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Some common fixed point results of graphs on b-metric space

Zead Mustafa[a,*], M. M. M. Jaradat[a], H. M. Jaradat[b,c]

[a]*Department of Mathematics, Statistics and Physics, Qatar University, Doha, Qatar.*
[b]*Department of Mathematics, Al al-Bayt University, Jordan.*
[c]*Department of Mathematics and Applied Sciences, Dhofar University, Salalah, Oman.*

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Abstract

The aim of this paper is to present some coincidence point results and common fixed points for pair of self-mappings satisfying generalized contractive condition in the framework of b-metric spaces endowed with a graph. We present applications and some examples to illustrate the main result.

Keywords: Coincidence point, common fixed point with graph, b-metric Space.
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1. Introduction and preliminaries

The area of the fixed point theory has very important application in applied mathematics and sciences. Recently, the common fixed points of mappings satisfying certain contractive conditions has been studied extensively by many authors. In 1976, Jungck [20] proved a common fixed point theorem for commuting maps, but his results required the continuity of one of the maps. Sessa [31] in 1982 first introduced a weaker version of commutativity for a pair of selfmaps, and it is shown by Sessa that weakly commuting pair of maps in metric space is commuting, but the converse may not be true. Later, in 1986, Jungck [21] introduced the notion of compatible mappings in order to generalize the concepts of weak commutativity and showed that weak commuting map is compatible, but the reverse implication may not hold. In 1996, Jungck [23] defined

*Corresponding author
Email addresses: `zead@qu.edu.qa` (Zead Mustafa), `mmjst4@qu.edu.qa` (M. M. M. Jaradat), `husseinjaradat@yahoo.com` (H. M. Jaradat)

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a pair of self-mappings to be weakly compatible if they commute at their coincidence points. Therefore, we have one-way implication, namely,

commuting maps $\Rightarrow$ weakly commuting maps $\Rightarrow$ compatible maps $\Rightarrow$ weakly compatible maps.

Various authors have introduced coincidence points results for various classes of mappings on metric spaces, for more details of coincidence point theory and related results see [22, 24, 29].

The concept of b-metric space was introduced by Bakhtin [5] in 1989 and extensively used by Czerwik in [12, 13]. Hereafter, several interesting results about the existence of a fixed point for single-valued and multi-valued operators in (ordered) b-metric spaces have been obtained (see, e.g., [1, 2, 10, 11, 18, 25–28, 30, 32]).

In 2005, Echenique [14] studied fixed point theory by using graphs to give a short and constructive proof of Tarski's fixed point theorem and of Zhou's extension of Tarsk's fixed point theorem to set-valued maps. Espinola and Kirk [15] obtained fixed point results in R-tree with application to graph theory. Recently, Jachymski [19] proved a sufficient condition for a selfmap f of a metric space (X, d) to be a Picard operator and applied this condition to the Kelisky-Rivlin theorem on iterates of the Bernstein operators on the space $C[0, 1]$. More recently, many authors introduced some fixed point theorems in metric spaces with a graph (see [3, 4, 6–8]).

The following definitions and results will be needed in the sequel.

Definition 1.1 ([12])**.** Let X be nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is said to be a b-metric on X if the following conditions hold:

(b1) $d(x, y) = 0$ if and only if $x = y$;
(b2) $d(x, y) = d(y, x)$ for all $x, y, \in X$;
(b3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

The pair (X, d) is called a b-metric space.

One can note that if $s = 1$, then the b-metric will be reduced to a usual metric, thus the class of b-metric space is larger than metric space.

Definition 1.2 ([11])**.** Let (X, d) be a b-metric space, $x \in X$ and (x_n) be a sequence in X. Then

1. The sequence (x_n) converges to x if and only if $\lim_{n\to\infty} d(x_n, x) = 0$, which is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
2. The sequence (x_n) is called Cauchy sequence if and only if
$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$
3. (X, d) is complete if and only if every Cauchy sequence in X is convergent.

Recently, Hussain et al. [17] have presented an example of a b-metric which is not continuous (see Example 2 of [17]).

The following example shows that, in general, a b-metric does not necessarily need to be a metric.

Example 1.3 ([2])**.** Let (X, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$. However, if (X, d) is a metric space, then (X, ρ) is not necessarily a metric space. For example, if $X = R$ is the set of real numbers and $d(x, y) = |x - y|$ is the usual Euclidean metric, then $\rho(x, y) = (x - y)^s$ is a b-metric on R with $s = 2$, but is not a metric on R.

Definition 1.4 ([1])**.** Let T and g be self mappings of a set X. If $Tv = gv = u$ for some $v \in X$, then v is called a coincidence point of T and g, and u is called a point of coincidence of T and g.

Definition 1.5 ([23])**.** The mappings $T, g : X \to X$ are weakly compatible if they commute at their coincidence point, i.e.,

$$T(gx) = g(Tx) \text{ whenever } gx = Tx.$$

Proposition 1.6 ([1])**.** *Let T and g be weakly compatible self-maps of a nonempty set X. If T and g have a unique point of coincidence $u = Tv = gv$, then u is the unique common fixed point of T and g.*

Next, we review some basic notions in graph theory. Let (X, d) be a b-metric space. We assume that G is a reflexive digraph with the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains no parallel edges. So we can identify G with the pair $(V(G), E(G))$. By G^{-1} we denote the graph obtained from G by reversing the direction of edges, i.e., $E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}$. Let $\tilde{G}$ denote the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat G as a digraph for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \bigcup E(G^{-1}).$$

Our graph theory notations and terminology are standard and can be found in any graph theory books, such as [9] and [16].

Definition 1.7 ([16])**.** For a given digraph G, any path from the vertex x to the vertex y in G of length n $(n \in N)$ is a sequence $(x_i)_{i=0}^{n}$ of $n + 1$ vertices such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \cdots, n$.

