t_k-\alpha(\mu)])}\right] + (n-m) * \left(\frac{1}{\sigma}\left[\frac{\tau e^{\mu_2-\mu_1}}{([t_1-\alpha(\mu)])} - 1\right]\right) * \tag{25}$$

$$\frac{\left(1-\Phi\left(\frac{\log[t_1-\alpha(\mu)]-\mu_2}{\sigma}\right)\right) * \varphi\left(\frac{1}{\sigma}log[t_1-\alpha(\mu)]-\mu_2\right) * - \frac{\tau e^{\mu_2-\mu_1}}{\sigma\left[[t_1-\alpha(\mu)]\right]} - \left(\left(\varphi\left(\frac{1}{\sigma}log[t_1-\alpha(\mu)]-\mu_2\right)\right)^2 * - \frac{-\tau e^{\mu_2-\mu_1}}{\sigma\left[[t_1-\alpha(\mu)]\right]}\right)}{\left(1-\Phi\left(\frac{\log[t_1-\alpha(\mu)]-\mu_2}{\sigma}\right)\right)^2} - \frac{\varphi\left(\frac{1}{\sigma}log[t_1-\alpha(\mu)]-\mu_2\right)}{1-\Phi\left[\frac{1}{\sigma}log[t_1-\alpha(\mu)]-\mu_2\right]} *$$

$$\frac{1}{\sigma^2}\left[\frac{\tau e^{\mu_2-\mu_1}}{([t_1-\alpha(\mu)])}\right]$$

$$\frac{\partial^2 l(\theta|t)}{\partial\sigma\partial\mu_2} = \sum_{k=N_1+1}^{m} \frac{-log[t_k-\alpha(\mu)]}{\sigma^2} * \frac{1}{\sigma}\left[\frac{\tau e^{\mu_2-\mu_1}}{([t_k-\alpha(\mu)])} - 1\right] - \left(\frac{log[t_k-\alpha(\mu)]}{\sigma}\right) * \frac{1}{\sigma^2}\left[\frac{\tau e^{\mu_2-\mu_1}}{(\sigma[t_k-\alpha(\mu)])} - 1\right] + (n-m) * \left(\frac{1}{\sigma}\left[\frac{\tau e^{\mu_2-\mu_1}}{([t_1-\alpha(\mu)])} - 1\right]\right) *$$

$$\frac{\left(1-\Phi\left(\frac{\log[t_1-\alpha(\mu)]-\mu_2}{\sigma}\right)\right) * \varphi\left(\frac{1}{\sigma}log[t_1-\alpha(\mu)]-\mu_2\right) * - \frac{1}{\sigma^2}[log[t_1-\alpha(\mu)]-\mu_2] - \left(\left(\varphi\left(\frac{1}{\sigma}log[t_1-\alpha(\mu)]-\mu_2\right)\right)^2 * \frac{1}{\sigma^2}log[t_1-\alpha(\mu)]-\mu_2\right)}{\left(1-\Phi\left(\frac{\log[t_1-\alpha(\mu)]-\mu_2}{\sigma}\right)\right)^2} - \tag{26}$$

$$\frac{\varphi\left(\frac{1}{\sigma}log[t_1-\alpha(\mu)]-\mu_2\right)}{1-\Phi\left[\frac{1}{\sigma}log[t_1-\alpha(\mu)]-\mu_2\right]} * \frac{1}{\sigma^2}\left[\frac{\tau e^{\mu_2-\mu_1}}{([t_1-\alpha(\mu)])} - 1\right]$$

The second partial derivatives of the log likelihood function for case II (type II censoring) are as follows:

$$\frac{\partial^2 l(\theta|t)}{\partial \mu^2{}_1} = -\sum_{k=N_1+1}^{r} \tau e^{\mu_2-\mu_1} - \frac{N_1}{\sigma^2} + \sum_{k=N_1+1}^{r}\left[\left(\frac{\tau e^{\mu_2-\mu_1}}{(\sigma[t_k-\alpha(\mu)])}\right)^2 + \left(\frac{log[t_k-\alpha(\mu)]}{\sigma}\right) * \frac{(-\sigma\tau e^{\mu_2-\mu_1}[t_k-\alpha(\mu)])+\sigma(\tau e^{\mu_2-\mu_1})^2}{(\sigma[t_k-\alpha(\mu)])^2}\right] - (n - r) * \left[\frac{1-\Phi\left(\frac{\log[t_r-\alpha(\mu)]-\mu_2}{\sigma}\right)*\varphi\left(\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right)*\frac{1-\tau e^{\mu_2-\mu_1}}{\sigma\,[t_r-\alpha(\mu)]}*\frac{\left(\tau e^{\mu_2-\mu_1}\right)}{\sigma[t_r-\alpha(\mu)]}}{\left(1-\Phi\left(\frac{\log[t_r-\alpha(\mu)]-\mu_2}{\sigma}\right)\right)^2}\right] + \left[\frac{\varphi\left(\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right)}{1-\Phi\left[\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right]} * \frac{(-\sigma[t_r-\alpha(\mu)]*\tau e^{\mu_2-\mu_1})+\left(\sigma(\tau e^{\mu_2-\mu_1})^2\right)}{(\sigma[t_r-\alpha(\mu)])^2}\right] \tag{27}$$

$$\frac{\partial^2 l(\theta|t)}{\partial \mu^2{}_2} = \sum_{k=N_1+1}^{r} \frac{[t_k-\alpha(\mu)]*\tau e^{\mu_2-\mu_1}-\left((\tau e^{\mu_2-\mu_1})^2\right)}{[t_k-\alpha(\mu)]^2} + \sum_{k=N_1+1}^{r}\left[\frac{\tau e^{\mu_2-\mu_1}}{(\sigma[t_k-\alpha(\mu)])} * \frac{1}{\sigma}\left[\frac{\tau e^{\mu_2-\mu_1}}{([t_k-\alpha(\mu)])} - 1\right]\right] + \left(\frac{log[t_k-\alpha(\mu)]}{\sigma}\right) * \frac{1}{\sigma}\frac{t_k-\alpha(\mu)*\tau e^{\mu_2-\mu_1}-\left((\tau e^{\mu_2-\mu_1}-1)*\tau e^{\mu_2-\mu_1}\right)}{[t_k-\alpha(\mu)]^2} + (n-r) * \tag{28}$$

$$\left[\frac{1-\Phi\left(\frac{\log[t_r-\alpha(\mu)]-\mu_2}{\sigma}\right)*\varphi\left(\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right)*\frac{1}{\sigma}\left(\frac{\tau e^{\mu_2-\mu_1}}{[t_r-\alpha(\mu)]}-1\right)*\frac{\left(\tau e^{\mu_2-\mu_1}\right)}{\sigma[t_r-\alpha(\mu)]}-1+\left(\left(\varphi\left(\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right)\right)^2*\frac{1}{\sigma}\left[\frac{\tau e^{\mu_2-\mu_1}}{(\sigma[t_r-\alpha(\mu)])}-1\right]\right)}{\left(1-\Phi\left(\frac{\log[t_r-\alpha(\mu)]-\mu_2}{\sigma}\right)\right)^2}\right]+$$

