

Joint replenishment model for multiple products with substitution



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ABSTRACT

We extend the classical economic order quantity model to address the joint replenishment of multiple products under substitution. The proposed model optimizes ordering quantities for each product under substitution effects with the objective of minimizing the total cost associated with the setup, holding, and shortage of products, while partially meeting demand. First, the special case of three substitutable products is examined in detail. Then, a nonlinear mathematical programming formulation is presented as a general-purpose solution approach for any number of substitutable products. The convexity of the model is discussed. We find that the objective function to be convex in the important special case of products having equal unit holding costs, which typically holds for substitutable products in practice. Sensitivity analysis is conducted in order to determine the impact of cost parameters variations on the ordering policy. We focus on identifying conditions that favor substitution among products. We find that allowing substitution among products is an effective vehicle for cost cutting in supply chain settings involving high fixed costs, low holding costs, and low shortage costs.

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1. Introduction

Joint replenishment of products is a common approach in supply chain management, which is sought to reduce fixed costs, e.g., by shipping several products on one truck. Joint replenishment may be also needed for practical logistical purposes. For example, many retailers prefer to receive certain types of goods (e.g., groceries) at a specific time of the day or the week. The popularity of joint replenishment practices is reflected by a wide academic research on the topic as indicated

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in the review paper of Khouja and Goyal [1]. Examples of recent works on joint replenishment include Hong and Kim [2], Porras and Dekker [3], Schulz and Telha [4], Silva and Gao [5] Zhang et al. [6], and Zhou et al. [7].

Demand substitutability commonly occurs for products that are similar in nature (e.g., different brands of coffee or soda) and may be jointly ordered/shipped. Under such settings, customers of one product can switch to another similar product due to price or availability. Analyzing the effect of demand substitution on the inventory management of several related products is also an important problem that has received wide attention in the literature. Detailed accounts of the demand substitution literature can be found in the review papers by Kök et al. [8], Mahajan and van Ryzin [9], and Maddah et al. [10]. Recent literature has focused on stochastic demand substitution driven by consumer choice models adapted from the economics and marketing literature (e.g., van Ryzin and Mahajan [11], Mahajan and van Ryzin [12], Smith and Agrawal [13], Cachon and Kök [14], Gaur and Honhon [15], Gurler and Yilmaz [16], Maddah and Bish [17], and Maddah et al. [10]).

Despite the relevance of the joint replenishment practice to substitutable products, the academic literature has devoted little attention to studying the joint replenishment of substitutable products. Notable exceptions include the studies by Drezner et al. [18], Gurnani and Drezner [19], and the recent work by Salameh et al. [20]. Drezner et al. [18] study the joint replenishment of two substitutable products having deterministic demand in an economic order quantity setting, assuming a one-to-one substitution, where the demand of a product is fully substituted by another product in the event of a stock-out of the first product. This type of substitution is possible in manufacturing settings. Gurnani and Drezner [19] extend the work of Drezner et al. [18] to analyze joint replenishment of multiple, two or more, products. Gurnani and Drezner [19] also assumed a one-to-one substitution and consider a type of one-way substitution where customers could “upgrade” to a set of higher quality products in the event that their most preferred product is stocked out.

Recently, Salameh et al. [20] consider a two-product joint replenishment model with substitution in an EOQ framework similar to Drezner et al. [18]. However, they allowed for partial substitution, meaning that in the event of a product stock-out, a fraction of its customers will substitute to the other product, while the remainder customers will chose not to buy, leading to loss sales, which incurs a penalty. Salameh et al. [20] also allowed for a two-way substitution. They suggest adopting the substitution direction which has the lowest cost, by solving two related problems with (i) the second product substituting the first, and (ii) vice versa. Krommyda et al. [21] consider a problem similar to Salameh et al. [20] of two products under two-way, stock-out, and partial substitution within the EOQ framework; but further assume that the demand of a product is stimulated by the inventory levels of *both* products, in an interesting extension of the single-product literature with stock-dependent demand.

In this paper, we consider the joint replenishment of multiple, three or more, substitutable products, in an EOQ framework, under a versatile substitution model where every product in-stock can partially substitute a stocked-out product. In addition, a fraction of customers may elect not to substitute their most preferred product, and a lost sales penalty is charged. We first develop our substitution model and cast it into a nonlinear programming model, and then draw useful managerial insights. Our work can be seen as an extension of the two-product work of Salameh et al. [20] to the more challenging case of three or more products. We differ from the work of Gurnani and Drezner [19] in that we consider partial two-way substitutions, while they consider one-to-one, one-way substitution. Moreover, our mathematical model is more general than that of Gurnani and Drezner [19], which can be seen as a special case of our model. More notably, our work is applicable to retailing, while that of Gurnani and Drezner [19] is more adequate for manufacturing contexts.

