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MOMENT APPROXIMATION OF A MONEY RETURN MODEL EMPLOYING A BIRTH AND DEATH DIFFUSION PROCESS WITH GENERAL EXTERNAL EFFECT

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ABSTRACT

This paper derives moment approximation as well as the mean and the variance of a money return model within an important diffusion return process by considering the stochastic analogs of classical differences and differential equations. This is accomplished by employing an interest rate process that follows a birth and death diffusion process with general external effect process and represented as a solution to a stochastic differential equation. The analysis introduces a generalization of a widely applied statistical distribution, the exponential distribution. In particular, the moment approximation for some external effect distributions of Beta and Exponential distributions, as well as for the case of no external effects are generated. Numerical examples for a sample path of such a money return process are also considered for the case of fixed annualized interest rate and for the case of no jumps as well as the case of the occurrence of jump process that follow a uniform and exponential distribution. The results are useful in studying the behavior of the process and in statistical inference problems. The model generalization should attract wider applicability.

Key Words: Money Return, Interest rate Process, Birth-Death Diffusion Process, General External Effect, Moment Approximation, Mean and Variance.

1. Introduction and Background

There is a voluminous theoretical literature in finance that relates to the return generating process which is usually formulated as that the conditional expected returns are always constant. The returns, however, are notoriously difficult to be modeled since harnessing information from these models has proven difficult (Simon 2021). On the other hand, financial analysts routinely supply information on the trend of the money returns in a dollar invested at time zero in different classes of investments, which is important to plan for financial activities. Nevertheless, Gu *et al.* (2020) point out the difficulty of predicting money returns and that the "kitchen sink" models suffer from substantial predicting errors. Today, finance researchers are interested in describing phenomena in theoretical models involving economic structure by considering the stochastic analogs of classical differences and differential equations. The literature documents that all money return models that are often used to test the incremental information content of a flow variable (e.g., market model, constant expected returns model, capital asset pricing model, dividend discount model) are statistically estimated by employing robust regressions (see for example Simon, 2021). These regressions employ an iterative procedure that weights each observation based on absolute residuals in an attempt to diminish the influence of outliers and to provide a better fit to the majority of the data; a procedure that should reduce variation in the coefficients across different models (De Simone 2016; Brownen-Trinh, 2019).

Motivation of the Study

However, this study proposes a new theoretical money return model based on the birth and death of a diffusion process and employs an interest rate process to test a value relevance by estimating a money return model, which provides a relatively straightforward test of the incremental information in a variable. As financial analysts are interested in measuring the stochastic effect of information as reflected in a flow variable such as interest rate, this might be difficult due to expected errors in the measurement of market expectation that is required to be specified and measured within a money return model, a feature that characterizes the success of money return models in general (Kothari and Zimmerman (1995)).

Several authors have studied different topics related to money return from different points of view. For example, Kothari and Zimmerman (1995) provided a framework for choosing between return models such as returns regressed on scaled earnings variables and the stock price regressed on earnings per share. Their empirical results confirmed that price models' earnings response coefficients are less biased. However, return models have less serious econometric problems than price models but in some research contexts, the combined use of both price and return models may be useful.

Carpinteyro *et al.* (2021) have developed a stochastic volatility model that is useful to explain the dynamics of the returns of gold, silver, and platinum during the period 1994–2019. They assumed that the precious metal returns are driven by fractional Brownian motions, combined with Poisson processes and modulated by continuous-time homogeneous Markov chains. The calibration is carried out by estimating the Jump Generalized Autoregressive Conditional Heteroscedasticity (Jump-GARCH) and Markov regime-switching models of each precious metal, as well as computing their Hurst exponents. In addition,

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they innovated the use of non-linear, non-normal, multi-factor, time-varying risk stochastic models, useful for an investors' decision- making process when they intend to include precious metals in their portfolios as safe-haven assets.