$$\left[\frac{\varphi\left(\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right)}{1-\Phi\left[\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right]}*\frac{1}{\sigma}\frac{([t_r-\alpha(\mu)]*\tau e^{\mu_2-\mu_1})-\left((\tau e^{\mu_2-\mu_1}-1)\tau e^{\mu_2-\mu_1}\right)}{([t_r-\alpha(\mu)])^2}\right]$$

$$\frac{\partial^2 l(\theta|t)}{\partial\sigma^2}=\frac{r}{\sigma^2}-\sum_{k=1}^{N_1}\frac{(log(t_k)-\mu_1)^2}{\sigma^4}-\sum_{k=N_1+1}^{r}\frac{(log[t_k-\alpha(\mu)]-\mu_2)^2}{\sigma^4}+(n-r)*\frac{log[t_r-\alpha(\mu)]-\mu_2}{\sigma^2}* \qquad (29)$$

$$\frac{-\left(1-\Phi\left(\frac{\log[t_r-\alpha(\mu)]-\mu_2}{\sigma}\right)*\varphi\left(\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right)*\left(\frac{1}{\sigma^2}log[t_r-\alpha(\mu)]-\mu_2\right)\right)}{\left(1-\Phi\left(\frac{\log[t_r-\alpha(\mu)]-\mu_2}{\sigma}\right)\right)^2}-\frac{\left(\frac{\log[t_r-\alpha(\mu)]-\mu_2}{\sigma}\right)*-\varphi\left(\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right)*-\frac{1}{\sigma^2}log[t_r-\alpha(\mu)]-\mu_2}{\left(1-\Phi\left(\frac{\log[t_r-\alpha(\mu)]-\mu_2}{\sigma}\right)\right)^2}+$$

$$\frac{\varphi\left(\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right)}{1-\Phi\left(\frac{\log[t_r-\alpha(\mu)]-\mu_2}{\sigma}\right)}*\frac{-log[t_r-\alpha(\mu)]-\mu_2}{\sigma^3}$$

$$\frac{\partial^2 l(\theta|t)}{\partial\mu_1\partial\mu_2} = \sum_{K=N_1+1}^{r} \frac{([t_k-\alpha(\mu)]*\tau e^{\mu_2-\mu_1})-((\tau e^{\mu_2-\mu_1})^2)}{([t_k-\alpha(\mu)])^2} + \sum_{k=N_1+1}^{r}\left(\left(\frac{1}{\sigma}\left[\frac{\tau e^{\mu_2-\mu_1}}{([t_k-\alpha(\mu)])}-1\right]\right)*\frac{\tau e^{\mu_2-\mu_1}}{([t_k-\alpha(\mu)])}\right)+\left(\left(\frac{log[t_k-\alpha(\mu)]-\mu_2}{\sigma}\right)*\right. \tag{30}$$

$$\left.\frac{1}{\sigma}\frac{((t_k-\alpha(\mu))*\tau e^{\mu_2-\mu_1})-((\tau e^{\mu_2-\mu_1})^2)}{[t_k-\alpha(\mu)]^2}\right)-(n-r)*$$

$$\left[\frac{1-\Phi\left(\frac{\log[t_r-\alpha(\mu)]-\mu_2}{\sigma}\right)*\varphi\left(\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right)*\frac{1}{\sigma}\left(\frac{\tau e^{\mu_2-\mu_1}}{[t_r-\alpha(\mu)]}-1\right)+\left(\left(\varphi\left(\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right)\right)^2*\frac{1}{\sigma}\left[\frac{\tau e^{\mu_2-\mu_1}}{(\sigma[t_r-\alpha(\mu)])}-1\right]\right)}{\left(1-\Phi\left(\frac{\log[t_r-\alpha(\mu)]-\mu_2}{\sigma}\right)\right)^2}\right]$$

$$\frac{\partial^2 l(\theta|t)}{\partial\sigma\partial\mu_1} = \sum_{k=1}^{N_1}\frac{-2}{\sigma^3}\left(log(t_k-\mu_1)\right)*-\sum_{k=N_1+1}^{r}\frac{2log[t_k-\alpha(\mu)]}{\sigma^3}*\left[\frac{\tau e^{\mu_2-\mu_1}}{([t_k-\alpha(\mu)])}\right]+(n-r)*\left(\frac{1}{\sigma}\left[\frac{\tau e^{\mu_2-\mu_1}}{([t_r-\alpha(\mu)])}-1\right]\right)* \tag{31}$$

$$\frac{\left(1-\Phi\left(\frac{\log[t_r-\alpha(\mu)]-\mu_2}{\sigma}\right)\right)*\varphi\left(\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right)*-\frac{\tau e^{\mu_2-\mu_1}}{\sigma[[t_r-\alpha(\mu)]]}-\left(\left(\varphi\left(\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right)\right)^2*-\frac{-\tau e^{\mu_2-\mu_1}}{\sigma[[t_r-\alpha(\mu)]]}\right)}{\left(1-\Phi\left(\frac{\log[t_r-\alpha(\mu)]-\mu_2}{\sigma}\right)\right)^2}-\frac{\varphi\left(\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right)}{1-\Phi\left[\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right]}*$$

$$\frac{1}{\sigma^2}\left[\frac{\tau e^{\mu_2-\mu_1}}{([t_r-\alpha(\mu)])}\right]$$

$$\frac{\partial^2 l(\theta|t)}{\partial\sigma\partial\mu_2} = \sum_{k=N_1+1}^{r} \frac{-log[t_k-\alpha(\mu)]}{\sigma^2} * \frac{1}{\sigma}\left[\frac{\tau e^{\mu_2-\mu_1}}{([t_k-\alpha(\mu)])} - 1\right] - \left(\frac{log[t_k-\alpha(\mu)]}{\sigma}\right) * \frac{1}{\sigma^2}\left[\frac{\tau e^{\mu_2-\mu_1}}{(\sigma[t_k-\alpha(\mu)])} - 1\right] + (n-r) * \left(\frac{1}{\sigma}\left[\frac{\tau e^{\mu_2-\mu_1}}{([t_r-\alpha(\mu)])} - 1\right]\right) *$$

$$\frac{\left(1-\Phi\left(\frac{\log[t_r-\alpha(\mu)]-\mu_2}{\sigma}\right)\right) * \varphi\left(\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right) * -\frac{1}{\sigma^2}[log[t_r-\alpha(\mu)]-\mu_2] - \left(\left(\varphi\left(\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right)\right)^2 * \frac{1}{\sigma^2}log[t_r-\alpha(\mu)]-\mu_2\right)}{\left(1-\Phi\left(\frac{\log[t_r-\alpha(\mu)]-\mu_2}{\sigma}\right)\right)^2} - \tag{32}$$

$$\frac{\varphi\left(\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right)}{1-\Phi\left[\frac{1}{\sigma}log[t_r-\alpha(\mu)]-\mu_2\right]} * \frac{1}{\sigma^2}\left[\frac{\tau e^{\mu_2-\mu_1}}{([t_r-\alpha(\mu)])} - 1\right]$$