2. Main results

In this paper, we assume that (X, d) is a b-metric space endowed with a reflexive digraph G containing no multiple edges such that $X = V(G)$ and the mappings $T, g : X \to X$ with $T(X) \subseteq g(X)$.

If $x_0 \in X$ is arbitrary, then there exists $x_1 \in X$ such that $Tx_0 = gx_1$. Continuing this process we can construct a sequence (gx_n) such that $gx_n = Tx_{n-1}$. Throughout this paper we set the following property:

The property $G_{(T,gx_n)}$: If $(gx_n)_{n\in N}$ is a sequence in X obtained as above such that $gx_n \to x$ and $(gx_n, gx_{n+1}) \in E(\tilde{G})$ for all $n \geq 1$, then there is a subsequence $(gx_{n_i})_{i\in N}$ of $(gx_n)_{n\in N}$ such that $(gx_{n_i}, x) \in E(\tilde{G})$ for all $i \geq 1$.

Furthermore, throughout this paper we use the notion $G_{gT} = \{x_0 \in X : (gx_n, gx_m) \in E(\tilde{G})$, where $m, n = 1, 2, 3, \cdots\}$.

Theorem 2.1. *Let (X, d) be a b-metric space (with $s \geq 1$) endowed with a graph G and the mappings $T, g : X \to X$ satisfy*

$$d(Tx, Ty) \leq ad(gx, Tx) + bd(gy, Ty) + cd(gx, Ty) + ed(gy, Tx) + kd(gx, gy) \tag{2.1}$$

for all $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$ when either ($a+b+2c+e+k < \frac{1}{s}$ and $a+se < \frac{1}{s}$) or ($a+b+c+2e+k < \frac{1}{s}$ and $a + sc < \frac{1}{s}$).

Suppose, $T(X) \subseteq g(X)$ and $g(X)$ is complete subspace of X. Then

1. *If $G_{gT} \neq \phi$ and the property $G_{(T,gx_n)}$ is satisfied, then T and g have a point of coincidence in X.*

2. *Moreover, if x and y are point of coincidence of T and g in X, it implies $(x, y) \in E(\tilde{G})$, then T and g have a unique point of coincidence in X.*

3. *Further, if T and g are weakly compatible, then T and g have a unique common fixed point in X.*

Proof. Assume that $G_{gT} \neq \phi$, then there exists $x_0 \in G_{gT}$. As a result of $T(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $gx_1 = Tx_0$, again we can find $x_2 \in X$ such that $gx_2 = Tx_1$. Continuing this process we can construct a sequence (gx_n) such that $gx_n = Tx_{n-1}$ for $n = 1, 2, 3, \cdots$ and $(gx_n, gx_m) \in E(\tilde{G})$.

Suppose that $gx_n = gx_{n+1}$ for some $n \in \mathbf{N}$. Then $gx_n = Tx_n$, which implies that x_n is a coincidence point. Therefore, we assume that $gx_n \neq gx_{n+1}$ for all $n \in \mathbf{N}$.

Now we will show that the sequence (gx_n) is a Cauchy sequence in $g(X)$. By using the condition (2.1) for $n \in N$ we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq ad(gx_{n-1}, Tx_{n-1}) + bd(gx_n, Tx_n) + cd(gx_{n-1}, Tx_n) + ed(gx_n, Tx_{n-1}) + kd(gx_{n-1}, gx_n) \\ &= ad(gx_{n-1}, gx_n) + bd(gx_n, gx_{n+1}) + cd(gx_{n-1}, gx_{n+1}) + ed(gx_n, gx_n) + kd(gx_{n-1}, gx_n). \end{aligned}$$

By $(b3)$ of b-metric axioms we have

$$d(gx_{n-1}, gx_{n+1}) \leq s[d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})].$$

Then,

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq ad(gx_{n-1}, gx_n) + bd(gx_n, gx_{n+1}) + csd(gx_{n-1}, gx_n) + csd(gx_n, gx_{n+1}) \\ &\quad + ed(gx_n, gx_n) + kd(gx_{n-1}, gx_n). \end{aligned}$$

Thus,

$$d(gx_n, gx_{n+1}) \leq \beta d(gx_{n-1}, gx_n), \tag{2.2}$$

where $\beta = \frac{a+k+cs}{1-b-cs}$. Since $a + k + 2cs + b < sa + sk + 2cs + sb + se < 1$, we get $\beta < 1$.

Continuing this process, equation (2.2) becomes,

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq \beta d(gx_{n-1}, gx_n) \\ &\leq \beta^2 d(gx_{n-2}, gx_{n-1}) \\ &\;\;\vdots \\ &\leq \beta^n d(gx_0, gx_1). \end{aligned}$$

Now, for $m, n \in \mathbf{N}$ such that $m > n$ and by $(b3)$ of b-metric axioms we have

$$\begin{aligned} d(gx_n, gx_m) &\leq sd(gx_n, gx_{n+1}) + sd(gx_{n+1}, gx_m) \\ &\leq sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + s^2 d(gx_{n+2}, gx_m) \\ &\;\;\vdots \\ &\leq sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + s^3 d(gx_{n+2}, gx_{n+3}) + \cdots \\ &\quad + s^{m-n-1} d(gx_{m-2}, gx_{m-1}) + s^{m-n-1} d(gx_{m-1}, gx_m) \\ &\leq sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + s^3 d(gx_{n+2}, gx_{n+3}) + \cdots \\ &\quad + s^{m-n-1} d(gx_{m-2}, gx_{m-1}) + s^{m-n} d(gx_{m-1}, gx_m). \end{aligned} \tag{2.3}$$

Hence, equations (2.2) and (2.3) give

$$\begin{aligned} d(gx_n, gx_m) &\leq s\beta^n d(gx_0, gx_1) + s^2 \beta^{n+1} d(gx_0, gx_1) + s^3 \beta^{n+2} d(gx_0, gx_1) + \cdots \\ &\quad + s^{m-n-1} \beta^{m-1} d(gx_0, gx_1) \\ &= s\beta^n [1 + (s\beta) + (s\beta)^2 + \cdots + (s\beta)^{m-n-1}] d(gx_0, gx_1) \\ &\leq s\beta^n \sum_{i=0}^{\infty} (s\beta)^i \; d(gx_0, gx_1) \\ &\leq s\beta^n \frac{1}{1 - s\beta} d(gx_0, gx_1) \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Therefore, (gx_n) is a Cauchy sequence and by the completeness of $g(X)$ there is $u \in g(X)$ such that $(gx_n) \to u = g(v)$ for some $v \in X$.