On the other hand, Naeem *et al.* (2019) have tested the existence of regime changes by using Markov-switching Generalized Autoregressive Conditional Heteroscedasticity (MS-GARCH) models to explain the volatility of the four precious metals and fit several models with different regimes to the log-returns of each precious metal to test the in-sample analysis of volatility.

Moreover, Vallejo–Jiménez and Venegas–Martinez (2017) have developed a model that explains the dynamics of an individual asset that is driven by multiple jumps, fractional Brownian motion, and regime-switching stochastic volatility.

Furthermore, Goh *et al*. (2015) investigate how investors price the fair value estimates of assets as required by Statement of Financial Accounting Standards No. 157 (SFAS 157) since the financial crisis in 2008. They observed that Level 3 fair value (value of illiquid assets that are determined by the related entity) estimates are typically priced lower than Level 1 fair value and Level 2 fair value (determined by market) estimates between 2008 and 2011. However, the difference between the pricing of the different estimates reduces over time, suggesting that as market conditions stabilize in the aftermath of the 2008 financial crisis, reliability concerns about Level 3 estimates dissipated to some extent. Also, they examined whether Level 3 gains affect the pricing of Level 3 estimates because managers have discretion to use Level 3 gains to manage earnings and asset values upwards. They found that differences in Level 3 gains do not lead investors to price Level 3 estimates differently. Finally, they found evidence that the pricing of the Level 1 and Level 2 fair value estimates of assets is lower for banks with lower capital adequacy.

In addition, Hasan and Al-Eideh (2002) have studied the modelling of default risk using a stochastic process approach. They have concerned with financial contracts since default risk is a component of essentially all contracts. They introduced a stochastic diffusion model for the annualized interest rate. Stocks and bonds offer contrasting advantages and disadvantages. History tells us that over the long run, stocks have a higher rate of return than bonds. Since 1926, stocks have enjoyed an average annual return almost twice that of bonds. At the same time, stocks come with more volatility. Bonds in a portfolio reduce the volatility, but at the cost of lower expected returns.

In addition, a key feature of insurance is that premiums are received in advance of the risk being borne and claim payments follow even later. In the meantime, provisions need to be established in respect of the expected future liabilities and additional solvency margins have to be maintained. These provisions and reserves should be backed by appropriate assets, having regard to the liabilities, and with a reasonable balance between security, liquidity and good investment return (including interest, dividend or rental income and capital gains). Thus, the investment process is an integral part of the financial management of an insurer.

Models projecting investment prices and returns have been developed in recent years, for example by Ibbotson and Sinquefield (1977), (1982) in the US and Wilkie (1984), (1986) in the UK. The properties and practical value of the Wilkie model were

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investigated by a financial management group working party of the institute and the faculty of Actuaries (C.f. Geoghegan *et al*. (1992)). The working party also, reviewed econometric and time series modelling procedures other than the Wilkie model, while is based on Box-Jenkins methods, and made some proposals with regard to possibilities for developing and improving the Wilkie model. For more details about applications in Actuaries, we refer the reader to Daykin *et al*. (1996).

This paper focusses on studying the moment approximations as well as the mean and the variance of the money return model using an interest rate process that is characterized by a birth and death diffusion process with general external effect at a constant rate, for some external effect of Beta and Exponential distributions, as well as the case of no effects. Diffusion processes have earned universal acceptance in the fields of finance and economics as their parametrization conforms to a widely accepted assumption of markets being efficient. Hence, testing this assumption centers on estimating coefficients describing the diffusion process that is relatively tractable, given the fact that the martingale processes (i.e., prices follow random walks) are very consistent with the concept of market efficiency. The objective of this research is to identify critical knowledge types required by economists of money return through building a stochastic money return model that has never been examined before as far as I know. The results should be very useful and will benefit the economists and others to study the behavior of the money return in a dollar invested at time zero through different applications.