The inverse of I denoted by $I^{-1}$ is the variance-covariance matrix for the MLEs.

$$I^{-1} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$$

The approximate confidence intervals are then given by:

$$\hat{\mu}_{1\pm} z{\alpha}/_{2} * \sqrt{d_{11}}$$

$$\hat{\mu}_{2\pm} z{\alpha}/_{2} * \sqrt{d_{22}}$$

$$\hat{\sigma}_{\pm} z{\alpha}/_{2} * \sqrt{d_{33}}$$

Numerical Methods

The ML equations for the lognormal distribution cannot be found in a closed form. Therefore, the MLEs are obtained by using a numerical optimization algorithm in order to maximize the  log likelihood function. We use the nlm function in R to numerically obtain the MLEs of the model parameters. The nlm function minimizes the log likelihood function and therefore, we need to multiply the log likelihood function of a negative sign to obtain the MLEs.

## 2.5 Bootstrap Inference

Bootstrapping is a computer intensive non-parametric method used for making statistical inference about the parameter of interest. Bootstrap sampling can also be carried out parametrically when some knowledge about the distribution of the population is available. To estimate the parameter of interest by bootstrap sampling, B samples of size n are drawn from the parametric estimate of the population and the statistic of interest is evaluated for each bootstrap sample (Efron, B. &

Tibshirani, R,1998).

Bias and Standard Error

Let $\theta$ be the parameter of some population, $\hat{\theta}$ is the estimate from the given data and $\hat{\theta}^*$ is the estimate for the bootstrap sample. The bootstrap bias estimate is given by:

$$E\left(\hat{\theta}^*\right) - \hat{\theta}$$

Standard Error

The parametric bootstrap estimate of the standard error is the standard deviation of the bootstrap sample.

Bootstrap Sample

The bootstrap sample from the simple step stress lognormal model with hybrid censoring is obtained using the following steps:

Step 1. Generating a random sample of size n from the uniform distribution and sorting it in an ascending order as shown in eq 33:

$$U_{(1)}<U_{(2)}<\ldots\ldots\ldots<U_{(N1)}<\tau<U_{(N1+1)}<\ldots<U_{(N1+N2)}<\min(t_{(r)},t_1) \quad (33)$$

Step 2. Finding $N_1$ such that:

$$U_{N_1:n} \leq \Phi\left(\frac{\log\tau - \hat{\mu}_1}{\hat{\sigma}}\right) \quad (34)$$

For i=1,………..,$N_1$

$$T_{i:n}^* = e^{\hat{\sigma}*\Phi^{-1}(u)+\hat{\mu}_1} \quad (35)$$

For j= $N_1$+1,……..,n

$$T_{j:n}^* = \alpha(\hat{\mu}) + e^{\hat{\sigma}*\Phi^{-1}(u)+\hat{\mu}_2}, where\ \alpha(\hat{\mu}) = \tau * (1 - e^{\hat{\mu}_2-\hat{\mu}_1}) \quad (36)$$

Step 3. Finding $N_2$ such that: $T_{j:n}^* \leq \min(t_{(r)},t_1)$

Then the sample will be in the following form:

$$T_{(1:n)}^*< T_{(2:n)}^*<\ldots\ldots\ldots< T_{(N_1:n)}^*< \tau<T_{(N_1+1:n)}^*<\ldots< T_{(N_1+N_2:n)}^* <\min(t_{(r)},t_1) \quad (37)$$

Step 4. Obtaining the MLEs $\hat{\mu}_1^*,\ \hat{\mu}_2^*$ and $\hat{\sigma}^*$ by using the nlm function in R.

Step 5. Repeating steps 2-5 R times.

Bootstrap Confidence Intervals

The parametric bootstrap procedures can be used as a replacement for the mathematical approximations that are difficult to compute and obtain by the means of Monte Carlo simulation. The parametric bootstrap procedures are used when the given data has a specified distribution and it usually gives good confidence intervals even for small sample sizes if the chosen distribution for the given data is the right distribution. There are several bootstrap confidence intervals that have been proposed so far. In our study we focus on two bootstrap intervals, the bootstrap-t interval, and the percentile interval.

Bootstrap-t Intervals

In the bootstrap-t method, we compute the statistic $t^* = \frac{\hat{\theta}^* - \hat{\theta}}{s_{\hat{\theta}^*}}$ for each generated bootstrap sample where $\hat{\theta}^*$ is the value of $\hat{\theta}$ for the bootstrap sample and $s_{\hat{\theta}^*}$ is the standard error for the bootstrap sample.

The bootstrap-t interval is constructed as follows:

$$\left(\hat{\theta} - q_{1-\alpha} s_{\hat{\theta}}, \hat{\theta} - q_{\alpha} s_{\hat{\theta}}\right)$$

Where $q_{1-\alpha}$ and $q_{\alpha}$ are the $(1-\alpha)$ & $\alpha$ percentiles of $t^*$ respectively.

Percentile Intervals

The percentile interval is the range of the middle $(1-\alpha)\%$ of a bootstrap distribution and it is given by:

$$\left(q_{\alpha/2}, q_{1-\alpha/2}\right)$$

# CHAPTER 3: SIMULATION STUDY

A Monte Carlo simulation has been performed for different sample sizes n, different predetermined number of failures r, and different stress changing time to assess the performance of the point and interval estimates for the model parameters and survival function. The simulation results are based on 2000 simulated samples and 1000 bootstrap samples. Table 1 shows the different choices of hybrid censoring schemes with different sample sizes and stress changing time.