As, $x_0 \in G_{gT}$, then $(gx_n, gx_m) \in E(\tilde{G})$ for $m, n = 1, 2, 3, \cdots$ and so, $(gx_n, gx_{n+1}) \in E(\tilde{G})$. By property $G_{(T,gx_n)}$, there is a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, u) \in E(\tilde{G})$. Now, by using $(b3)$ we have

$$d(Tv, gv) \leq sd(Tv, Tx_{n_i}) + sd(Tx_{n_i}, gv). \tag{2.4}$$

On the other hand, condition (2.1) implies that

$$d(Tv, Tx_{n_i}) \leq ad(gv, Tv) + bd(gx_{n_i}, Tx_{n_i}) + cd(gv, Tx_{n_i}) + ed(gx_{n_i}, Tv) + kd(gv, gx_{n_i}).$$

Thus equation (2.4) becomes

$$\begin{aligned} d(Tv, gv) \leq & sad(gv, Tv) + sbd(gx_{n_i}, Tx_{n_i}) + scd(gv, Tx_{n_i}) \\ & + sed(gx_{n_i}, Tv) + skd(gv, gx_{n_i}) + sd(Tx_{n_i}, gv). \end{aligned} \tag{2.5}$$

Replacing Tx_{n_i} by gx_{n_i+1} in equation (2.5) we get

$$\begin{aligned} d(Tv, gv) \leq & sad(gv, Tv) + sbd(gx_{n_i}, gx_{n_i+1}) + scd(gv, gx_{n_i+1}) \\ & + sed(gx_{n_i}, Tv) + skd(gv, gx_{n_i}) + sd(gx_{n_i+1}, gv). \end{aligned}$$

Hence,

$$\begin{aligned} d(Tv, gv) \leq & \frac{sb}{1 - sa} d(gx_{n_i}, gx_{n_i+1}) + \frac{sc}{1 - sa} d(gv, gx_{n_i+1}) \\ & + \frac{se}{1 - sa} d(gx_{n_i}, Tv) + \frac{sk}{1 - sa} d(gv, gx_{n_i}) + \frac{s}{1 - sa} d(gx_{n_i+1}, gv). \end{aligned} \tag{2.6}$$

From equation (2.2) we have $d(gx_{n_i}, gx_{n_i+1}) \leq \beta^{n_i} d(gx_0, gx_1)$ where $\beta < 1$, and from $(b3)$ of b-metric axioms we have

$$d(gx_{n_i}, Tv) \leq sd(gx_{n_i}, gv) + sd(gv, Tv).$$

Therefore, equation (2.6) becomes

$$\begin{aligned} d(Tv, gv) \leq & \frac{sb}{1 - sa} \beta^{n_i} d(gx_0, gx_1) + \frac{sc + s}{1 - sa} d(gv, gx_{n_i+1}) \\ & + \frac{s^2 e + sk}{1 - sa} d(gx_{n_i}, gv) + \frac{s^2 e}{1 - sa} d(gv, Tv). \end{aligned} \tag{2.7}$$

Taking the limit of (2.7) as $i \to \infty$ and using the fact that $\lim_{i\to\infty} d(gv, gx_{n_i}) = 0$, we get

$$d(Tv, gv) \leq \frac{s^2 e}{1 - sa} d(gv, Tv).$$

Since $\frac{s^2 e}{1-sa} < 1$, it implies that $d(Tv, gv) = 0$. Therefore, $Tv = gv = u$, and so v is a coincidence point of T and g, and u is a point of coincidence.

To show that the point of coincidence is unique, suppose that the assumption (2) of Theorem 2.1 is satisfied, i.e., there is $u^* \in X$ such that $Tx = gx = u^*$ for some $x \in X$ and $(u, u^*) \in E(\tilde{G})$. Now, condition (2.1) implies that

$$\begin{aligned} d(u, u^*) = d(Tv, Tx) & \leq ad(gv, Tv) + bd(gx, Tx) + cd(gv, Tx) + ed(gx, Tv) + kd(gv, gx) \\ & = ad(u, u) + bd(u^*, u^*) + cd(u, u^*) + ed(u^*, u) + kd(u, u^*). \end{aligned} \tag{2.8}$$

Hence, equation (2.8) becomes $d(u, u^*) < (c + e + k)d(u, u^*)$ which gives that $u = u^*$, since $c + e + k < 1$.

If T and g are weakly compatible, then by Proposition 1.6, T and g have a unique common fixed point. □

3. Examples

The following example shows that the mappings T and g satisfying condition (2.1) on a graph G does not need to be true on the whole space X.

Example 3.1. Let $X = [0, \infty)$ and $T, g : X \to X$, such that

$$Tx = \begin{cases} \frac{x^2}{2}, & \text{when } x \neq \sqrt{8}, \\ 0, & \text{when } x = \sqrt{8}, \end{cases}$$

and

$$gx = \begin{cases} x^2, & \text{when } x \neq 2, \\ 1, & \text{when } x = 2. \end{cases}$$

Let (X, d) be a b-metric space where $d(x, y) = |x - y|^2$, G be the graph with $V(G) = X$ and $E(G) = \{(x, x); x \in X\} \cup \{(0, \frac{1}{2^n}); n \in \mathbf{N}\}$, and the constants $a = b = \frac{1}{64}, c = e = \frac{1}{16}$ and $k = \frac{1}{4}$. Note that $(gx, gy) \in E(\tilde{G})$ occurs only in either one of the following cases:

1. $x = y$;
2. one of x and y is zero and the other is $\frac{1}{2^{(\frac{n}{2})}}$ for some $n \in \mathbf{N}$.