The adopted money return model in this study is distinct from other money return models generated in prior studies (see for example Easton *et al.* (1991) in two respects. First, the parametrization underlying the current money return model is believed to be more realistic, transparent in its properties, and economically meaningful in that the interest rate process is assumed to simulate that of the general process in economy. Second, the employed diffusion model tends to be comprehensive in a way that it considers all types of stochastic process of interest rate formulations in any economy, to the point that there tends to be a closer tie between the theoretical predictions of the adopted model and the different general structures of interest rate models. Besides, the model allows efficient implementation.

2. A Money Return Model Using a Birth and Death Diffusion Process with Downward External Effect

In finance, the return is defined as a profit (loss), assuming the amount invested is greater than zero on an investment. It covers any change in value of the investment that may be measured either in dollars as an absolute term or as a percentage of the amount invested in a holding period return, which the investor receives whether as interest payments or dividends. It is always desirable to consider the return over a period of a standard length in order to compare returns over periods of different lengths on an equal basis. The result of the conversion is termed as the rate of return. Usually, the period is a year, in which case the rate of return is also called the annualized return. The rate of return is the percentage change from the beginning of the period until the end. Money returns models are tremendously employed by finance researchers to model the return generating process.

Birth and death processes are considered to be flexible class of continuous time Markov chains where jumps are allowed but with restrictions. Births usually are in terms of

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"particles" in a space that usually take non-negative integers. In returns models, however, returns can be positive or negative. Hence, the following model is distinct. After defining the rules of births and deaths in the return system by postulating the behavior of the return process, probabilities of births and deaths can be specified accordingly.

Consider an economy over the time interval $[0,T]$, with uncertainty represented by probability space (Ω, F, P), and the arrival of information modelled by a filtration $F = \{F_t \mid 0 \le t \le T\}$ satisfying the usual continuous-time Markov chain. The economy has a money market account paying the continuously compounded annualized interest rate r_t at time t, for an F adapted a stochastic process

$$r = \{r_t \mid 0 \le t \le T\}.$$

The return at time t on a dollar invested at time 0 is given by the stochastic process $B = \{B_t \mid 0 \le t \le T\}$ Where

$$B_t = \exp\left(\int_0^t r_u du\right) \tag{1}$$

Let

$$F(x) = \int_0^t r_u du \tag{2}$$

Then equation (1) can be written as

$$B_t = \exp(F(t)) \tag{3}$$

Assume that interest rates follow a stochastic diffusion process $\{r_t; t \ge 0\}$ in which the diffusion coefficient a and the drift coefficient b are both proportional to interest rate r_t. The diffusion process r_t is assumed to be interrupted by jumps occurring at a constant rate c and having magnitude with distribution function $H_x(\cdot)$(waiting time of the jump is irrelevant as it is considered when calculating the return). Then $\{r_t; t \ge 0\}$ is a Markov process with state space $[0, \infty)$ and can be regarded as the solution of the stochastic differential equation given the initial interest rate at time 0 is r_0 :

$$dr_t = br_t dt + ar_t dW_t - r_{t^-} dZ_t \tag{4}$$

Where $\{W_t : t \ge 0\}$ is a standard Wiener process with zero mean and variance $\sigma^2 t$. Also, the process $\{Z_t : t \ge 0\}$ is a compound Poisson process, with external jump rate $c > 0$ and jump size distribution H_x , is given by

$$Z_t = \sum_{i=1}^{N_t} Y_i$$

Here, $\{N_t\}$ is a Poisson process with mean rate c, where c is the external jump rate, and $Y_1, Y_2, Y_3, \ldots,$ are independent and identically distributed random variables with distribution function $H_x(\cdot)$, with mean $\mu = E(Y_1)$ and variance $v^2 = Var(Y_1)$. Note that the moments of Z_t can be determined from the random sums' formulas, and are

$$E[Z_t] = c\mu t$$

And

$$Var[Z_t] = c(v^2 + \mu^2)t$$

Assume that the existence and uniqueness conditions are satisfied (Cf. Gihman and Skorohod (1974)). Then the solution to the above SDE in equation (4) is given by:

$$r_t = r_0 \exp\{bt + aW_t - Z_t\} \tag{5}$$

Now, solving $F(t)$ in equation (2), using the concept of Ito integrals assists in solving the integral (Taylor and Karlin (1981, pp. 177)) by applying a Taylor expansion to r_t. After some algebraic manipulations, it is easily shown that (cf. Al-Eideh and Al-Hussainan (2002))

$$F(t) = \int_0^t r_u du = \int_0^t r_0 \exp\{bu + aW_u - Z_u\} du$$
$$= \frac{2(1-b)}{2a + a^2 - b^2} r_0 \exp\{bt + aW_t - Z_t\} \quad (6)$$

Therefore, the money return model $B(t)$ at time t using interest rate r_t follows the birth and death diffusion process, as defined in equation (5), is then given by

$$B(t) = \exp\left\{\frac{2(1-b)}{2a + a^2 - b^2} r_0 \exp\{bt + aW_t - Z_t\}\right\} \quad (7)$$

Where r_0 is the initial interest rate at time zero.

3.Moment Approximation of the Life Table Survival Model *l*(*x*) Using the Birth and Death Diffusion Force of Mortality with External Jump Processes

In this section, the moment approximation as well as the mean and the variance for the money return model $B(t)$, using the interest rate r_t with external jump process $H(\cdot)$, are derived.

Let $M_n(t) = E[B^n(t)]$ for all $n = 1,2,3,\ldots,$ be the $n-th$ moment of the money return $B(t)$. Then

$$M_n(t) = E[B^n(t)] = E[\exp(nF(t))] \quad (8)$$

Note that

$$E[\exp(nF(t))] \approx 1 + nE[F(t)] + \frac{n^2}{2} E[F^2(t)] \quad (9)$$

Using the results of finding the moment approximation of a birth and death diffusion process with general rate jump process (cf. Al-Eideh (2001)), it is easily shown that

$$E[F^n(t)] = \left(\frac{2(1-b)}{2a + a^2 - b^2}\right)^n r_0^n E[\exp\{nbt + anW_t - nZ_t\}]$$
$$= \left(\frac{2(1-b)}{2a + a^2 - b^2}\right)^n r_0^n E(\exp(nbt)) E[\exp(naW_t)] E[\exp(-nZ_t)] \quad (10)$$

Note that

$$E[\exp(naW_t)] = \exp\left(\frac{n^2}{2} a^2 \sigma^2 t\right) \quad (11)$$

In addition, because of equation (9), we obtain

$$E[\exp(-nZ_t)] \approx 1 - nE[Z_t] + \frac{n^2}{2} E[Z_t^2] \quad (12)$$

Using equations (3) and (4), equation (12) becomes

$$E[\exp(-nZ_t)] \approx 1 - ncmt + \frac{n^2}{2} c(\nu^2 + m^2)t \quad (13)$$

Now, by direct substitution of equations (11) and (12), we obtain

$$E[F^n(x)] \approx r_0^n \left(\frac{2(1-b)}{2a+a^2-b^2} \right)^n \exp\left\{ \left(nb + \frac{n^2}{2} a^2 \sigma^2 \right) t \right\} \cdot \left\{ 1 - ncmt + \frac{n^2}{2} c(v^2+m^2)t \right\} \quad (14)$$

Consequently,

$$E[F(x)] \approx r_0 \left(\frac{2(1-b)}{2a+a^2-b^2} \right) \exp\left\{ \left(b + \frac{1}{2} a^2 \sigma^2 \right) t \right\} \cdot \left\{ 1 - cmt + \frac{1}{2} c(v^2+m^2)t \right\} \quad (15)$$