Table 1. Different Hybrid Censoring Schemes

| Scheme | n | r | $(\tau, t_1)$ |
|---|---|---|---|
| 1 | 30 | $r_1$ =15 | (30,60) |
| 2 | 30 | $r_2$ =23 | (30,60) |
| 3 | 30 | $r_3$ =27 | (30,60) |
| 4 | 50 | $r_1$ =25 | (30,60) |
| 5 | 50 | $r_2$ =38 | (30,60) |
| 6 | 50 | $r_3$ =45 | (30,60) |
| 7 | 80 | $r_1$ =40 | (30,60) |
| 8 | 80 | $r_2$ =60 | (30,60) |
| 9 | 80 | $r_3$ =72 | (30,60) |
| 10 | 100 | $r_1$ =50 | (30,60) |
| 11 | 100 | $r_2$ =75 | (30,60) |
| 12 | 100 | $r_3$ =90 | (30,60) |
| 13 | 30 | $r_1$ =15 | (50,80) |
| 14 | 30 | $r_2$ =23 | (50,80) |
| 15 | 30 | $r_3$ =27 | (50,80) |
| 16 | 50 | $r_1$ =25 | (50,80) |
| 17 | 50 | $r_2$ =38 | (50,80) |
| 18 | 50 | $r_3$ =45 | (50,80) |
| 19 | 80 | $r_1$ =40 | (50,80) |
| 20 | 80 | $r_2$ =60 | (50,80) |
| 21 | 80 | $r_3$ =72 | (50,80) |
| 22 | 100 | $r_1$ =50 | (50,80) |
| 23 | 100 | $r_2$ =75 | (50,80) |
| 24 | 100 | $r_3$ =90 | (50,80) |

The simulation study is performed based on two stress changing time values $\tau =$ 30 and 50 respictively and test times $T_1^* = \min(t_r, t_1 = 60)$ and $T_2^* = \min(t_r, t_1 = 80)$ with different sample sizes n and predetermined number of failures r. The sample was generated considering the true values of $\mu_1, \mu_2$ and $\sigma$ as log(200), log(5) and 3, respectively. The true values of the survival function for $t_1$= 4.278758, $t_2$=41.47604 and $t_3$= 200 are .9, .7 and .5, respectively.

The performance of the MLEs of the model parameters and survival function is studied by calculating the Bias and MSE for different censoring schemes.

Approximate confidence interval, bootstrap-t interval and percentile confidence intervals have been constructed to obtain the interval estimates of the model parameters and the survival function for three different values of t. To compare the performance of these confidence intervals, the coverage probabilities and average interval lengths are calculated.

Table 2 shows the Bias and MSE of the point estimators of $\mu_1, \mu_2$ and $\sigma$ for different hybrid censoring schemes.

Table 2. The Bias and MSE of the MLE of the Model Parameters for Different Hybrid Censoring Schemes

| n | r | $(\tau, t_1)$ | Bias | | | MSE | | |
|---|---|---|---|---|---|---|---|---|
| | | | $\hat{\mu}_1$ | $\hat{\mu}_2$ | $\hat{\sigma}$ | $\hat{\mu}_1$ | $\hat{\mu}_2$ | $\hat{\sigma}$ |
| 30 | 15 | (30,60) | -0.145 | -0.013 | -0.247 | 0.815 | 0.521 | 0.621 |
| | 23 | | -0.042 | 0.094 | -0.140 | 0.748 | 0.414 | 0.474 |
| | 27 | | -0.021 | 0.100 | -0.116 | 0.725 | 0.408 | 0.446 |
| 50 | 25 | | -0.082 | 0.013 | -0.149 | 0.511 | 0.307 | 0.380 |
| | 38 | | -0.016 | 0.075 | -0.082 | 0.441 | 0.253 | 0.269 |
| | 45 | | -0.002 | 0.079 | -0.068 | 0.431 | 0.249 | 0.257 |
| 80 | 40 | | -0.045 | 0.013 | -0.098 | 0.341 | 0.182 | 0.231 |
| | 60 | | -0.009 | 0.047 | -0.060 | 0.278 | 0.151 | 0.159 |

| n | r | $(\tau, t_1)$ | Bias | | | MSE | | |
|---|---|---|---|---|---|---|---|---|
| | | | $\hat{\mu}_1$ | $\hat{\mu}_2$ | $\hat{\sigma}$ | $\hat{\mu}_1$ | $\hat{\mu}_2$ | $\hat{\sigma}$ |
| | 72 | | -0.0001 | 0.050 | -0.050 | 0.270 | 0.149 | 0.153 |
| 100 | 50 | | -0.056 | 0.008 | -0.082 | 0.272 | 0.141 | 0.192 |
| | 75 | | -0.027 | 0.035 | -0.051 | 0.214 | 0.118 | 0.127 |
| | 90 | | -0.020 | 0.037 | -0.043 | 0.210 | 0.117 | 0.123 |
| 30 | 15 | (50,80) | -0.035 | -0.011 | -0.177 | 0.638 | 0.594 | 0.548 |
| | 23 | | -0.020 | 0.100 | -0.128 | 0.667 | 0.447 | 0.459 |
| | 27 | | -0.003 | 0.107 | -0.109 | 0.653 | 0.440 | 0.438 |
| 50 | 25 | | -0.026 | 0.009 | -0.108 | 0.410 | 0.357 | 0.328 |
| | 38 | | -0.0003 | 0.078 | -0.074 | 0.397 | 0.279 | 0.263 |
| | 45 | | 0.010 | 0.083 | -0.062 | 0.389 | 0.275 | 0.254 |
| 80 | 40 | | -0.005 | 0.001 | -0.064 | 0.273 | 0.216 | 0.203 |
| | 60 | | 0.010 | 0.042 | -0.046 | 0.243 | 0.169 | 0.156 |
| | 72 | | 0.017 | 0.046 | -0.037 | 0.238 | 0.167 | 0.152 |
| 100 | 50 | | -0.012 | -0.004 | -0.046 | 0.218 | 0.165 | 0.170 |
| | 75 | | -0.003 | 0.027 | -0.034 | 0.186 | 0.131 | 0.124 |
| | 90 | | 0.004 | 0.029 | -0.027 | 0.183 | 0.129 | 0.121 |

From table 2, we can see that the bias decreases as the sample size n and r increase, however the bias for $\hat{\mu}_2$ increases slightly with increasing the predetermined number of failures r. The MSE decreases as the sample size n and r increase. It can be noticed that the bias for $\hat{\sigma}$ is always negative which indicates that the MLE for $\hat{\sigma}$ seems to underestimate the parameter value of $\sigma$.

Table 3 shows the Bias and MSE of the point estimators of the survival function for different hybrid censoring schemes.