At this point, it is worth clarifying what is exactly meant by “substitutable products” in this paper. For our purpose in this paper, the substitutable products we consider belong to a set of products that serves the same basic need for the consumer (e.g., drinking coffee, brushing teeth, or washing clothes.) but differ in some secondary aspect such as color, flavor, or smell. This is, for example, the case of several fast-moving consumer goods (FMCG) categories that are offered by super markets, e.g., coffee, toothpaste, and washing detergent. The economics literature refers to this type of substitutable products as “horizontally differentiated”, and generally considers such products to have equal or approximately equal unit costs and different demand rates (e.g., Anderson et al. [22]).

Finally, it is worth commenting on the EOQ setting utilized in this paper. The EOQ model is among the most popular inventory systems, especially in academic studies (see, for example, Silver et al. [23] and Zipkin [24] for overviews). While some authors defend the applicability of the EOQ model in practice (e.g., Osteryoung et al. [25] and Silver [26]), many criticize its applicability, mainly due to the difficulty is estimating its related costs (e.g., Jones [27], Selen and Wood [28], and Sprague and Sardy [29]). Jaber et al. [30] attempt to rectify the limitations of the EOQ model by appealing to thermodynamics principles with an “entropy cost” capturing hidden costs. Subsequently, several extensions of the base model in Jaber et al. [30] took place to account for different complicating factors such as delays in payments, deterioration effects, and supply chain effects, among other things, as nicely summarized in Jaber [31]. It is worth noting than none of the works surveyed in Jaber [31] considers product substitution effects such as the ones mentioned in this paper, which could be an interesting area for future work, especially that it might be difficult to estimate the substitution rates and the stock-out costs of our model in some settings.

The remainder of this paper is organized as follows. Section 2 provides a formal problem statement and our model for the case of three substitutable products. Section 3 extends the model to $n \geq 4$ products, presents the solution methodology in order to get the optimal ordering quantities, and establishes convexity properties of the cost function. In Section 4, a numerical study is presented which leads to several managerial insights. Finally, in Section 5, we conclude the paper with a summary of our findings and directions for future research.

2. Model formulation for the three-product case

The focus of this section is on formulating our joint replenishment model with substitution for the three-product case. We consider three products only in order to simplify the presentation and facilitate the understanding of the general model presented in the next section. The main objective of this model is to specify how large the order quantities of the three products should be in order to minimize the total system cost. The three products are ordered jointly (at the same time) over repetitive cycles in an EOQ-type setting. In order to further reduce the cost, we allow for stock-outs and substitution among products. That is, during an ordering cycle, a product may run out of stock and part of its demand from that point onward, is substituted to other products, which are still in stock, while the remaining demand is lost. The ordering quantities of the products dictate the stock-out pattern in the inventory cycle, which ends when the last product, among the three, runs out of stock. We next present the notation, assumption, and mathematical formulation of this model.

Consider an inventory system with a set $j \in N = \{1, 2, 3\}$ of substitutable products having ordering quantities y_j and deterministic demand rates D_j per unit time. As aforementioned, at the beginning of an ordering cycle the three products are ordered simultaneously up to y_1, y_2 , and y_3 . Then, when product j , denoted by P_j , is totally consumed, a proportion of its customers will substitute to $P_k, k \neq j$. Let γ_{jk} be the percentage of the demand of P_j that will be substituted by P_k after the stock-out of P_j , when the third product in the assortment is still in stock. For example, the parameter γ_{12} is the substitution rate between P_1 and P_2 , when P_1 runs out and P_2 and P_3 are both in stock. Similarly, γ_{13} is the substitution rate between P_1 and P_3 , in this same situation. Therefore, when P_1 is out of stock, the demands for P_2 and P_3 are respectively $D_{12} = D_2 + \gamma_{12}D_1$ and $D_{13} = D_3 + \gamma_{13}D_1$. We adopt a numbering of the products such that P_1 runs out of stock first, followed by P_2 , then P_3 . The notation $D_{jk}, j < k$ denotes the demand for P_k when P_1, \dots, P_j are out of stock. This notation is useful when we describe the general multi-product model in the next section.

When two products are out of stock, the substitution rate between a product P_j , which is stocked out, and a product P_k , which is still in stock, exceeds γ_{jk} . Specifically, we assume that “second-choice” substitution is also done according to γ_{jk} . That is, when some customers fail to find their most preferred product, P_j in stock, a γ_{jk} fraction of them will switch to looking for their second most preferred product, P_k . Then, if P_k is also out of stock, a fraction γ_{kl} , of the secondary demand for P_k being diverted to P_j , will switch to the least-preferred product P_l , with the remaining P_j demand being lost. For example, when P_1 and P_2 are both out of stock, the total demand for P_3 is composed of its own demand D_3 , the first-order substitution demands, $\gamma_{13}D_1$ and $\gamma_{23}D_2$, and the second-order substitution demands $\gamma_{12}\gamma_{23}D_1$ and $\gamma_{21}\gamma_{13}D_2$. Therefore, the total demand for P_3 when P_1 and P_2 are both out of stock is $D_{23} = D_3 + (\gamma_{13} + \gamma_{12}\gamma_{23})D_1 + (\gamma_{23} + \gamma_{21}\gamma_{13})D_2$. We point out, finally, that our demand model capturing second-choice demand is consistent with the literature on multi-product substitution (e.g., Smith and Agrawal [13] and the references, therein).