If $x = y = 0$, then $d(Tx, Ty) = d(0, 0) = 0$ which satisfies condition (2.1). For the second case, without loss of generality we may assume $x = 0$ and $y = \frac{1}{2^{\frac{n}{2}}}$ for some $n \in \mathbf{N}$, then

$$\begin{aligned} d(Tx, Ty) =& d(0, \frac{1}{2^{n+1}}) = \frac{1}{4^{n+1}}, \\ d(gx, Tx) =& d(0, 0) = 0, \\ d(gy, Ty) =& d(\frac{1}{2^{2n}}, \frac{1}{2^{n+1}}) = \frac{1}{4^{n+1}}, \\ d(gx, Ty) =& d(0, \frac{1}{2^{n+1}}) = \frac{1}{4^{n+1}}, \\ d(gy, Tx) =& d(\frac{1}{2^{2n}}, 0) = \frac{4}{4^{n+1}}, \\ d(gx, gy) =& d(0, \frac{1}{2^{2n}}) = \frac{4}{4^{n+1}}. \end{aligned}$$

Now,

$$\begin{aligned} & ad(gx, Tx) + bd(gy, Ty) + cd(gx, Ty) + ed(gy, Tx) + kd(gx, gy) \\ &= \frac{1}{64}(0) + \frac{1}{64}(\frac{1}{4^{n+1}}) + \frac{1}{16}(\frac{1}{4^{n+1}}) + \frac{1}{16}(\frac{4}{4^{n+1}}) + \frac{1}{4}(\frac{4}{4^{n+1}}) \\ &= \frac{1}{4^{n+1}}(\frac{1}{64} + \frac{1}{16} + \frac{1}{4} + 1) \\ &> \frac{1}{4^{n+1}} = d(Tx, Ty). \end{aligned}$$

On the other hand, let $x = 0$ and $y = 2$. Then $(gx, gy) = (0, 1) \notin E(\tilde{G})$ and

$$\begin{aligned} d(Tx, Ty) &= d(T0, T2) = d(0, 2) = 4, \\ d(gx, Tx) &= d(0, 0) = 0, \\ d(gy, Ty) &= d(1, 2) = 1, \\ d(gx, Ty) &= d(0, 2) = 4, \end{aligned}$$

$$\begin{aligned} d(gy, Tx) &= d(1, 0) = 1, \\ d(gx, gy) &= d(0, 1) = 1. \end{aligned}$$

Thus,

$$\begin{aligned} &ad(gx, Tx) + bd(gy, Ty) + cd(gx, Ty) + ed(gy, Tx) + kd(gx, gy) \\ &= \frac{1}{64}(0) + \frac{1}{64}(1) + \frac{1}{16}(4) + \frac{1}{16}(1) + \frac{1}{4}(1) \\ &= (\frac{1}{64} + \frac{16}{64} + \frac{4}{64} + \frac{16}{64}) = \frac{37}{64} \\ &< 4 = d(Tx, Ty). \end{aligned}$$

Therefore, the mappings T and g satisfy the condition (2.1) on the graph G but do not on the whole space X.

The following example illustrates Theorem 2.1.

Example 3.2. Let $X = [0, \infty)$ and $T, g : X \to X$, such that $Tx = \frac{x^2}{3}$ and $gx = x^2$. Let (X, d) be a b-metric space where $d(x, y) = |x - y|^2$, G is the graph with $V(G) = X$ and $E(G) = \{(x, x); x \in X\} \cup \{(0, \frac{1}{3^n}); n \in \mathbf{N}\}$ (see Figure 1), and the constants of (2.1) are $a = b = \frac{9}{108}$ and $c = e = k = \frac{1}{24}$. Note that $(gx, gy) \in E(\tilde{G})$ occurs only in either one of the following cases:

1. $x = y$;
2. one of x and y is zero and the other is $\frac{1}{3^{(\frac{n}{2})}}$.

If $x = y = 0$, then $d(Tx, Ty) = d(0, 0) = 0$ which satisfies condition (2.1). For the second case, without loss of generality we may assume $x = 0$ and $y = \frac{1}{3^{\frac{n}{2}}}$ for some $n \in \mathbf{N}$, then

$$\begin{aligned} d(Tx, Ty) &= d(0, \frac{1}{3^{n+1}}) = \frac{1}{9^{n+1}}, \\ d(gx, Tx) &= d(0, 0) = 0, \\ d(gy, Ty) &= d(\frac{1}{3^n}, \frac{1}{3^{n+1}}) = \frac{4}{9^{n+1}}, \\ d(gx, Ty) &= d(0, \frac{1}{3^{n+1}}) = \frac{1}{9^{n+1}}, \\ d(gy, Tx) &= d(\frac{1}{2^{2n}}, 0) = \frac{1}{9^n}, \\ d(gx, gy) &= d(0, \frac{1}{3^n}) = \frac{1}{9^n}. \end{aligned}$$

Now,

$$\begin{aligned} &ad(gx, Tx) + bd(gy, Ty) + cd(gx, Ty) + ed(gy, Tx) + kd(gx, gy) \\ &= \frac{9}{108}(0) + \frac{9}{108}(\frac{4}{9^{n+1}}) + \frac{1}{24}(\frac{1}{9^{n+1}}) + \frac{1}{24}(\frac{1}{9^n}) + \frac{1}{24}(\frac{1}{9^n}) \\ &= \frac{1}{9^n}(\frac{1}{8}) \\ &\geq \frac{1}{9^n}\frac{1}{9} = d(Tx, Ty). \end{aligned}$$

Therefore, for all $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$, condition (2.1) of Theorem 2.1 is satisfied. Now, we discuss all sequences of the form (gx_n) such that $gx_n = Tx_{n-1}$. Let $x_0 \in X$. If $x_0 = 0$, then there is x_1

such that $gx_1 = Tx_0 = T0 = 0$, which implies that $x_1 = 0$. Similarly, there is x_2 such that $gx_2 = Tx_1 = 0$ which gives that $x_2 = 0$. By continuing this process we can get that $gx_n = 0$ for $n = 1, 2, \cdots$, hence $(gx_n, gx_m) = (0, 0) \in E(\tilde{G})$ for $m, n = 1, 2, 3, \cdots$. Thus $G_{gT} \neq \phi$.