Table 3. The Bias and MSE of the MLE of the Survival Function for Different Hybrid Censoring Schemes

| n | r | $(\tau, t_1)$ | Bias | | | MSE | | |
|---|---|---|---|---|---|---|---|---|
| | | | $S(t_1)$ | $S(t_2)$ | $S(t_3)$ | $S(t_1)$ | $S(t_2)$ | $S(t_3)$ |
| 30 | 15 | (30,60) | 0.008 | -0.014 | -0.044 | 0.002 | 0.007 | 0.019 |
| | 23 | | 0.005 | -0.006 | -0.023 | 0.002 | 0.007 | 0.015 |
| | 27 | | 0.004 | -0.004 | -0.018 | 0.002 | 0.007 | 0.014 |
| 50 | 25 | | 0.005 | -0.008 | -0.025 | 0.001 | 0.004 | 0.011 |
| | 38 | | 0.003 | -0.002 | -0.012 | 0.001 | 0.004 | 0.008 |
| | 45 | | 0.003 | -0.001 | -0.009 | 0.001 | 0.004 | 0.008 |
| 80 | 40 | | 0.004 | -0.004 | -0.015 | 0.001 | 0.003 | 0.007 |
| | 60 | | 0.003 | -0.001 | -0.007 | 0.001 | 0.003 | 0.005 |
| | 72 | | 0.003 | -0.0004 | -0.005 | 0.001 | 0.003 | 0.005 |
| 100 | 50 | | 0.002 | -0.006 | -0.015 | 0.001 | 0.002 | 0.005 |
| | 75 | | 0.001 | -0.003 | -0.008 | 0.001 | 0.002 | 0.004 |
| | 90 | | 0.001 | -0.002 | -0.007 | 0.001 | 0.002 | 0.004 |
| 30 | 15 | (50,80) | 0.010 | 0.001 | -0.022 | 0.002 | 0.005 | 0.012 |
| | 23 | | 0.006 | -0.002 | -0.017 | 0.002 | 0.006 | 0.012 |
| | 27 | | 0.005 | -0.001 | -0.014 | 0.002 | 0.006 | 0.012 |
| 50 | 25 | | 0.006 | -0.001 | -0.014 | 0.001 | 0.004 | 0.008 |
| 50 | 38 | (50,80) | 0.004 | -.00003 | -0.008 | 0.001 | 0.004 | 0.007 |
| | 45 | | 0.004 | 0.001 | -0.007 | 0.001 | 0.004 | 0.007 |
| 80 | 40 | | 0.004 | .0001 | -0.007 | 0.001 | 0.002 | 0.005 |
| | 60 | | 0.003 | 0.001 | -0.004 | 0.001 | 0.002 | 0.004 |
| | 72 | | 0.003 | 0.002 | -0.002 | 0.001 | 0.002 | 0.004 |
| 100 | 50 | | 0.002 | -0.001 | -0.007 | 0.001 | 0.002 | 0.004 |
| | 75 | | 0.002 | -0.0003 | -0.004 | 0.001 | 0.002 | 0.003 |
| | 90 | | 0.001 | .00004 | -0.003 | 0.001 | 0.002 | 0.003 |

Based on the above table, the bias and MSE decrease with increasing the sample size n and the predetermined number of failures r.

Table 4 shows the coverage probability and average length for the three confidence intervals for estimating the model parameters.

Table 4. The Average Length and Coverage Probability for the Three Confidence Intervals for Estimating the Model Parameters

| | n | r | $(\tau, t_1)$ | Approx. CI | | | Bootstrap-t CI | | | Percentile CI | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | $\mu_1$ | $\mu_2$ | $\sigma$ | $\mu_1$ | $\mu_2$ | $\sigma$ | $\mu_1$ | $\mu_2$ | $\sigma$ |
| E.L | 30 | 15 | (30,60) | 3.580 | 2.693 | 2.876 | 4.109 | 3.193 | 3.964 | 3.295 | 2.946 | 2.750 |
| C.P | | | | 0.892 | 0.934 | 0.862 | 0.955 | 0.957 | 0.957 | 0.936 | 0.932 | 0.853 |
| E.L | | 23 | | 3.307 | 2.474 | 2.530 | 3.824 | 2.590 | 3.202 | 3.390 | 2.559 | 2.541 |
| C.P | | | | 0.917 | 0.936 | 0.900 | 0.943 | 0.947 | 0.951 | .919 | .929 | .882 |
| E.L | | 27 | | 3.312 | 2.486 | 2.522 | 3.573 | 2.571 | 2.877 | 3.394 | 2.549 | 2.510 |
| C.P | | | | 0.924 | 0.938 | 0.909 | 0.937 | 0.950 | 0.945 | 0.93 | 0.93 | 0.908 |
| E.L | 50 | 25 | | 2.858 | 2.104 | 2.322 | 2.969 | 2.266 | 2.658 | 2.661 | 2.210 | 2.208 |
| C.P | | | | 0.918 | 0.939 | 0.896 | 0.926 | 0.947 | 0.935 | 0.921 | 0.938 | 0.872 |
| E.L | | 38 | | 2.574 | 1.925 | 1.978 | 2.826 | 1.946 | 2.118 | 2.586 | 1.947 | 1.970 |
| C.P | | | | 0.933 | 0.945 | 0.917 | 0.946 | 0.941 | 0.933 | 0.927 | 0.935 | 0.898 |
| E.L | | 45 | | 2.574 | 1.930 | 1.970 | 2.703 | 1.925 | 2.047 | 2.570 | 1.940 | 1.938 |
| C.P | | | | 0.939 | 0.946 | 0.931 | 0.947 | 0.942 | 0.933 | 0.937 | 0.936 | 0.915 |
| E.L | 80 | 40 | | 2.293 | 1.668 | 1.874 | 2.442 | 1.731 | 2.029 | 2.206 | 1.713 | 1.820 |
| C.P | | | | 0.930 | 0.951 | 0.922 | 0.948 | 0.945 | 0.945 | 0.944 | 0.944 | 0.9 |
| E.L | | 60 | | 2.031 | 1.521 | 1.563 | 2.098 | 1.639 | 1.662 | 2.051 | 1.538 | 1.576 |
| C.P | | | | 0.936 | 0.946 | 0.933 | 0.948 | 0.957 | 0.939 | 0.946 | 0.940 | 0.921 |
| E.L | | 72 | | 2.028 | 1.523 | 1.555 | 2.105 | 1.618 | 1.608 | 2.034 | 1.534 | 1.549 |
| C.P | | | | 0.942 | 0.946 | 0.940 | 0.953 | 0.957 | 0.942 | 0.956 | 0.941 | 0.928 |
| E.L | 100 | 50 | | 2.052 | 1.496 | 1.684 | 2.119 | 1.550 | 1.784 | 2.008 | 1.528 | 1.661 |
| C.P | | | | 0.928 | 0.951 | 0.919 | 0.942 | 0.948 | 0.933 | 0.944 | 0.942 | 0.916 |
| E.L | | 75 | | 1.809 | 1.362 | 1.396 | 1.918 | 1.425 | 1.439 | 1.840 | 1.381 | 1.415 |
| C.P | | | | 0.943 | 0.950 | 0.935 | 0.955 | 0.953 | 0.938 | 0.945 | 0.943 | 0.935 |
| E.L | 100 | 90 | (30,60) | 1.807 | 1.364 | 1.389 | 1.870 | 1.419 | 1.393 | 1.823 | 1.376 | 1.392 |
| C.P | | | | 0.948 | 0.949 | 0.937 | 0.951 | 0.956 | 0.936 | 0.948 | 0.948 | 0.94 |
| E.L | 30 | 15 | (50,80) | 3.302 | 2.937 | 2.783 | 3.401 | 3.583 | 3.910 | 2.918 | 3.238 | 2.668 |

