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Research Article

Logarithmic Bounds for Oscillatory Singular Integrals on Hardy Spaces

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We establish a logarithmic bound for oscillatory singular integrals with quadratic phases on the Hardy space $H^1(\mathbb{R}^n)$. The logarithmic rate of growth is the best possible.

1. Introduction

For $n \in \mathbb{N}$, let $K(x)$ be a Calderón-Zygmund kernel on $\mathbb{R}^n$ and let $P(x)$ be a polynomial of n variables with real coefficients. Consider the following oscillatory singular integral operator:

$$T_P : f \longrightarrow \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x-y)} K(x-y) f(y)\, dy. \quad (1)$$

It is well known that T_P is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $1 < p < \infty$ and also from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. Additionally, $L^p \to L^p$ and $L^1 \to L^{1,\infty}$ bounds are dependent on the degree of the phase polynomial P only, not its coefficients (see [1, 2]).

However, for $H^1(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$ boundedness of T_P, the answers are not nearly as clear-cut. First, it was shown in [3] that, in general, T_P may fail to be bounded on $H^1(\mathbb{R}^n)$ and when the coefficients of the first-order terms of P vanish, T_P is bounded from $H^1(\mathbb{R}^n)$ to itself with a bound independent of the higher order coefficients of P.

More recent work can be found in [4, 5], including the following.

Theorem 1 (see [5]). *Let $n \in \mathbb{N}$, $m \geq 2$, and $P(x) = \sum_{0\le|\alpha|\le m} a_\alpha x^\alpha$ be a polynomial of degree m in $\mathbb{R}^n$ with real coefficients. Let K be a Calderón-Zygmund kernel and let T_P be given as in (1). Then, there exists a positive constant C such that*

$$\|T_P f\|_{H^1(\mathbb{R}^n)} \le C\left(1 + \frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{2\le|\alpha|\le m} |a_\alpha|^{1/|\alpha|}}\right)\|f\|_{H^1(\mathbb{R}^n)} \quad (2)$$

for all $f \in H^1(\mathbb{R}^n)$. The constant C may depend on n, m, and K but is independent of the coefficients $\{a_\alpha\}$ of P.

In order to determine the optimal bound on $\|T_P\|_{H^1\to H^1}$, an example was given in [5] to show that, as $\sum_{|\alpha|=1} |a_\alpha| / \sum_{2\le|\alpha|\le m} |a_\alpha|^{1/|\alpha|} \to \infty$, any bound on $\|T_P\|_{H^1\to H^1}$ must increase at least at the rate of $\log(\sum_{|\alpha|=1} |a_\alpha| / \sum_{2\le|\alpha|\le m} |a_\alpha|^{1/|\alpha|})$. This naturally leads to the following question.

Does

$$\|T_P f\|_{H^1(\mathbb{R}^n)} \le C_{n,m}\left(1 + \log^+\left(\frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{2\le|\alpha|\le m} |a_\alpha|^{1/|\alpha|}}\right)\right)\|f\|_{H^1(\mathbb{R}^n)} \quad (3)$$

hold for all $f \in H^1(\mathbb{R}^n)$?

In this paper, we will prove that the answer to the above question is affirmative for all quadratic polynomials. Namely, we have the following.

Theorem 2. *Let $n \in \mathbb{N}$ and $P(x) = \sum_{0\le|\alpha|\le 2} a_\alpha x^\alpha$ be a quadratic polynomial in $\mathbb{R}^n$ with real coefficients. Let K be a Calderón-Zygmund kernel and let T_P be given as in (1). Then, there exists a positive constant C such that*

$$\|T_P f\|_{H^1(\mathbb{R}^n)} \le C\left(1+\log^+\left(\frac{\sum_{|\alpha|=1}|a_\alpha|}{\sum_{|\alpha|=2}|a_\alpha|^{1/2}}\right)\right)\|f\|_{H^1(\mathbb{R}^n)} \quad (4)$$

for all $f \in H^1(\mathbb{R}^n)$. The constant C may depend on n and K but is independent of the coefficients $\{a_\alpha\}$ of P.

We point out that C denotes an absolute constant whose value may change from line to line.