Our cost model follows that of the classical EOQ model with a unit holding cost of h_j (\$/unit/unit time), a fixed ordering cost of K_j (\$/order), and a lost sales cost π_j (\$/unit) for $P_j, j \in N$.

Next, we derive the inventory cost per unit time, which is the sum of the ordering, holding and lost sales costs of all three products. Fig. 1 shows the behavior of the inventory level over one ordering cycle. This figure follows our convention that the order of product stock-outs is P_1 , then P_2 , then P_3 . The following auxiliary decisions variables are defined in Fig. 1.

1. The time until P_1 is stocked-out is $\bar{T}_0 = t_1 = y_1/D_1$.
2. The inventory level of P_2 and P_3 , when P_1 runs out of stock are respectively $w_{12} = y_2 - D_2t_1$ and $w_{13} = y_3 - D_3t_1$.
3. The time between the stock-outs of P_1 and P_2 is $\bar{T}_1 = w_{12}/D_{12}$, where $D_{12} = D_2 + \gamma_{12}D_1$, as discussed above. (Equivalently, and to better relate to the notation, this is the duration of time when only P_1 is stocked-out.)
4. The inventory level of P_3 when P_2 runs out of stock is $w_{23} = w_{13} - \bar{T}_1D_{13}$.
5. The time between the stock-outs of P_2 and P_3 is $\bar{T}_2 = w_{23}/D_{23}$, where $D_{23} = D_3 + (\gamma_{13} + \gamma_{12}\gamma_{23})D_1 + (\gamma_{23} + \gamma_{21}\gamma_{13})D_2$, as defined above. (Equivalently, this is the duration of time when both P_1 and P_2 are stocked-out.)

We denote by $TC_u^1(y_1, y_2, y_3)$ as the total cost per unit time for products P_1, P_2 and P_3 under the scenario that the order of product stock-out is P_1 , then P_2 , then P_3 . We name this pattern of stock-outs as Scenario 1. We elaborate on other stock-out scenarios below. We next develop the cost per ordering cycle for each product and the ordering cycle duration.

The total cost of P_1 , under Scenario 1 of stock-outs, is composed of the fixed ordering, holding and shortage cost of P_1 over one cycle. Therefore, its total cost per cycle is

$$TC_1^1(y_1, y_2, y_3) = K_1 + \frac{h_1 t_1 y_1}{2} + \pi_1 D_1 \bar{T}_1 [1 - \gamma_{12} - \gamma_{13}] + \pi_1 D_1 \bar{T}_2 [1 - \gamma_{12}\gamma_{23} - \gamma_{13}]. \quad (1)$$

The last two terms in (1) are the shortage cost of P_1 over \bar{T}_1 , when only P_1 is stocked-out, and over \bar{T}_2 when both P_1 and P_2 are stocked-out.

Ordering Quantities

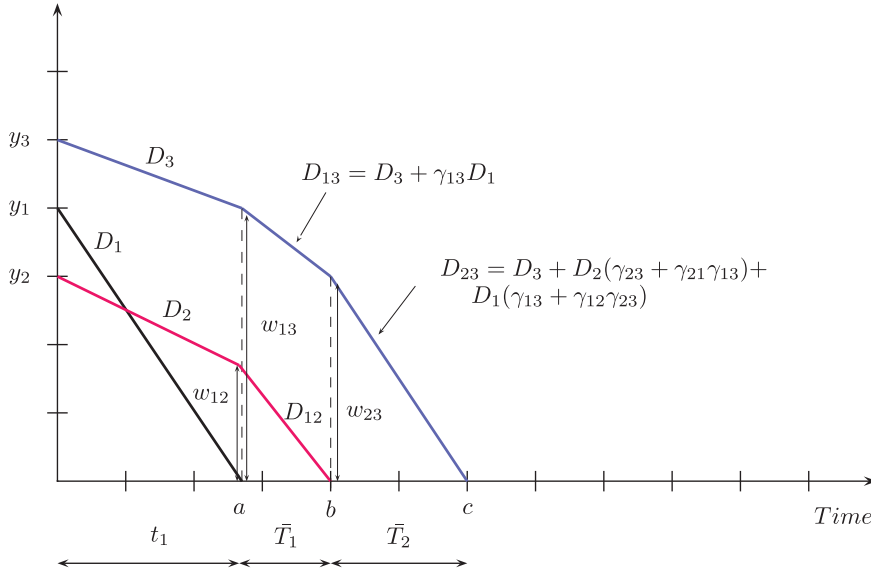


Fig. 1. J RMS model under Scenario 1: P_1 runs out of stock, then P_2 then P_3 .