| | n | r | $(\tau, t_1)$ | Approx. CI | | | Bootstrap-t CI | | | Percentile CI | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | $\mu_1$ | $\mu_2$ | $\sigma$ | $\mu_1$ | $\mu_2$ | $\sigma$ | $\mu_1$ | $\mu_2$ | $\sigma$ |
| C.P | | | | 0.938 | 0.929 | 0.870 | 0.969 | 0.959 | 0.970 | 0.982 | 0.934 | 0.889 |
| E.L | | 23 | | 3.066 | 2.561 | 2.485 | 3.368 | 2.803 | 3.240 | 3.109 | 2.663 | 2.480 |
| C.P | | | | 0.922 | 0.926 | 0.898 | 0.944 | 0.955 | 0.963 | 0.927 | 0.924 | 0.893 |
| E.L | | 27 | | 3.072 | 2.570 | 2.480 | 3.227 | 2.727 | 3.064 | 3.120 | 2.637 | 2.465 |
| C.P | | | | 0.925 | 0.929 | 0.908 | 0.941 | 0.952 | 0.96 | 0.938 | 0.928 | 0.915 |
| E.L | 50 | 25 | | 2.599 | 2.279 | 2.216 | 2.511 | 2.606 | 2.472 | 2.361 | 2.491 | 2.114 |
| C.P | | | | 0.932 | 0.940 | 0.901 | 0.936 | 0.961 | 0.941 | 0.953 | 0.938 | 0.895 |
| E.L | | 38 | | 2.390 | 1.992 | 1.946 | 2.406 | 2.046 | 2.037 | 2.390 | 2.026 | 1.932 |
| C.P | | | | 0.930 | 0.936 | 0.915 | 0.926 | 0.94 | 0.925 | 0.926 | 0.934 | 0.903 |
| E.L | | 45 | | 2.392 | 1.996 | 1.940 | 2.376 | 2.021 | 1.989 | 2.382 | 2.014 | 1.912 |
| C.P | | | | 0.931 | 0.941 | 0.921 | 0.935 | 0.939 | 0.925 | 0.938 | 0.939 | 0.914 |
| E.L | 80 | 40 | | 2.082 | 1.783 | 1.785 | 2.170 | 1.887 | 1.953 | 1.962 | 1.905 | 1.723 |
| C.P | | | | 0.947 | 0.948 | 0.928 | 0.958 | 0.952 | 0.953 | 0.951 | 0.945 | 0.915 |
| E.L | | 60 | | 1.894 | 1.577 | 1.545 | 1.975 | 1.697 | 1.616 | 1.896 | 1.594 | 1.545 |
| C.P | | | | 0.944 | 0.938 | 0.934 | 0.956 | 0.961 | 0.949 | 0.946 | 0.938 | 0.922 |
| E.L | | 72 | | 1.892 | 1.579 | 1.538 | 1.936 | 1.677 | 1.611 | 1.885 | 1.586 | 1.526 |
| C.P | | | | 0.944 | 0.938 | 0.939 | 0.954 | 0.959 | 0.956 | 0.955 | 0.936 | 0.931 |
| E.L | 100 | 50 | | 1.866 | 1.597 | 1.606 | 1.922 | 1.625 | 1.676 | 1.791 | 1.683 | 1.564 |
| C.P | | | | 0.946 | 0.952 | 0.932 | 0.952 | 0.943 | 0.935 | 0.944 | 0.944 | 0.921 |
| E.L | | 75 | | 1.692 | 1.413 | 1.382 | 1.747 | 1.460 | 1.445 | 1.698 | 1.427 | 1.387 |
| C.P | | | | 0.943 | 0.947 | 0.939 | 0.954 | 0.955 | 0.942 | 0.943 | 0.944 | 0.927 |
| E.L | | 90 | | 1.691 | 1.415 | 1.377 | 1.722 | 1.460 | 1.406 | 1.688 | 1.422 | 1.371 |
| C.P | | | | 0.949 | 0.950 | 0.937 | 0.954 | 0.953 | 0.941 | 0.949 | 0.945 | 0.931 |

From table 4, the performance of the three confidence intervals for estimating the model parameters is evaluated based on the average length and coverage probability.

Based on the average length, it can be seen that as the sample size n and predetermined number of failures r increase, the average lengths for the three confidence intervals decrease. As it can be seen from table 4, the bootstrap-t interval has the highest average length compared to the approximate and percentile CIs.

Based on the coverage probability, when the sample size is small (n=30,n=50), the bootstrap-t interval gives the best results. For large sample sizes (n=80,n=100), all three confidence intervals give considerably good coverage probabilities.

Comparing the performance of the three CI based on the coverage probabilities, the bootstrap-t interval gives the best results at all considered sample sizes and r values. Both the approximate and percentile intervals give good coverage probabilities when the sample size is large.

Comparing the coverage probability for the different sample sizes considered in this simulation study, the coverage probability is closer to the nominal level for the three confidence intervals when the sample size is 100.

Table 5 shows the coverage probability and average length for the three confidence intervals for estimating the survival function.

Table 5. The Average Length and Coverage Probability for the Three Confidence Intervals for Estimating the Survival Function