2. Some Definitions and Lemmas

Many of the tools we use are known. For readers who wish to see the definitions and some of their properties, the following references are suggested: [6–12].

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}$ and $|B(x,r)|$ denote the Euclidean volume of $B(x,r)$.

Let ϕ be a function in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n}\phi(x)dx = 1$. For each $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we let

$$M_\phi f(x) = \sup_{s>0}|(f*\phi_s)(x)|, \quad (5)$$

where $\phi_s(x) = s^{-n}\phi(x/s)$.

Definition 3. For a nonnegative, locally integrable function w on $\mathbb{R}^n$, the Hardy space $H^1(\mathbb{R}^n)$ is given by

$$H^1(\mathbb{R}^n) = \left\{f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|M_\phi f\|_{L^1(\mathbb{R}^n)} < \infty\right\}, \quad (6)$$

with $\|f\|_{H^1(\mathbb{R}^n)} = \|M_\phi f\|_{L^1(\mathbb{R}^n)}$.

Definition 4. A measurable function f on $\mathbb{R}^n$ is called H^1 atom if there exist $\zeta \in \mathbb{R}^n$ and $r > 0$ such that

$$\operatorname{supp}(f) \subseteq B(\zeta,r); \quad (7)$$

$$\|f\|_\infty \le \frac{1}{|B(\zeta,r)|}; \quad (8)$$

$$\int_{\mathbb{R}^n} f(y)\,dy = 0. \quad (9)$$

Lemma 5 (see [9, 10]). *For each $f \in H^1(\mathbb{R}^n)$, there exist H^1 atoms $\{f_\nu\}$ and coefficients $\{\omega_\nu\}$ such that*

$$f = \sum_\nu \omega_\nu f_\nu, \qquad \|f\|_{H^1(\mathbb{R}^n)} \approx \inf\sum_\nu|\omega_\nu|. \quad (10)$$

Definition 6. A C^1 function $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is called a Calderón-Zygmund kernel if the following are true:

(i) There exists $C > 0$ such that

$$|K(x)| + |x||\nabla K(x)| \le A|x|^{-n} \quad (11)$$

holds for all $x \in \mathbb{R}^n \setminus \{0\}$.

(ii) For all $0 < a < b$,

$$\int_{B(0,b)\setminus B(0,a)} K(x)\,dx = 0. \quad (12)$$

Lemma 7. *Let $P(x) = \sum_{0\le|\alpha|\le 2} a_\alpha x^\alpha$ for $x \in \mathbb{R}^n$ and $\lambda \ge 0$. Define operator $U_{P,\lambda}$ by*

$$(U_{P,\lambda}f)(x) = \frac{\chi_{B(0,\lambda)^c}(x)}{|x|^n}\int_{B(0,1)} e^{iP(x-y)} f(y)\,dy. \quad (13)$$

Then, there exists $C > 0$ independent of P such that

$$\|U_{P,\lambda}f\|_{L^1(\mathbb{R}^n)} \le C\|f\|_{L^\infty(B(0,1))} \quad (14)$$

holds for all $f \in L^\infty(B(0,1))$ and $\lambda \ge (\sum_{|\alpha|=2}|a_\alpha|^{1/2})^{-2}$.

Proof. We start by treating the more difficult case $n \ge 2$. The other case, $n = 1$, will be briefly considered later.

Write

$$\sum_{|\alpha|=2} a_\alpha x^\alpha = \sum_{j=1}^n\sum_{k=1}^n b_{jk}x_j x_k, \quad (15)$$

with $b_{jk} = b_{kj}$ for $1 \le j,\ k \le n$. Then, there exist $l, s \in \{1,\dots,n\}$ such that

$$|b_{ls}| = \max\left\{|b_{jk}| : 1 \le j,\ k \le n\right\}. \quad (16)$$

Thus, we have

$$2n^4|b_{ls}|\lambda \ge \lambda\left(\sum_{|\alpha|=2}|a_\alpha|^{1/2}\right)^2 > 1. \quad (17)$$

For $x, y \in \mathbb{R}^n$, let

$$x' = (x_1,\dots,x_{l-1},x_{l+1},\dots,x_n), \qquad \tilde{y} = (y_1,\dots,y_{s-1},y_{s+1},\dots,y_n). \quad (18)$$