Similarly, the total cost per cycle of P_2 under Scenario 1 of stock-out is

$$TC_2^1(y_1, y_2, y_3) = K_2 + h_2 \left[\frac{t_1(y_2 + w_{12})}{2} + \frac{(w_{12}\bar{T}_1)}{2} \right] + \pi_2 D_2 \bar{T}_2 [1 - \gamma_{21}\gamma_{13} - \gamma_{23}], \tag{2}$$

and that P_3 is

$$TC_3^1(y_1, y_2, y_3) = K_3 + h_3 \left[\frac{t_1(y_3 + w_{13})}{2} + \frac{(w_{13} + w_{23})\bar{T}_1}{2} + \frac{w_{23}\bar{T}_2}{2} \right]. \tag{3}$$

Next, we find the ordering cycle duration, denoted by T . This can be found as follows:

$$T = t_1 + \bar{T}_1 + \bar{T}_2. \tag{4}$$

Then, the total cost per unit time, under Scenario 1 of stock-outs, is found based on (1)–(4) as

$$TC_u^1(y_1, y_2, y_3) = \frac{\sum_{j=1}^3 TC_j^1(y_1, y_2, y_3)}{T}. \tag{5}$$

The optimal order quantities, y_1^{1*} , y_2^{1*} , and y_3^{1*} , under Scenario 1 of stock-outs, can then be found by minimizing $TC_u^1(y_1, y_2, y_3)$ in (5). We discuss the convexity of this cost function in the next section in a more general setting. Our numerical experimentation indicates, however, that the optimal order quantities can be found easily with any nonlinear solver.

Finally, in this section, we discuss the issue of finding the optimal ordering policy under all possible stock-out scenarios. Recall that in our analysis, thus far, we have assumed that the sequence of product stock-out is P_1 , then P_2 , then P_3 . In reality, this scenario of stock-outs may not be the optimal one, depending on the values of demand and cost parameters of each product. In order to find the optimal stock-out scenario, and corresponding optimal order quantities, one needs to evaluate the following six stock-out scenarios. These scenario can be analyzed similar to Scenario 1 with an appropriate renumbering of products.

- Scenario 1.** Stock-out sequence: 1, 2, then 3.
- Scenario 2.** Stock-out sequence: 1, 3, then 2.
- Scenario 3.** Stock-out sequence: 2, 1, then 3.
- Scenario 4.** Stock-out sequence: 2, 3, then 1.
- Scenario 5.** Stock-out sequence: 3, 1, then 2.
- Scenario 6.** Stock-out sequence: 3, 2, then 1.

Once these scenarios have been analyzed, as described above, the order quantities from each scenario, y_1^{j*} , y_2^{j*} , and y_3^{j*} , and corresponding optimal cost per unit time, $TC_u^{j*} = TC_u^j(y_1^{j*}, y_2^{j*}, y_3^{j*})$, $J = 1, 2, \dots, 6$, can be found. Then, the optimal scenario can be identified as the one having the least cost, TC_u^{j*} , and the corresponding optimal order quantities are determined based on the optimal scenario.

While enumerating six scenarios with three products may be seen as a manageable task, the number of scenarios, unfortunately, increases substantially with more products. In general, the number of scenarios to be considered is $n!$, which may be hard to work with. In the case of a large number of products, n , which is discussed in the next section, we recommend, utilizing simple heuristics to identify promising scenarios. For example, a heuristic that seems to be promising is to order the products based on a stock-out desirability factor (SDF) equal to the ratio of the unit shortage cost to the unit holding cost. For Product j , this factor is $SDF_j = h_j/\pi_j$. This heuristic is sought to allow products having a low shortage cost and a high holding cost to run out of stock first.

3. General formulation and convexity for the joint replenishment model with substitution

In this section, utilizing similar notation and assumptions to Section 2, we extend our model to more than three products, $N \geq 4$, and present some further convexity analysis. In Section 3.1, we present our general model as a nonlinear program, specifically, a quadratic program. Then, in Section 3.2 we discuss the convexity of the model, establishing convexity in an important special case.

3.1. General model formulation

Consider a set $N = \{1, \dots, n\}$ of substitutable products, which are ordered jointly, at the beginning of an ordering cycle, and then run out of stock sequentially, similar to the three-product case described on Section 2. In the following, we present the necessary formulation for finding the optimal order quantities, assuming a stock-out scenario where P_1 runs out of stock, then P_2, \dots , then P_n . The notation we utilize in this section is essentially the same as that in Section 2. We summarize our notation below for completeness.

Model parameters

- K_j : Fixed setup cost per order of Product $j \in N$.
- h_j : Holding cost per unit per unit time of Product $j \in N$.
- π_j : Shortage cost per unit of Product $j \in N$.
- D_k : Demand rate for Product $k \in N$ when all other products in N are in-stock.
- D_{jk} : Demand rate for Product $k \in N$, when all products in Set $\{1, 2, \dots, j\} \subset N$, $j < k$, are out of stock.
- γ_{jk} : Percentage of demand substitution from Product $j \in N$ to Product $k \in N$, after the stock-out of Product j , when the all other products in N are still in stock.