| | n | r | $(\tau, t_1)$ | Approx. CI | | | Bootstrap-t CI | | | Percentile CI | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | $S_1$ | $S_2$ | $S_3$ | $S_1$ | $S_2$ | $S_3$ | $S_1$ | $S_2$ | $S_3$ |
| E.L | 30 | 15 | (30,60) | 0.371 | 0.582 | 0.482 | 0.435 | 0.513 | 0.962 | 0.156 | 0.336 | 0.529 |
| C.P | | | | 0.970 | 0.983 | 0.828 | 0.947 | 0.950 | 0.985 | 0.877 | 0.997 | 0.986 |
| E.L | | 23 | | 0.325 | 0.501 | 0.443 | 0.282 | 0.492 | 0.771 | 0.160 | 0.336 | 0.478 |
| C.P | | | | 0.979 | 0.976 | 0.864 | 0.946 | 0.962 | 0.975 | 0.908 | 0.946 | 0.929 |
| E.L | | 27 | | 0.322 | 0.496 | 0.443 | 0.254 | 0.488 | 0.687 | 0.160 | 0.330 | 0.465 |
| C.P | | | | 0.978 | 0.979 | 0.880 | 0.943 | 0.966 | 0.971 | 0.914 | 0.946 | 0.941 |
| E.L | 50 | 25 | | 0.305 | 0.455 | 0.383 | 0.176 | 0.389 | 0.549 | 0.129 | 0.263 | 0.402 |
| C.P | | | | 0.990 | 0.985 | 0.867 | 0.932 | 0.973 | 0.953 | 0.896 | 0.976 | 0.961 |
| E.L | | 38 | | 0.258 | 0.386 | 0.345 | 0.149 | 0.358 | 0.484 | 0.128 | 0.256 | 0.362 |
| C.P | | | | 0.994 | 0.987 | 0.905 | 0.931 | 0.976 | 0.962 | 0.911 | 0.93 | 0.927 |
| E.L | | 45 | | 0.257 | 0.384 | 0.344 | 0.150 | 0.356 | 0.450 | 0.128 | 0.252 | 0.352 |
| C.P | | | | 0.995 | 0.986 | 0.917 | 0.932 | 0.977 | 0.965 | 0.912 | 0.938 | 0.937 |
| E.L | 80 | 40 | | 0.249 | 0.362 | 0.307 | 0.124 | 0.296 | 0.405 | 0.106 | 0.208 | 0.313 |
| C.P | | | | 0.998 | 0.994 | 0.894 | 0.949 | 0.978 | 0.966 | 0.918 | 0.955 | 0.949 |
| E.L | | 60 | | 0.207 | 0.304 | 0.272 | 0.112 | 0.258 | 0.334 | 0.104 | 0.200 | 0.281 |
| C.P | | | | 0.996 | 0.995 | 0.918 | 0.939 | 0.982 | 0.963 | 0.929 | 0.949 | 0.946 |
| E.L | 80 | 72 | (30,60) | 0.205 | 0.302 | 0.272 | 0.114 | 0.251 | 0.318 | 0.104 | 0.197 | 0.274 |

| | n | r | $(\tau, t_1)$ | Approx. CI | | | Bootstrap-t CI | | | Percentile CI | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | $S_1$ | $S_2$ | $S_3$ | $S_1$ | $S_2$ | $S_3$ | $S_1$ | $S_2$ | $S_3$ |
| C.P | | | | 0.996 | 0.993 | 0.926 | 0.943 | 0.980 | 0.958 | 0.931 | 0.950 | 0.956 |
| E.L | 100 | 50 | | 0.225 | 0.323 | 0.274 | 0.108 | 0.261 | 0.337 | 0.095 | 0.185 | 0.279 |
| C.P | | | | 0.999 | 0.995 | 0.891 | 0.95 | 0.983 | 0.955 | 0.916 | 0.955 | 0.945 |
| E.L | | 75 | | 0.186 | 0.271 | 0.242 | 0.101 | 0.231 | 0.274 | 0.094 | 0.178 | 0.249 |
| C.P | | | | 0.998 | 0.994 | 0.918 | 0.944 | 0.984 | 0.955 | 0.933 | 0.952 | 0.945 |
| E.L | | 90 | | 0.185 | 0.269 | 0.241 | 0.101 | 0.230 | 0.266 | 0.093 | 0.176 | 0.244 |
| C.P | | | | 0.998 | 0.993 | 0.926 | 0.945 | 0.982 | 0.955 | 0.933 | 0.955 | 0.949 |
| E.L | 30 | 15 | (50,80) | 0.336 | 0.524 | 0.452 | 0.994 | 0.416 | 0.590 | 0.151 | 0.279 | 0.448 |
| C.P | | | | 0.965 | 1 | 0.893 | 0.957 | 0.966 | 0.978 | 0.872 | 0.969 | 0.998 |
| E.L | | 23 | | 0.303 | 0.465 | 0.416 | 0.372 | 0.411 | 0.586 | 0.157 | 0.306 | 0.432 |
| C.P | | | | 0.970 | 0.978 | 0.885 | 0.951 | 0.963 | 0.968 | 0.895 | 0.935 | 0.932 |
| E.L | | 27 | | 0.302 | 0.462 | 0.416 | 0.356 | 0.415 | 0.558 | 0.158 | 0.303 | 0.423 |
| C.P | | | | 0.969 | 0.982 | 0.892 | 0.952 | 0.962 | 0.966 | 0.901 | 0.931 | 0.943 |
| E.L | 50 | 25 | | 0.276 | 0.408 | 0.352 | 0.191 | 0.339 | 0.393 | 0.128 | 0.229 | 0.344 |
| C.P | | | | 0.988 | 0.994 | 0.897 | 0.942 | 0.979 | 0.947 | 0.906 | 0.969 | 0.986 |
| E.L | | 38 | | 0.243 | 0.360 | 0.322 | 0.156 | 0.317 | 0.369 | 0.127 | 0.236 | 0.329 |
| C.P | | | | 0.991 | 0.987 | 0.909 | 0.937 | 0.971 | 0.940 | 0.903 | 0.934 | 0.926 |
| E.L | | 45 | | 0.241 | 0.358 | 0.322 | 0.153 | 0.316 | 0.360 | 0.127 | 0.234 | 0.323 |
| C.P | 80 | 40 | (50,80) | 0.991 | 0.986 | 0.915 | 0.932 | 0.973 | 0.947 | 0.905 | 0.935 | 0.938 |
| E.L | | | | 0.226 | 0.325 | 0.280 | 0.123 | 0.252 | 0.311 | 0.105 | 0.186 | 0.273 |
| C.P | | | | 0.996 | 0.997 | 0.919 | 0.937 | 0.966 | 0.957 | 0.919 | 0.978 | 0.984 |

| | n | r | $(\tau, t_1)$ | Approx. CI | | | Bootstrap-t CI | | | Percentile CI | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | $S_1$ | $S_2$ | $S_3$ | $S_1$ | $S_2$ | $S_3$ | $S_1$ | $S_2$ | $S_3$ |
| E.L | | 60 | | 0.195 | 0.284 | 0.254 | 0.112 | 0.248 | 0.298 | 0.103 | 0.186 | 0.258 |
| C.P | | | | 0.995 | 0.994 | 0.927 | 0.939 | 0.981 | 0.961 | 0.926 | 0.950 | 0.946 |
| E.L | | 72 | | 0.194 | 0.283 | 0.254 | 0.113 | 0.242 | 0.276 | 0.103 | 0.185 | 0.253 |
| C.P | | | | 0.995 | 0.993 | 0.933 | 0.938 | 0.977 | 0.958 | 0.932 | 0.95 | 0.955 |
| E.L | 100 | 50 | | 0.204 | 0.291 | 0.250 | 0.106 | 0.232 | 0.277 | 0.095 | 0.168 | 0.245 |
| C.P | | | | 0.998 | 0.998 | 0.923 | 0.937 | 0.982 | 0.959 | 0.922 | 0.975 | 0.968 |
| E.L | | 75 | | 0.176 | 0.254 | 0.226 | 0.103 | 0.220 | 0.251 | 0.093 | 0.166 | 0.230 |
| C.P | | | | 0.998 | 0.994 | 0.936 | 0.945 | 0.984 | 0.958 | 0.927 | 0.948 | 0.943 |
| E.L | | 90 | | 0.175 | 0.252 | 0.226 | 0.102 | 0.217 | 0.240 | 0.093 | 0.165 | 0.226 |
| C.P | | | | 0.998 | 0.995 | 0.938 | 0.943 | 0.982 | 0.957 | 0.930 | 0.954 | 0.949 |

From table 5, it can be seen that the average length of the confidence intervals decreases as the sample size n and r increase.