For $x_0 \neq 0$, there is $x_1 \in X$ such that $gx_1 = Tx_0 = \frac{x_0^2}{3}$ which implies $x_1 = \frac{x_0}{\sqrt{3}}$. Similarly, there is $x_2 \in X$ such that $gx_2 = Tx_1 = \frac{x_0^2}{3^2}$, hence $x_2 = \frac{x_0}{3}$. By continuing this process we get the sequence (gx_n) such that $gx_n = Tx_{n-1} = \frac{x_0^2}{3^n}$.

Note that $(gx_n, gx_m) = (\frac{x_0^2}{3^n}, \frac{x_0^2}{3^m}) \notin E(\tilde{G})$. Thus, the only convergent sequence such that $(gx_n, gx_m) \in E(\tilde{G})$ is the constant sequence $gx_n = 0$ for all $n = 1, 2, 3, \cdots$, which converges to 0. So for every subsequence (gx_{n_i}) of (gx_n) we have $(gx_{n_i}, 0) \in E(\tilde{G})$.

Moreover, the mappings T and g are weakly compatible, since $\frac{x^2}{3} = x^2$ only at $x = 0$ and $T0 = g0 = 0$, also, $Tg0 = T0 = 0 = gT0$, so all conditions of Theorem 2.1 are satisfied and 0 is the unique common fixed point of T and g in X.

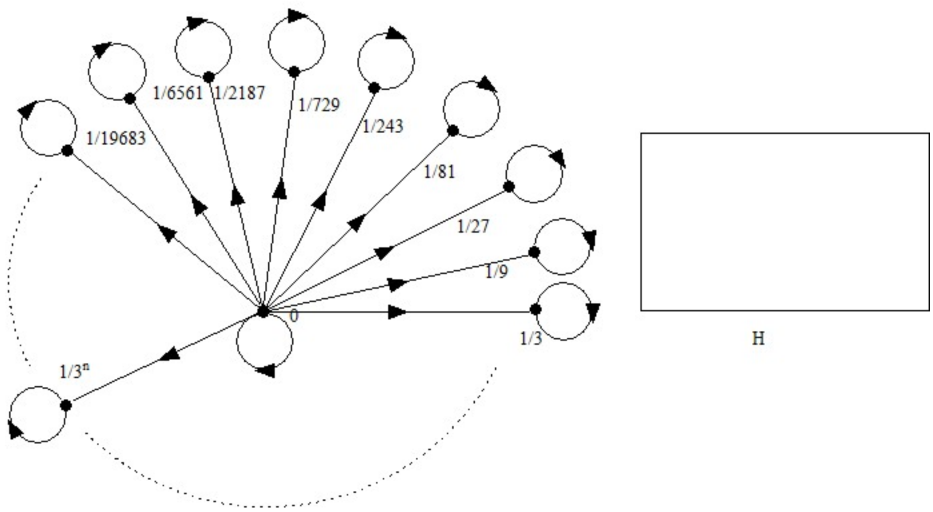

Figure 1: This Figure depicts graph G in Example 3.2 where H is the subgraph of G with vertex set $V(H) = [0, \infty) - \{0, \frac{1}{3^n} : n \in N\}$ and edge set $E(H) = \{(x, x); x \in V(H)\}$.

The following example shows that the property $G_{(T, gx_n)}$ is essential.

Example 3.3. Let $X = [0, 1]$ and $T, g : X \to X$, such that

$$Tx = \begin{cases} \frac{x^4}{16}, & \text{when } x \neq 0, \\ \frac{1}{2}, & \text{when } x = 0, \end{cases}$$

and

$$gx = \frac{x^2}{2}.$$

Let (X, d) be a b-metric space where $d(x, y) = |x - y|^3$, G be a graph with $V(G) = X$ and $E(G) = \{(0, 0)\} \cup \{(x, y) \in (0, 1] \times (0, 1]\}$, and the constants $a = b = c = e = k = \frac{1}{64}$.

It is clear that $T(X) \subseteq g(X)$ and $g(X)$ is complete subspace of X. Note that $(gx, gy) \in E(\tilde{G})$ occurs only in either one of the following cases:

1. $x = y = 0$;

2. $x \neq 0 \neq y$.

If $x = 0 = y$, then

$$d(Tx, Ty) = d(\frac{1}{2}, \frac{1}{2}) = 0 \leq ad(gx, Tx) + bd(gy, Ty) + cd(gx, Ty) + ed(gy, Tx) + kd(gx, gx).$$

If $x \neq 0 \neq y$, then

$$d(Tx, Ty) = |\frac{x^4}{16} - \frac{y^4}{16}|^3 = \frac{1}{16^3}|x^2 - y^2|^3|x^2 + y^2|^3 \leq \frac{1}{512}|x^2 - y^2|^3$$

and

$$d(gx, gy) = |\frac{x^2}{2} - \frac{y^2}{2}|^3 = \frac{1}{8}|x^2 - y^2|^3.$$

Thus,

$$\begin{aligned} d(Tx, Ty) &\leq \frac{1}{512}|x^2 - y^2|^3 = \frac{1}{64}\frac{1}{8}|x^2 - y^2|^3 = \frac{1}{64}d(gx, gy) \\ &\leq \frac{1}{64}d(gx, Tx) + \frac{1}{64}d(gy, Ty) + \frac{1}{64}d(gx, Ty) + \frac{1}{64}d(gy, Tx) + \frac{1}{64}d(gx, gy). \end{aligned}$$

Let $x_0 \in (0, 1]$ be an arbitrary element. Then $gx_1 = Tx_0 = \frac{x_0^4}{16} \neq 0$, which implies that $x_1 = \frac{\sqrt{2}x_0^2}{4} \in (0, 1]$. By the same process we can find $x_2 \in (0, 1]$ such that $gx_2 = Tx_1$.