Then, there are polynomials $Q_1(\cdot)$, $Q_2(\cdot)$ on $\mathbb{R}^n$, $Q_3(\cdot)$, $Q_4(\cdot)$ on $\mathbb{R}^{n-1}$, and $Q_5(\cdot)$ on $\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$ such that

$$\sum_{|\alpha|=2} a_\alpha(x-y)^\alpha = -2b_{ls}x_l y_s + Q_1(x) + Q_2(y) + x_l Q_3(\tilde{y}) + y_s Q_4(x') + Q_5(x',\tilde{y}). \quad (19)$$

Let $g(x) = f(x)$ for $x \in B(0,1)$ and $g(x) = 0$ if $x \in B(0,1)^c$. Then,

$$\begin{aligned}\|U_{P,\lambda}f\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n\setminus B(0,\lambda)} \left|\int_{\mathbb{R}^n} e^{iP(x-y)} g(y)\,dy\right| \frac{dx}{|x|^n} \\ &= \int_{\mathbb{R}^n\setminus B(0,\lambda)} \left|\int_{\tilde{y}\in\mathbb{R}^{n-1}} e^{i(P(0)+\sum_{|\alpha|=1} a_\alpha x^\alpha + Q_1(x) + x_l Q_3(\tilde{y}) + Q_5(x',\tilde{y}))} \left(\int_{y_s\in\mathbb{R}} e^{i(-2b_{ls}x_l y_s - \sum_{|\alpha|=1} a_\alpha y^\alpha + Q_2(y) + y_s Q_4(x'))} g(y)\,dy_s\right) d\tilde{y}\right| \frac{dx}{|x|^n} \\ &\le C \int_{x'\in\mathbb{R}^{n-1}} \int_{\tilde{y}\in\mathbb{R}^{n-1}} \int_{x_l\in\mathbb{R}} h_{x'}(x_l) \left|\int_{y_s\in\mathbb{R}} e^{-i(2b_{ls}x_l)y_s} g_{x',\tilde{y}}(y_s)\,dy_s\right| dx_l\,d\tilde{y}\,dx',\end{aligned} \tag{20}$$

where

$$\begin{aligned} g_{x',\tilde{y}}(y_s) &= e^{i(-\sum_{|\alpha|=1} a_\alpha y^\alpha + Q_2(y) + y_s Q_4(x'))} g(y), \\ h_{x'}(x_l) &= \frac{\chi_{[\lambda^2,\infty)}\left(|x_l|^2 + |x'|^2\right)}{\left(|x_l|^2 + |x'|^2\right)^{n/2}}. \end{aligned} \tag{21}$$

Since $|g_{x',\tilde{y}}(y_s)| = |g(y)|$ and $\operatorname{supp}(g_{x',\tilde{y}}) \subseteq [-1,1]$, we have