Based on the coverage probability, when the sample size is small, the approximate and percentile CI give poor performance while the bootstrap-t interval gives satisfactory performance. When the sample size increases, the bootstrap-t and percentile CI give good results while the approximate CI still gives poor results.

It can be noted that when comparing the bootstrap-t CI and the percentile CI, the bootstrap-t interval gives better coverage probability for small sample sizes while the percentile CI gives better results for large sample sizes.

# CHAPTER 4: AN ILLUSTRATIVE EXAMPLE

To illustrate the application of the methods presented and discussed in this research, we present a numerical example from simulated data.

A step stress lognormal sample of size 30 with hybrid censoring was generated. The predetermined number of failures r was chosen to be 15 with stress changing time of 30 and experiment time of $T_1^* = \min(t_{15}, 60)$. The MLE of $\mu_1, \mu_2$ ,$\sigma$ and their corresponding standard errors were obtained along with the hessian and variance-covariance matrix. The approximate, bootstrap percentile and bootstrap-t confidence intervals were obtained, and their corresponding length was also calculated.

The simulated sample and the obtained MLEs are shown in table 6 and 7.

Table 6. Generated Lognormal Simple Step Stress Sample of Size 30 with τ=30 and Fixed Time $t_1 = 60$ with True Parameter Values $\mu_1 = log(200)$, $\mu_2 = log(5)$ and $\sigma = 3$

| Lifetimes | | | | | | |
|---|---|---|---|---|---|---|
| Under normal stress level | | | | | | |
| 0.5259 | 11.2563 | 12.0893 | 21.9279 | | | |
| Under higher stress level | | | | | | |
| 30.0079 | 30.0537 | 30.9522 | 31.13089 | 31.3562 | 32.0215 | 32.15492 |
| 32.7659 | 32.8116 | 33.8831 | 35.6947 | 36.1690 | 36.4736 | 43.9303 |
| 48.6155 | 49.9542 | 52.2285 | 61.9079 | 68.0899 | 157.7506 | 158.6676 |
| 223.0157 | 432.5115 | 1338.9093 | 3101.1047 | 3276.2830 | | |

Table 7. The MLEs for the Parameters $\mu_2, \mu_2$and $\sigma$ and Their Corresponding Standard Errors

| $\hat{\mu}_1$ | $\hat{\mu}_2$ | $\hat{\sigma}$ |
|---|---|---|
| 6.553779 | 1.869605 | 2.870388 |
| $\mathrm{SE}(\hat{\mu}_1)$ | $\mathrm{SE}(\hat{\mu}_2)$ | $\mathrm{SE}(\hat{\sigma})$ |
| 1.995996 | 0.4186727 | 0.8634164 |

The hessian matrix was found as below:

$$\begin{pmatrix} 1.8111902 & -0.9171184 & -2.1090687 \\ -0.9171184 & 2.9909645 & 0.6566688 \\ -2.1090687 & 0.6566688 & 3.6810780 \end{pmatrix}$$

The variance-covariance matrix was then derived by finding the inverse of the hessian matrix and is given below:

$$\begin{pmatrix} 1.9959957 & .3756656 & 1.0765879 \\ .3756656 & .4186727 & .1405500 \\ 1.0765879 & .1405500 & .8634164 \end{pmatrix}$$

The confidence intervals for the model parameters and their corresponding lengths are calculated and shown in table 8.

Table 8. 95% Confidence Intervals for the Model Parameters

| | Approx. CI | Length | Bootstrap-t CI | Length | Percentile CI | Length |
|---|---|---|---|---|---|---|
| $\mu_1$ | (3.785, 9.323) | 5.538 | (2.787, 10.237) | 7.451 | (4.314, 8.844) | 4.530 |
| $\mu_2$ | (0.601, 3.138) | 2.536 | (0.481, 3.426) | 2.945 | (0.568, 3.030) | 2.461 |
| $\sigma$ | (1.049, 4.692) | 3.642 | (1.158, 5.223) | 4.065 | (1.092, 4.165) | 3.073 |

The length of the Bootstrap-t confidence interval is the longest followed by the approximate interval. The percentile interval for estimating the model parameters $\mu_1, \mu_2$ and $\sigma$ has the shortest length.

# CHAPTER 5: CONCLUSIONS AND FURTHER RESEARCH

## 5.1 Conclusions

In this research, we work with step stress lognormal life test data with hybrid censoring where the cumulative exposure model has been fit to the data. The study's main interest is finding good estimators for the model parameters in addition to estimating the survival function. We obtained the maximum likelihood estimators numerically by using the nlm function in R since the likelihood equations for the lognormal distribution cannot be found explicitly. To study and examine the performance of the point and interval estimates, a simulation study was performed. The Bias and MSE were calculated to assess the performance of the point estimators of the model parameters and survival function and the coverage probabilities and average lengths were calculated to examine the performance of the confidence intervals. After analyzing the simulation results, we observe the following points:

- When assessing and comparing the performance of the interval estimates of the model parameters based on the coverage probability, it was found that the bootstrap-t interval gives the best results for all the considered sample sizes in this simulation study compared to the approximate and percentile confidence intervals. The approximate and percentile confidence intervals can be used with large sample sizes (n=80, n=100).

- When evaluating and comparing the performance of the interval estimates of the survival function based on the coverage probability, it can be noticed that the approximate confidence interval gives the worst results for the small and large sample sizes considered in this simulation study and hence is not recommended to use for estimating the survival probability. The bootstrap-t interval gives satisfactory performance for all considered sample sizes, while the percentile intervals give satisfying coverage probability for large sample sizes.

## 5.2 Further Research

We now mention some problems related to our research and has yet to be considered:

- We considered a special case of the SSTLT experiments. therefore, a study about multiple step stress lognormal model can be considered.
- Other hybrid censoring schemes can also be considered.
- Other models for extrapolating step stress data can be also considered.

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