Continuing the same process we get $gx_n = Tx_{n-1} \neq 0$. Therefore $(gx_n, gx_m) \in E(\tilde{G})$. As a result of the above $x_0 \in G_{gT}$, and so $G_{gT} \neq \phi$.

Now, let $x_0 = 1$. We will construct a sequence (gx_n) by $gx_n = Tx_{n-1}$. So, $gx_1 = Tx_0 = \frac{1}{16}$, hence $x_1 = \frac{\sqrt{2}}{4}$. Similarly, there is x_2 such that $gx_2 = Tx_1 = \frac{(\sqrt{2})^4}{4^6}$, thus $x_2 = \frac{(\sqrt{2})^3}{4^3}$. Continuing this process we get

$$gx_n = Tx_{n-1} = \frac{(\sqrt{2})^{2^{n+1}-4}}{4^{2^{n+1}-2}} \to 0 \text{ as } n \to \infty.$$

By definition of $E(G)$ we get that $(gx_n, gx_m) \in E(\tilde{G})$ and $gx_n \to 0$ but $(gx_n, 0) \notin E(\tilde{G})$. So there is no subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, 0) \in E(\tilde{G})$.

One can easily see that the mappings T and g have no coincidence point, so there is no common fixed point.

4. Corollaries and consequences in b-metric space

The following corollaries are consequences of Theorem 2.1, by specifying some of the constants a, b, c, e, k to be zero as needed.

Corollary 4.1. *Let (X, d) be a b-metric space (with $s \geq 1$) endowed with a graph G and the mappings $T, g : X \to X$ satisfy*

$$d(Tx, Ty) \leq kd(gx, gy)$$

for all $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$ when $k < \frac{1}{s}$. Suppose, $T(X) \subseteq g(X)$ and $g(X)$ is complete subspace of X. Then

1. *If $G_{gT} \neq \phi$ and the property $G_{(T,gx_n)}$ is satisfied, then T and g have a point of coincidence in X.*
2. *Moreover, if x and y are points of coincidence of T and g in X, it implies $(x, y) \in E(\tilde{G})$, then T and g have a unique point of coincidence in X.*
3. *Further, if T and g are weakly compatible, then T and g have a unique common fixed point in X.*

Corollary 4.2. *Let (X, d) be a b-metric space (with $s \geq 1$) endowed with a graph G and the mappings $T, g : X \to X$ satisfy*

$$d(Tx, Ty) \leq ad(gx, Tx) + bd(gy, Ty)$$

for all $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$ when $a+b < \frac{1}{s}$. Suppose, $T(X) \subseteq g(X)$ and $g(X)$ is complete subspace of X. Then

1. *If $G_{gT} \neq \phi$ and the property $G_{(T,gx_n)}$ is satisfied, then T and g have a point of coincidence in X.*
2. *Moreover, if x and y are points of coincidence of T and g in X, it implies $(x, y) \in E(\tilde{G})$, then T and g have a unique point of coincidence in X.*
3. *Further, if T and g are weakly compatible, then T and g have a unique common fixed point in X.*

Corollary 4.3. *Let (X, d) be a b-metric space (with $s \geq 1$) endowed with a graph G and the mappings $T, g : X \to X$ satisfy*

$$d(Tx, Ty) \leq cd(gx, Ty) + ed(gy, Tx)$$

for all $x, y \in X$ with $(gx, gy) \in E(\tilde{G})$ when either $(2c + e < \frac{1}{s}$ and $e < \frac{1}{s^2})$ or $(c + 2e < \frac{1}{s}$ and $c < \frac{1}{s^2})$. Suppose, $T(X) \subseteq g(X)$ and $g(X)$ is complete subspace of X. Then

1. *If $G_{gT} \neq \phi$ and the property $G_{(T,gx_n)}$ is satisfied, then T and g have a point of coincidence in X.*
2. *Moreover, if x and y are point of coincidence of T and g in X, it implies $(x, y) \in E(\tilde{G})$, then T and g have a unique point of coincidence in X.*
3. *Further, if T and g are weakly compatible, then T and g have a unique common fixed point in X.*

The property G_{T,x_n}: If $x \in X$ be arbitrary and $(x_n)_{n\in N}$ is a sequence in X, where $x_n = T^n x$ such that $x_n \to x$ and $(x_n, x_{n+1}) \in E(\tilde{G})$ for all $n \geq 1$, then there is a subsequence $(x_{n_i})_{i\in N}$ of $(x_n)_{n\in N}$ such that $(x_{n_i}, x) \in E(\tilde{G})$ for all $i \geq 1$.

Furthermore, we mean by $G_T = \{x_0 \in X : (T^n x_0, T^m x_0) \in E(\tilde{G})$ where $m, n = 1, 2, 3, \cdots\}$. In previous corollaries, taking $g = I$ in Theorem 2.1 and Corollaries 4.1, 4.2 and 4.3 we get the following results:

Corollary 4.4. *Let (X, d) be a complete b-metric space (with $s \geq 1$) endowed with a graph G and the mapping $T : X \to X$ satisfies*

$$d(Tx, Ty) \leq a(x, Tx) + bd(y, Ty) + cd(x, Ty) + ed(y, Tx) + kd(x, y)$$

for all $x, y \in X$ with $(x, y) \in E(\tilde{G})$ when either $(a+b+2c+e+k < \frac{1}{s}$ and $a+se < \frac{1}{s})$ or $(a+b+c+2e+k < \frac{1}{s}$ and $a + sc < \frac{1}{s})$. Then,