Main and auxiliary decision variables

- y_j : Order quantity of Product $j \in n$. These are our main decision variables.
- \bar{T}_j : Time when products $\{1, 2, \dots, j\} \subset N$ are out of stock and substitution to the remaining stocked product occurs. That is, \bar{T}_j is the time between the stock-out of Product j and Product $j + 1$. By definition, let $\bar{T}_0 = y_1/D_1$.
- w_{jk} : Inventory level of a Product k when products in Set $\{1, 2, \dots, j\} \subset N$, $j < k$, are out of stock. By definition, let $w_{0k} = y_k$.
- H_j : Holding cost of Product $j \in N$ per ordering cycle.
- S_j : Shortage cost of Product $j \in N$ per ordering cycle.
- T : Ordering cycle duration

Then, assuming a stock-out scenario where P_1 runs out of stock, then P_2, \dots , then P_n , the optimal order quantities and corresponding cost per unit time can be found by solving the following nonlinear program.

$$(JRMS) : \text{Minimize } \min_{y_1, y_2, \dots, y_n} TC_u = \frac{\sum_{j=1}^n K_j + \sum_{j=1}^n (H_j + S_j)}{T}, \quad (6)$$

Subject to

$$\bar{T}_0 = \frac{y_1}{D_1}, \quad (7)$$

$$\bar{T}_j = \frac{w_{j,j+1}}{D_{j,j+1}}, \quad j = 1, \dots, n-1 \quad (8)$$

$$T = \sum_{j=0}^{n-1} \bar{T}_j, \quad (9)$$

$$w_{0k} = y_k, \quad k = 1, 2, \dots, n \quad (10)$$

$$w_{kk} = 0, \quad k = 1, 2, \dots, n \quad (11)$$

$$w_{jk} = w_{j-1,k} - \bar{T}_{j-1}D_{j-1,k}, \quad k = 2, \dots, n, j = 1, \dots, k - 1 \tag{12}$$

$$D_{0k} = D_k, \quad k = 1, 2, \dots, n \tag{13}$$

$$D_{jk} = D_{j-1,k} + \left(\gamma_{jk} + \sum_{l=1}^{j-1} \gamma_{lj} \gamma_{lk} \right) D_j, \quad k = 2, \dots, n, j = 1, \dots, k - 1 \tag{14}$$

$$H_j = h_j \sum_{l=0}^{j-1} \bar{T}_l \frac{(w_{lj} + w_{l+1,j})}{2}, \quad j = 1, 2, \dots, n \tag{15}$$

$$S_j = \pi_j D_j \sum_{k=j}^{n-1} \bar{T}_k \left(1 - \sum_{l=k+1}^n \gamma_{jl} - \sum_{l_1=1, l_1 \neq j}^k \sum_{l_2=k+1}^n \gamma_{jl_1} \gamma_{l_1 l_2} \right), \quad j = 1, 2, \dots, n - 1 \tag{16}$$

The objective function (6) minimizes the total cost composed of ordering, holding and shortage costs of all products in N . Constraint (7) gives the time until P_1 runs out of stock function of the order quantity and demand rate for this product. Constraint (8) gives the time between the consecutive product stock-outs as function of the inventory levels and demand. Specifically, the variable $w_{j,j+1}$ denotes the inventory level of Product $j + 1$ when P_j runs out of stock, and $D_{j,j+1}$ is the demand rate of P_{j+1} when products in the set $\{1, 2, \dots, j\}$ are out of stock. As such, the time \bar{T}_j defines the duration between the stock-out of P_j and P_{j+1} . Constraint (9) gives the order cycle duration as the sum of the incremental stock-out times, \bar{T}_j . Constraint (10) initializes the initial inventory level of each product in a cycle to its ordering quantity. Constraint (11) sets the final inventory level of each product, in an ordering cycle, to zero. Constraint (12) gives the inventory levels of a product, e.g., P_k , at the all products that stock-out before it, e.g., P_1, P_2, \dots, P_{k-1} . Constraint (13) initializes the demand rate of each product. Constraint (14) updates the demand rate of each product by incrementally adding the first- and second-choice substitution demand, similar to what we discuss in detail in Section 2, for the case of three products. Constraint (15) gives the holding costs per ordering cycle for P_j . It is based on aggregating the total inventory over incremental stock-out periods, \bar{T}_j . Finally, constraint (16) gives the shortage cost per ordering cycle for P_j , by aggregating the total number of P_j stock-outs over the incremental stock-out times, $\bar{T}_j, \bar{T}_{j+1}, \dots, \bar{T}_n$.

3.2. Convexity

In the mathematical program (JRMS) of Section 3.1, it can be easily seen that the constraints are linear in the order quantities, y_1, y_2, \dots, y_n . In the objective function, the cost per cycle (in the numerator), $\sum_{j=1}^n (K_j) + \sum_{j=1}^n (H_j + S_j)$ is a quadratic function since it can be easily seen that the holding cost, $\sum_{j=1}^n H_j$, is quadratic while the shortage cost, $\sum_{j=1}^n S_j$, is linear. The order cycle duration, T , is also linear. Following, a result in Avriel [32], it follows that the objective function is pseudoconvex if the holding cost is convex. It has been generally difficult to establish the convexity of the holding cost. However, in the following lemma, we show that in the special case where all the unit holding costs are equal, it can be shown that the holding cost is indeed convex.