$$\begin{aligned}\|U_{P,\lambda}f\|_{L^1(\mathbb{R}^n)} &\le C \int_{x'\in\mathbb{R}^{n-1}} \int_{\tilde{y}\in\mathbb{R}^{n-1}} \int_{x_l\in\mathbb{R}} h_{x'}(x_l) \\ &\quad \cdot \left|\widehat{g_{x',\tilde{y}}}(2b_{ls}x_l)\right| dx_l\,d\tilde{y}\,dx' \\ &\le C \int_{x'\in\mathbb{R}^{n-1}} \int_{\tilde{y}\in\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} |h_{x'}(x_l)|^2\,dx_l\right)^{1/2} \\ &\quad \cdot \left(\int_{\mathbb{R}} \left|\widehat{g_{x',\tilde{y}}}(2b_{ls}x_l)\right|^2 dx_l\right)^{1/2} d\tilde{y}\,dx' \\ &= C|b_{ls}|^{-1/2} \int_{x'\in\mathbb{R}^{n-1}} \int_{\tilde{y}\in\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} |h_{x'}(x_l)|^2\,dx_l\right)^{1/2} \\ &\quad \cdot \left(\int_{\mathbb{R}} |g_{x',\tilde{y}}(y_s)|^2\,dy_s\right)^{1/2} d\tilde{y}\,dx' \\ &\le C|b_{ls}|^{-1/2} \|f\|_{L^\infty(B(0,1))} \\ &\quad \cdot \left(\int_{|x'|\ge\lambda} \left(\int_{\mathbb{R}} \frac{dx_l}{\left(|x_l|^2 + |x'|^2\right)^n}\right)^{1/2} dx' \right. \\ &\quad \left. + \int_{|x'|<\lambda} \left(\int_{|x_l|\ge\sqrt{\lambda^2-|x'|^2}} \frac{dx_l}{\left(|x_l|^2 + |x'|^2\right)^n}\right)^{1/2} dx'\right) \\ &\le C|b_{ls}|^{-1/2} \|f\|_{L^\infty(B(0,1))} \left[\int_{|x'|\ge\lambda} \frac{dx'}{|x'|^{n-1/2}} \right. \\ &\quad \left. + \lambda^{(1-2n)/2} \int_{|x'|<\lambda} \left(\int_1^\infty \frac{dt}{t^n\sqrt{t-|x'/\lambda|^2}}\right)^{1/2} dx'\right] \\ &\le C|b_{ls}|^{-1/2} \|f\|_{L^\infty(B(0,1))} \left(\lambda^{-1/2} + \lambda^{-n+1/2} \int_{|x'|<\lambda} \left(1 - \left|\frac{x'}{\lambda}\right|^2\right)^{-1/4} dx'\right) \le C(\lambda|b_{ls}|)^{-1/2} \\ &\quad \cdot \|f\|_{L^\infty(B(0,1))} \le C\|f\|_{L^\infty(B(0,1))}.\end{aligned} \tag{22}$$

The treatment of the case $n = 1$ only involves the Fourier transform step of the preceding argument. Details are omitted. □

Lemma 8. *Let $n \in \mathbb{N}$ and $P(x) = \sum_{0\le|\alpha|\le 2} a_\alpha x^\alpha$ be a quadratic polynomial in $\mathbb{R}^n$ with real coefficients. Let K be a Calderón-Zygmund kernel satisfying (11)-(12) and let T_P be given as in (1). Then, there exists a positive constant C such that*

$$\|T_P f\|_{L^1(\mathbb{R}^n)} \le C\left(1 + \log^+\left(\frac{\sum_{|\alpha|=1}|a_\alpha|}{\sum_{|\alpha|=2}|a_\alpha|^{1/2}}\right)\right) \tag{23}$$

for every H^1 atom f which satisfies (7)–(9) with $\zeta = 0$ and $r = 1$. The constant C may depend on n and A but is independent of $\{a_\alpha\}$, K, and f.

Proof. By the uniform boundedness of T_P on $L^2(\mathbb{R}^n)$ and (7)-(8),

$$\begin{aligned}\int_{B(0,2)} |T_P f(x)|\,dx &\le |B(0,2)|^{1/2} \|T_P f\|_{L^2(\mathbb{R}^n)} \\ &\le C\|f\|_{L^2(\mathbb{R}^n)} \le C.\end{aligned} \tag{24}$$

By (11), we have

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus B(0,2)} \left|T_P f(x) - K(x)\int_{B(0,1)} e^{iP(x-y)} f(y)\,dy\right| dx \\ &\le \int_{\mathbb{R}^n\setminus B(0,2)} \int_{B(0,1)} |K(x-y) - K(x)|\,|f(y)|\,dy\,dx \\ &\le C\|f\|_{L^1(\mathbb{R}^n)} \int_{\mathbb{R}^n\setminus B(0,2)} |x|^{-n-1}\,dx \le C.\end{aligned} \tag{25}$$

Let $\lambda = (\sum_{|\alpha|=2} |a_\alpha|^{1/2})^{-2}$. It follows from (11) and (7)-(8) and Lemma 7 that

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus B(0,\max\{2,\lambda\})} \left|T_P f(x)\right| dx \\ &\quad \le C + \int_{\mathbb{R}^n \setminus B(0,\lambda)} |K(x)| \left| \int_{B(0,1)} e^{iP(x-y)} f(y)\, dy \right| dx \\ &\quad \le C + C \left\|U_{P,\lambda} f\right\|_{L^1(\mathbb{R}^n)} \le C. \end{aligned} \tag{26}$$

If $\lambda \le 2$, then (23) follows from (24) and (26).