1. *If $G_T \neq \phi$ and the property G_{T,x_n} is satisfied, then T has a fixed point in X.*
2. *Moreover, if x and y are two fixed points of T in X, it implies $(x, y) \in E(\tilde{G})$, then $x = y$.*

Corollary 4.5. *Let (X, d) be a complete b-metric space (with $s \geq 1$) endowed with a graph G and the mapping $T : X \to X$ satisfies*

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$ with $(x, y) \in E(\tilde{G})$ when $k < \frac{1}{s}$. Then,

1. *If $G_T \neq \phi$ and the property $G_{(T,x_n)}$ is satisfied, then T has a fixed point in X.*

2. *Moreover, if x and y are two fixed points of T in X, it implies $(x,y) \in E(\tilde{G})$, then $x = y$.*

Corollary 4.6. *Let (X,d) be a complete b-metric space (with $s \geq 1$) endowed with a graph G and the mapping $T : X \to X$ satisfies*

$$d(Tx,Ty) \leq ad(x,Tx) + bd(y,Ty)$$

for all $x,y \in X$ with $(x,y) \in E(\tilde{G})$ when $a + b < \frac{1}{s}$. Then,

1. *If $G_T \neq \phi$ and the property $G_{(T,x_n)}$ is satisfied, then T has a fixed point in X.*
2. *Moreover, if x and y are two fixed points of T in X, it implies $(x,y) \in E(\tilde{G})$, then $x = y$.*

Corollary 4.7. *Let (X,d) be a complete b-metric space (with $s \geq 1$) endowed with a graph G and the mapping $T : X \to X$ satisfies*

$$d(Tx,Ty) \leq cd(x,Ty) + ed(y,Tx)$$

for all $x,y \in X$ with $(x,y) \in E(\tilde{G})$ when either $(2c + e < \frac{1}{s}$ and $e < \frac{1}{s^2})$, or $(c + 2e < \frac{1}{s}$ and $c < \frac{1}{s^2})$. Then,

1. *If $G_T \neq \phi$ and the property $G_{(T,x_n)}$ is satisfied, then T has a fixed point in X.*
2. *Moreover, if x and y are two fixed points of T in X, it implies $(x,y) \in E(\tilde{G})$, then $x = y$.*

5. Application

In this section, we will use Theorem 2.1 to show that there is a solution to the following integral equation:

$$u(t) = \int_a^b G(t,s)f(s,u(s))ds + h(t); \quad t \in [a,b]. \tag{5.1}$$

Let $X = (C[a,b], \mathbf{R})$ denote the set of all continuous functions from $[a,b]$ to $\mathbf{R}$. Define a mapping $T : X \to X$ by

$$Tu(t) = \int_a^b G(t,s)f(s,u(s))ds + h(t); \quad t \in [a,b], \tag{5.2}$$

where $h(t) \in X$.

Theorem 5.1. *Consider equation (5.1) and suppose that*

1. *$G : [a,b] \times [a,b] \to [0,\infty)$ is a continuous function.*
2. *$f : [a,b] \times \boldsymbol{R} \to \boldsymbol{R}$, where $f(s,.)$ is monotone nondecreasing mapping for all $s \in [a,b]$.*
3. *$\max_{t\in[a,b]} \int_a^b G(t,s)ds < \alpha$, where $\alpha < \frac{1}{3}$.*
4. *There exists function $x_0(t) \in X$ such that*

$$x_0(t) \leq \int_a^b G(t,s)f(s,x_0(s))ds + h(t),\ t \in [a,b].$$

5. *For all $x(s), y(s) \in X$ with $x(s) \leq y(s)$ (or $y(s) \leq x(s)$), and $s \in [a,b]$ we have*

$$|f(s,x(s)) - f(s,y(s))|^3 \leq \max \left\{ \begin{array}{c} |x(s) - Tx(s)|^3, |y(s) - Ty(s)|^3, \\ |x(s) - Ty(s)|^3, |y(s) - Tx(s)|^3, \\ |x(s) - y(s)|^3 \end{array} \right\}.$$

Then equation (5.1) has a solution.

Proof. Let X and T be as defined above. For all $u, v \in X$ define the b-metric on X by

$$d(u,v) = \Big(\max_{t\in[a,b]} |u(t) - v(t)|\Big)^3.$$

Clearly (X, d) is a complete b-metric space with constant $(s^{'} = 4)$. Define the graph G with $V(G) = X$ and

$$E(G) = \{(x(t), y(t)) : \text{ either } x(t) \le y(t) \text{ or } y(t) \le x(t) \text{ for all } t \in [a,b]\}.$$

From the assumption (4) and (5.2) one can notice that $x_0(t) \le Tx_0(t)$. Also from the assumption (2) we get $f(s, x_0(t)) \le f(s, Tx_0(t))$. Since $G(t,s) \ge 0$ for all $t, s \in [a,b]$, so clearly

$$\int_a^b G(t,s)f(s,x_0(s))ds + h(t) \le \int_a^b G(t,s)f(s,Tx_0(s))ds + h(t),$$

which implies that $Tx_0(t) \le T(Tx_0(t)) = T^2x_0(t)$. Continuing this process we get

$$x_o(t) \le Tx_o(t) \le \cdots \le T^n x_0(t) \le T^{n+1}x_0(t) \le \cdots \le T^m x(t) \le \cdots.$$

Hence, $(T^n x_0(t), T^m x_0(t)) \in E(G)$ for $n, m = 1, 2, \cdots$. Therefore $G_T \ne \phi$.

Moreover, based on definitions of $f(s, .)$ and T, any convergent sequence $(x_n(t)) \to x^*(t)$, where $x_n(t) = T^n x(t)$ for $t \in [a,b]$ and $n = 1, 2, 3, \cdots$, we have $x_n(t) \le x^*(t)$ for $n = 1, 2, 3, \cdots$. Thus $(x_n(t), x^*(t)) \in E(\tilde{G})$. Therefore, for any subsequence $(x_{n_i}(t))$ of $(x_n(t))$ we have $(x_{n_i}(t), x^*(t)) \in E(\tilde{G})$.