Lemma 1. *If $h_i = h_j = h$, for $i = 1, \dots, n$, and $j = 1, \dots, n$, then the objective function of the JRMS problem is pseudoconvex.*

Proof. See Appendix. □

The special case in Lemma 1 is important, as in practice the unit costs of substitutable products are generally close, which leads to approximately equal holding cost. This implies that in most practical settings, the objective function of the JRMS problem is pseudoconvex, which together with the linear constraints makes solving JRMS quite easy with many available commercial solvers.

4. Numerical examples and managerial insights

In this section, we present numerical examples which illustrate the application of the proposed model where we derive the optimal ordering quantities. We then perform sensitivity analysis and develop useful managerial insights. Section 4.1 presents our base example, and Section 4.2 presents extensive sensitivity analysis on this example.

Table 1
Initial data.

Parameters	P_1	P_2	P_3
Consumption rate (D_j)	25	35	40
Holding cost (h_j)	5	5	5
Fixed setup cost (K_j)	350	400	250
Shortage cost (π_j)	4	5	3

Table 2
Percentage of substitution.

γ_{jk}	P_1	P_2	P_3
P_1	*	0.13	0.17
P_2	0.16	*	0.14
P_3	0.10	0.20	*

Table 3
Results for all six scenarios of the base example.

Scenario	y_1	y_2	y_3	TCU_j
1	20	35	125	899.56
2	17	125	28	846.98
3	23	32	125	899.93
4	106	28	32	821.74
5	20	125	24	846.57
6	105	35	24	820.41

4.1. Base example

Consider a situation with three products whose demand rate, holding cost, setup cost, percentage of substitution, and shortage cost are known, as shown in Table 1.

The percentage of substitution γ_{jk} are given in Table 2.

First, we solve the corresponding optimization problems for the six possible stock-out sequence scenarios discussed in Section 2 using AMPL/KNITRO. Table 3 presents the results of optimizing all scenarios. Comparing all scenarios, Scenario 6, involving P_3 running out of stock first, then P_2 , then P_1 , gives the minimum total cost per unit time. Thus, the optimal ordering quantities are those of Scenario 6, i.e., $y_1^* = 24$ units, $y_2^* = 35$ units, $y_3^* = 105$, and the optimal cost is \$820.41/unit time.

As a side note, and as a follow-up on the idea of utilizing a simple heuristic to select good scenarios, instead of enumerating all possible scenarios, consider the heuristic based on the stock-out desirability briefly discussed in Section 2. This heuristic directly yields Scenario 5 for this example (as $SDF_1 = 5/4$, $SDF_2 = 1$, $SDF_3 = 5/3$, and $SDF_3 > SDF_1 > SDF_2$), with a cost of \$846.57/unit time having an optimality gap of 3.2%.

In order to assess the benefit of allowing substitution in this example, we compare our results with those of the classical joint replenishment model, involving no substitution. In this model, all products in N are ordered at the beginning of the ordering cycle, and they all run out of stock simultaneously at the end of the ordering cycle. This model has a simple closed-form solution, which is described in the Appendix, for completeness. The optimal order quantities of the joint replenishment model with no substitution is $y_1 = 50$, $y_2 = 70$, and $y_3 = 80$ with a total cost of \$1000. Therefore, allowing substitution in this example reduces the total cost from \$1000 to \$820.41/unit time, which represents a relative “improvement” of $(1000 - 820.41)/820.41 = 21.89\%$. Note that we measure the improvement from substitution in terms of the relative increase in cost if substitution is not allowed. This improvement measure is the main focus of the sensitivity analysis in Section 4.2.

4.2. Sensitivity analysis and managerial insights

In this section, we evaluate the impact of changing parameter values on optimal solutions of the JRMS model. Specifically, we start with the base example of Section 4.1 and perform a one-way sensitivity analysis on all the cost parameters. For each instance of the parameters, we find an optimal solution for all possible six substitution scenarios, and select the one that has the lowest cost, similar to our analysis in Section 4.1. However, in this section we only report on the results of the optimal scenario for each instance. For comparison purpose, we also report results on the classic joint replenishment model without substitution (henceforth JRM). Details of the JRM are presented in the Appendix.

The results in Tables 4 reports on the optimal solution of JRMS as the fixed cost of P_1 is varied while holding all other model parameters fixed at their base values (given in Section 4.1). Table 4 also reports on the improvement from allowing substitution by comparison to the joint replenishment model with no substitution, as explained in Section 4.1. Table 4

Table 4
Sensitivity analysis on the fixed setup costs of P_1 .

Joint replenishment with substitution					Joint replenishment without substitution					Improvement (%)
K_1	y_1	y_2	y_3	TC_u	y_1	y_2	y_3	TC_u		
100	88	35	24	734.42	45	63	72	866.67	18.01	
150	92	35	24	752.80	45	63	72	894.44	18.82	
250	99	35	24	787.67	50	70	80	950.00	20.61	
350	105	35	24	820.41	50	70	80	1000.00	21.89	
400	108	35	24	836.10	50	70	80	1025.00	22.59	
450	111	35	24	851.38	55	77	88	1050.00	23.33	

Table 5
Sensitivity analysis on the unit holding costs of P_1 .