Thus, we may assume that $\lambda > 2$. To finish the proof, it suffices to show that

$$\begin{aligned} &\int_{B(0,\lambda)\setminus B(0,2)} \left|T_P f(x)\right| dx \\ &\quad \le C \left(1 + \log^+ \left(\frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{|\alpha|=2} |a_\alpha|^{1/2}} \right)\right). \end{aligned} \tag{27}$$

We will establish (27) by discussing two cases.

Case 1 ($\sum_{|\alpha|=1} |a_\alpha| \ge 1/2$). In this case, we have

$$\begin{aligned} &\int_{B(0,\lambda)\setminus B(0,2)} \left|T_P f(x)\right| dx \\ &\quad \le C \int_{B(0,\lambda)\setminus B(0,2)} \int_{B(0,1)} |x-y|^{-n} |f(y)|\, dy\, dx \\ &\quad \le C \|f\|_{L^1(\mathbb{R}^n)} \int_{B(0,\lambda)\setminus B(0,2)} |x|^{-n}\, dx \\ &\quad \le C \ln \left(\frac{1}{2\left(\sum_{|\alpha|=2} |a_\alpha|^{1/2}\right)^2} \right) \\ &\quad \le C \left(\ln 2 + 2 \ln \left(\frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{|\alpha|=2} |a_\alpha|^{1/2}} \right) \right) \\ &\quad \le C \left(1 + \log^+ \left(\frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{|\alpha|=2} |a_\alpha|^{1/2}} \right) \right). \end{aligned} \tag{28}$$

Case 2 ($\sum_{|\alpha|=1} |a_\alpha| < 1/2$). In this case, we let

$$Q(x) = P(0) + \sum_{|\alpha|=2} a_\alpha x^\alpha. \tag{29}$$

It follows from Theorem 1 of [3] that

$$\left\|T_Q f\right\|_{L^1(\mathbb{R}^n)} \le C. \tag{30}$$

For $x \in \mathbb{R}^n$ and $y \in B(0,1)$, we have

$$\left| e^{iP(x-y)} - e^{i(\sum_{|\alpha|=1} a_\alpha x^\alpha + Q(x-y))} \right| \le \sum_{|\alpha|=1} |a_\alpha|. \tag{31}$$

By (30)-(31) and

$$\sup_{0<t<1/2} t \ln \left(\frac{1}{t}\right) = \frac{1}{e}, \tag{32}$$

we have

$$\begin{aligned} &\int_{B(0,\lambda)\setminus B(0,2)} \left|T_P f(x)\right| dx \le \left\|T_Q f\right\|_{L^1(\mathbb{R}^n)} \\ &\quad + \int_{B(0,\lambda)\setminus B(0,2)} \left|T_P f(x) - e^{i(\sum_{|\alpha|=1} a_\alpha x^\alpha)} T_Q f(x)\right| dx \\ &\quad \le C + C \left(\sum_{|\alpha|=1} |a_\alpha|\right) \|f\|_{L^1(\mathbb{R}^n)} \int_{B(0,\lambda)\setminus B(0,2)} |x|^{-n}\, dx \\ &\quad \le C + C \left(\sum_{|\alpha|=1} |a_\alpha|\right) \ln \left(\frac{1}{\sum_{|\alpha|=2} |a_\alpha|^{1/2}} \right) = C \\ &\quad + C \left(\sum_{|\alpha|=1} |a_\alpha|\right) \\ &\quad \cdot \left[\ln \left(\frac{1}{\sum_{|\alpha|=1} |a_\alpha|} \right) + \ln \left(\frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{|\alpha|=2} |a_\alpha|^{1/2}} \right) \right] \\ &\quad \le C \left(1 + \log^+ \left(\frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{|\alpha|=2} |a_\alpha|^{1/2}} \right) \right). \end{aligned} \tag{33}$$

Thus, (27) holds in both cases. □

3. Proof of Main Theorem

To finish the proof, we recall the following result concerning Riesz transforms and Hardy spaces.