Now, Let $x(t), y(t) \in X$ such that $(x(t), y(t)) \in E(\tilde{G})$, without loss of generality we may assume $x(t) \le y(t)$ for each $t \in [a,b]$. From definition (5.2) we have

$$\begin{aligned} |Tx(t) - Ty(t)| &= \Big|\int_a^b G(t,s)(f(s,x(s)) - f(s,y(s))ds\Big| \\ &\le \int_a^b G(t,s)|f(s,x(s)) - f(s,y(s)|ds. \end{aligned} \tag{5.3}$$

Hence, from condition (5) and (5.3)

$$\begin{aligned} &d(Tx(t), Ty(t)) \\ &\quad = \Big(\max_{t\in[a,b]} |Tx(t) - Ty(t)|\Big)^3 \\ &\quad \le \Big(\max_{t\in[a,b]} \int_a^b G(t,s)|f(s,x(s)) - f(s,y(s)|ds\Big)^3 \\ &\quad \le \Big[\max_{t\in[a,b]} \int_a^b G(t,s)\Big(\max\left\{\begin{array}{c} |x(s) - Tx(s)|^3, |y(s) - Ty(s)|^3, |x(s) - Ty(s)|^3, \\ |y(s) - Tx(s)|^3, |x(s) - y(s)|^3 \end{array}\right\}\Big)^{\frac{1}{3}} ds\Big]^3 \\ &\quad \le \Big[\max_{t\in[a,b]} \int_a^b G(t,s)\Big(\max\left\{\begin{array}{c} \max_{s\in[a,b]} |x(s) - Tx(s)|^3, \max_{s\in[a,b]} |y(s) - Ty(s)|^3, \\ \max_{s\in[a,b]} |x(s) - Ty(s)|^3, \max_{s\in[a,b]} |y(s) - Tx(s)|^3, \\ \max_{s\in[a,b]} |x(s) - y(s)|^3 \end{array}\right\}\Big)^{\frac{1}{3}} ds\Big]^3 \\ &\quad = \Big(\max_{t\in[a,b]} \int_a^b G(t,s)ds\Big)^3 \max\left\{\begin{array}{c} d(x(s), Tx(s)), d(y(s), Ty(s)), d(x(s), Ty(s)), \\ d(y(s), Tx(s)), d(x(s), y(s)) \end{array}\right\} \\ &\quad \le (\alpha)^3 \max\{d(x(s), Tx(s)), d(y(s), Ty(s)), d(x(s), Ty(s)), d(y(s), Tx(s), d(x(s), y(s)))\} \\ &\quad \le (\alpha)^3 d(x(s), Tx(s)) + (\alpha)^3 d(y(s), Ty(s)) + (\alpha)^3 d(x(s), Ty(s)) + (\alpha)^3 d(y(s), Tx(s)) \\ &\qquad + (\alpha)^3 d(x(s), y(s)). \end{aligned}$$

Therefore, all conditions of Corollary 4.4 are satisfied for $a = b = c = e = k = (\alpha)^3$, where $a + b + 2c + e + k = 6(\alpha)^3) < \frac{6}{27} < \frac{1}{4}$ and $a + se = (\alpha)^3 + 4(\alpha)^3) < \frac{4}{27} < \frac{1}{4}$.

As a result of Corollary 4.4 the mapping T has fixed point in X which is a solution of (5.1). □

The following example illustrates the validity of Theorem 5.1.

Example 5.2. The following integral equation has a solution in $X = (C[0,1], \mathbf{R})$.

$$u(t) = \int_0^1 \frac{ts}{2}\, u(s)ds + (1 - \frac{t^3}{4}); \quad t \in [0,1]. \tag{5.4}$$

Proof. Let $T : X \to X$ be defined as $Tu(t) = \int_0^1 \frac{ts}{2}\, u(s)ds + (1 - \frac{t^3}{4})$ for $t \in [0,1]$. By specifying $G(t,s) = \frac{ts}{2}$, $f(s,t) = t$, $h(t) = 1 - \frac{t^3}{4}$ and $x_0(t) = 1$ in Theorem 5.1 we get that:

1. The function $G(t,s)$ is continuous on $[0,1] \times [0,1]$.
2. $f(s,t)$ is monotone increasing on $[0,1] \times \mathbf{R}$ for all $s \in [0,1]$.
3. $\max_{t\in[0,1]} \int_0^1 G(t,s)ds = \max_{t\in[0,1]} \int_0^1 \frac{ts}{2}ds = \max_{t\in[0,1]} \frac{t}{4} = \frac{1}{4} < \frac{1}{3}$.
4. $$\begin{aligned}\int_0^1 G(t,s)f(s,x_0(s))ds + h(t) &= \int_0^1 \frac{ts}{2}ds + (1 - \frac{t^3}{4})\\ &= \frac{ts^2}{4}\Big]_0^1 + (1 - \frac{t^3}{4})\\ &= \frac{t}{4} + (1 - \frac{t^3}{4})\\ &= \frac{t - t^3}{4} + 1 > 1 = x_0(t).\end{aligned}$$
5. For all $x(s), y(s) \in X$ with $x(s) \le y(s)$ or $y(s) \le x(s)$ for all $s \in [0,1]$ we have
$$|f(s,x(s)) - f(s,y(s)|^3 = |x(s) - y(s)|^3 \le \max \left\{ \begin{array}{c} |x(s) - Tx(s)|^3, |y(s) - Ty(s)|^3, \\ |x(s) - Ty(s)|^3, |y(s) - Tx(s)|^3, \\ |x(s) - y(s)|^3 \end{array} \right\}.$$

Therefore, all conditions of Theorem 5.1 are satisfied, hence the mapping T has a fixed point in X, which is a solution to equation (5.4). □

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