Joint replenishment with substitution					Joint replenishment without substitution					Improvement (%)
h_1	y_1	y_2	y_3	TC_u	y_1	y_2	y_3	TC_u		
3	140	32	23	693.32	55	77	88	949.55	36.96	
4	119	34	23	761.12	50	70	80	975.00	28.10	
5	105	35	24	820.41	50	70	80	1000.00	21.89	
6	16	126	25	848.79	50	70	80	1025.00	20.76	
7	13	127	25	850.19	50	70	80	1050.00	23.50	
8	11	127	26	851.16	45	63	72	1073.06	26.07	

Table 6
Sensitivity analysis on the shortage cost of P_1 .

Joint replenishment with substitution					Joint replenishment without substitution					Improvement (%)
π_1	y_1	y_2	y_3	TC_u	y_1	y_2	y_3	TC_u		
0.5	2	131	24	785.61	50	70	80	1000.00	27.29%	
1	5	130	24	795.26	50	70	80	1000.00	25.74%	
2	10	129	24	813.64	50	70	80	1000.00	22.90%	
3	105	35	24	820.41	50	70	80	1000.00	21.89%	
4	105	35	24	820.41	50	70	80	1000.00	21.89%	
5	105	35	24	820.41	50	70	80	1000.00	21.89%	

validates the common intuition that higher fixed cost lead to higher order quantities and costs. It is interesting to note in Table 4 that the order quantities of P_2 and P_3 (remaining fixed at 45 and 63, respectively) are not changed as K_1 increase in JRMS. This is in contrast with the JRM, where the order quantities of all products are increased as K_1 increases. This can be interpreted as the condition in which allowing substitution, products get “decoupled” under joint replenishment, with a product having an order quantity close to its own EOQ in an ordering cycle, and any additional demand for that product in the ordering cycle is either lost or substituted. Another interesting, and probably more important, observation in Table 4 is that the improvement from allowing substitution (i.e., of JRMS over JRM) increases as the fixed cost increase to around 23% for high fixed costs. This is related to the key insight that substitution improves on joint replenishment by allowing longer cycle and reducing the fixed ordering cost. This key insight has been observed by Salameh et al. [20] for the case of two products only. Here, we generalize this insight to multiple products.

Table 5 reports on the optimal solution of the JRMS and JRM models as the holding cost of P_1 is varied, with all other parameters held at their base values. The results conform the intuition that higher holding costs lead to lower order quantities and higher cost, in what concerns P_1 only. However, it is interesting to note that as h_1 increases the order quantity of P_2 , and of P_3 , to an extent, increase. This can be interpreted that substitution is allowing a fraction demand of the high-holding cost product, P_1 , by the low-holding cost product, P_2 , leading to a higher demand for P_2 . Note that for the JRM model involving no substitution, the order quantities of all products increase as h_1 increases. Table 5 also indicates that that substitution is most useful when the holding cost is low, with the improvement of JRMS over JRM reaching 37% for low h_1 .

Finally, Table 6 reports on the optimal solution of the JRMS and JRM models as the shortage cost of P_1 is varied, with all other parameters held at their base values. Table 6 conforms the intuition that higher shortage costs lead to higher order quantities and cost for P_1 only. Interestingly, Table 6 indicates that high shortage costs of P_1 lead to lower order quantities for P_2 . This can be interpreted that having high-order quantities for P_1 makes it appealing to have some P_2 customers switch to it, which cuts the costs for P_2 . Table 6 also indicates that substitution is most useful when the shortage cost is low, with the improvement of JRMS over JRM reaching 27% for low π_1 .

Tables 4–6 indicate that JRMS works well (outperforming) JRM when the fixed cost is high and the holding and shortage costs are low. While this cost pattern is not common in typical retailing setting, it is observed, for example, for retailers that import goods from overseas, where the shipping cost is high. Moreover, many of these retailers import low-cost products such as car and mobile phone accessories. For these kind of products the holding and shortage costs are also typically low. The holding cost is low as it is tied-up to low unit cost, and the shortage cost is low, due to non-critical nature of the products, the lack of brand loyalty, and the availability of several alternatives in the open market for such products. This type of retailing is common in the Middle East, with imports typically coming from China.

The main managerial insight from Tables 4–6 is that JRMS attempt to improve on JRM by having more flexibility for the product inventory cycle. To elaborate, note that the main advantage of joint replenishment (in JRM) is the reduction of shipping and other logistical costs, as discussed in Section 1. Note also that the main down side of joint replenishment is its restrictive nature, as it requires all products to have the same common inventory cycle, which could increase costs for a product if the common cycle is far from its “own” (optimal) cycle, which balances its own costs. (A product’s own cycle can be obtained in the context of our model from the classic EOQ model, applied to the product individually.) By allowing for stock-outs, JRMS is in a sense allowing some products to have a cycle different from the common cycle, and close to these products own cycles, while continuing to benefit from joint replenishment. This is done by allowing partial substitution and shortages of the products, which might, in turn, be costly.