Lemma 9 (see [10, 13]). *For $1 \le j \le n$, let R_j denote the jth Riesz transform; that is,*

$$\widehat{R_j f}(\xi) = \frac{i\xi_j}{|\xi|} \hat{f}(\xi). \tag{34}$$

Then, there exist $C, C_1, C_2 > 0$ such that

$$\left\|R_j f\right\|_{H^1(\mathbb{R}^n)} \le C \|f\|_{H^1(\mathbb{R}^n)} \tag{35}$$

for $1 \le j \le n$, and

$$\begin{aligned} C_1 \|f\|_{H^1(\mathbb{R}^n)} &\le \|f\|_{L^1(\mathbb{R}^n)} + \sum_{j=1}^n \left\|R_j f\right\|_{L^1(\mathbb{R}^n)} \\ &\le C_2 \|f\|_{H^1(\mathbb{R}^n)} \end{aligned} \tag{36}$$

for all $f \in H^1(\mathbb{R}^n)$.

We will now give the proof of Theorem 2.

Proof. For $f \in H^1(\mathbb{R}^n)$, let $\{\omega_\nu\}$ be a sequence of complex numbers and let $\{f_\nu\}$ be a sequence of H^1 atoms such that

$$f = \sum_\nu \omega_\nu f_\nu. \tag{37}$$

For each ν, let $\zeta_\nu \in \mathbb{R}^n$ and $r_\nu > 0$ such that $\operatorname{supp}(f_\nu) \subseteq B(\zeta_\nu, r_\nu)$ and $\|f_\nu\|_\infty \le |B(\zeta_\nu, r_\nu)|^{-1} = |B(0,1)|^{-1} r_\nu^{-n}$. Then,

$$r_\nu^n T_P f_\nu (r_\nu x + \zeta_\nu) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP_\nu(x-y)} K_\nu (x-y) (r_\nu^n f_\nu (r_\nu y + \zeta_\nu))\, dy, \quad (38)$$

where $P_\nu(x) = P(r_\nu x)$ and $K_\nu(x) = r_\nu^n K(r_\nu x)$. Observe that, for each ν, K_ν satisfies (11)-(12) with the same constant A and $r_\nu^n f_\nu(r_\nu y + \zeta_\nu)$ satisfies (7)–(9) with $\zeta = 0$, $r = 1$. Since

$$P_\nu (x) = \sum_{0\le|\alpha|\le 2} r_\nu^{|\alpha|} a_\alpha x^\alpha, \quad (39)$$

by Lemma 8,

$$\|T_P f_\nu\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |r_\nu^n T_P f_\nu (r_\nu x + \zeta_\nu)|\, dx = C\left(1 + \log^+\left(\frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{|\alpha|=2} |a_\alpha|^{1/2}}\right)\right), \quad (40)$$

which implies that

$$\|T_P f\|_{L^1(\mathbb{R}^n)} \le C\left(1 + \log^+\left(\frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{|\alpha|=2} |a_\alpha|^{1/2}}\right)\right)\left(\sum_\nu |\omega_\nu|\right). \quad (41)$$

It follows from Lemma 5 that

$$\|T_P f\|_{L^1(\mathbb{R}^n)} \le C\left(1 + \log^+\left(\frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{|\alpha|=2} |a_\alpha|^{1/2}}\right)\right)\|f\|_{H^1(\mathbb{R}^n)}. \quad (42)$$

By the translation invariance of T_P and (42) and (35), we have

$$\begin{aligned}\sum_{j=1}^n \|R_j T_P f\|_{L^1(\mathbb{R}^n)} &= \sum_{j=1}^n \|T_P R_j f\|_{L^1(\mathbb{R}^n)} \\ &\le C\left(1 + \log^+\left(\frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{|\alpha|=2} |a_\alpha|^{1/2}}\right)\right) \cdot \left(\sum_{j=1}^n \|R_j f\|_{H^1(\mathbb{R}^n)}\right) \\ &\le C\left(1 + \log^+\left(\frac{\sum_{|\alpha|=1} |a_\alpha|}{\sum_{|\alpha|=2} |a_\alpha|^{1/2}}\right)\right)\|f\|_{H^1(\mathbb{R}^n)}.\end{aligned} \quad (43)$$

By applying (36), (42), and (43), we obtain (4). □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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