5. Conclusion

In this paper, we address a challenging problem of extending the classical joint replenishment model (JRM) to account for stock-out based substitution of multiple products. To simplify the presentation, we first presented our model for joint replenishment with substitution (JRMS) in the context of three products only and then provided a general quadratic programming formulation with linear constraints. One complicating aspect of our analysis is that the form of the cost function depends on the sequence of product stock-outs. This implies that one needs to enumerate a large number of substitution scenarios (specifically, $n!$ scenarios for n products). However, with the available modern-day computing capabilities, and some desirable convexity properties of our JRMS model, the total enumeration scheme of stock-out scenarios proved to be manageable, and allowed to gain useful managerial insights by numerically analyzing a three-product illustrative example.

Our numerical analysis indicated some interesting counter-intuitive behavior, where JRMS departs from JRM. For example, a high holding cost of one product could lead to a *higher* order quantity of another product, as it becomes appealing to induce customers to switch from the former to the later product. A similar interesting observation is made on the shortage cost, as increasing these costs for one product could lead to *lower* order quantities for another product. Moreover, our numerical analysis reports significant cost improvements of JRMS over JRM in the order of 20–30%. These substitution-driven improvements are observed to occur in high fixed costs, and low holding and shortage costs environments.

An explicit enumeration of stock-out scenarios is not effective to solve larger problems involving possibly tens of products in practice. Future extensions of our work could address developing efficient search schemes in order to explore or curtail the space of stock-out scenarios.

Finally, one may see our JRMS strategy in this paper as a form of partial “inventory pooling,” with more demand pooled to certain products via substitution. The classical study of inventory pooling is focused on the risk aspect, as demand variability could be reduced due to pooling (e.g., Eppen [33]). In this paper, we consider an EOQ system with deterministic demand. It might be interesting to study the risk aspect of the inventory pooling encapsulated in JRMS-type systems in future works considering stochastic demand.

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Appendix

Proof of Lemma 1

It is enough to show the holding cost per ordering cycle is convex. Note that with equal unit holding costs, the inventory level of all products can be aggregated, and the inventory level when product j runs out of stock can be written as $\beta_j y$ where $y = \sum_{k=1}^n y_k$ and $0 \leq \beta_j < \beta_{j+1} \leq 1$. Letting the total demand when Products 1, \dots , j are out of stock be D^j , $j = 1, \dots, n-1$. It follows that $D^j = \sum_{k=j+1}^n D_{jk}$. Also, define D^0 as the total demand, $D^0 = \sum_{k=1}^n D_k$. Then, it follows that the holding cost is

$$H = h \sum_{j=0}^{n-1} \frac{(\beta_j y + \beta_{j+1} y)}{2} \frac{(\beta_j y - \beta_{j+1} y)}{D^j} = h \left(\sum_{k=1}^n y_k \right)^2 \sum_{j=0}^{n-1} \frac{(\beta_j^2 - \beta_{j+1}^2)}{2D^j},$$

where by definition $\beta_0 = 1$, and $\beta_n = 0$. Since $\beta_{j+1} < \beta_j$, it can be easily shown that the function $g(y_1, \dots, y_n) = (\sum_{k=1}^n y_k)^2$ is convex in y_1, y_2, \dots, y_n . It follows that H is convex.

To show that $g(y_1, \dots, y_n)$ is convex, note that $g(y_1, \dots, y_n) = h(l(y_1, \dots, y_n))$ where $l(y) = \sum_{k=1}^n y_k$ and $h(l) = l^2$. The fact that $h(\cdot)$ is convex and increasing and $l(\cdot)$ is linear and increasing completed the proof, since the composition of two monotone convex functions is also convex (e.g., Avriel [32]).

The Joint Replenishment Model (JRM)

We consider a simplified version of the classical JRM model, where both products have the same cycle time, t_0 . Then,

$$t_0 = \frac{y_1}{D_1} = \frac{y_2}{D_2} = \frac{y_3}{D_3}. \quad (17)$$

The total cost per unit time is then:

$$TC_u(t_0) = \frac{\sum_{j \in N} (K_j + h_j \frac{D_j t_0^2}{2})}{t_0}. \quad (18)$$

The first-order optimality conditions then give:

$$\frac{\partial TC_u(t_0)}{\partial t_0} = -\frac{\sum_{j \in N} K_j}{t_0^2} + \sum_{j \in N} \frac{h_j D_j}{2}. \quad (19)$$

Further, $\frac{\partial^2 TC_u t_0}{\partial^2 t_0} = 2 \frac{\sum_{j \in N} K_j}{t_0^3} > 0$, which establishes the convexity of $TC_u(t_0)$. Then, the optimal cycle length is given from the first-order conditions as:

$$t_0^* = \sqrt{\frac{2 \sum_{j \in N} K_j}{\sum_{j \in N} h_j D_j}}. \quad (20)$$

Then, the optimal order quantities for all products $j \in N$ are:

$$y_j^* = D_j t_0^*. \quad (21)$$

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