And

$$E[F^2(x)] \approx r_0^2 \left(\frac{2(1-b)}{2a+a^2-b^2} \right)^2 \exp\{ (2b + 2a^2\sigma^2) t \} \cdot \{ 1 - 2cmt + 2c(v^2+m^2)t \} \quad (16)$$

(15) And (16) in equation (9), we obtain from equation (10) the $n-th$ moment of the survival function $M_n(t) = E[B^n(t)]$ for all $n = 1,2,3,\ldots,$ that is given by

$$\begin{aligned} M_n(t) = E[B^n(t)] \approx 1 &+ nr_0 \left(\frac{2(1-b)}{2a+a^2-b^2} \right) \exp\left\{ \left(b + \frac{1}{2} a^2 \sigma^2 \right) t \right\} \left(1 - cmt + \frac{1}{2} c(v^2+m^2)t \right) \\ &+ \frac{n^2}{2} r_0^2 \left(\frac{2(1-b)}{2a+a^2-b^2} \right)^2 \exp\{ (2b + 2a^2\sigma^2) t \} (1 - 2cmt + 2c(v^2+m^2)t) \end{aligned} \quad (17)$$

Where r_0 is the initial interest rate at time zero.

In particular, let $M_1(t) = E[B(t)]$ and $V(t) = V[B(t)]$ be the mean and the variance of $B(t)$ respectively. Then using equation (17), it is easily shown that

$$\begin{aligned} M_1(t) = E[B(t)] \approx 1 &+ r_0 \left(\frac{2(1-b)}{2a+a^2-b^2} \right) \exp\left\{ \left(b + \frac{1}{2} a^2 \sigma^2 \right) t \right\} \left(1 - cmt + \frac{1}{2} c(v^2+m^2)t \right) \\ &+ \frac{1}{2} r_0^2 \left(\frac{2(1-b)}{2a+a^2-b^2} \right)^2 \exp\{ (2b + 2a^2\sigma^2) t \} (1 - 2cmt + 2c(v^2+m^2)t) \end{aligned} \quad (18)$$

And

$$\begin{aligned} M_2(t) = E[B^2(t)] \approx 1 &+ 2r_0 \left(\frac{2(1-b)}{2a+a^2-b^2} \right) \exp\left\{ \left(b + \frac{1}{2} a^2 \sigma^2 \right) t \right\} \left(1 - cmt + \frac{1}{2} c(v^2+m^2)t \right) \\ &+ 2r_0^2 \left(\frac{2(1-b)}{2a+a^2-b^2} \right)^2 \exp\{ (2b + 2a^2\sigma^2) t \} (1 - 2cmt + 2c(v^2+m^2)t) \end{aligned} \quad (19)$$

Therefore, the variance of $B(t)$ is then given by

$$V(t) = V[B(t)] = M_2(t) - (M_1(t))^2 \quad (20)$$

Where $M_1(t)$ and $M_2(t)$ are defined in equations (19) and (20) respectively.

4. Examples of External Effect Distributions

In this section, formulas for moment approximation for some external effect distributions are derived.

4.1 Beta Distribution

Let $H(y)$ be the distribution function of beta random variable with parameters α and β, i.e.

$$H(y) = \begin{cases} 0, & y < 0 \\ \int_0^y \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} dx & 0 \le y < 1 \\ 1, & y \ge 1 \end{cases}$$

Where $B(\alpha, \beta)$ is the beta function, defined as:

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$$B(\alpha,\beta)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$$

Note that

$$m=\frac{\alpha}{\alpha+\beta}$$

And

$$\nu^2=\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Now, substituting in equation (17), then the $n-th$ moment approximation of the money return model $M_n(t)=E\left[B^n(t)\right]$ for all $n=1,2,3,\ldots,$ is given by

$$\begin{aligned}M_n(t)=E\left[B^n(t)\right]&\approx 1\\&+nr_0\left(\frac{2(1-b)}{2a+a^2-b^2}\right)\exp\left\{\left(b+\frac{1}{2}a^2\sigma^2\right)t\right\}\left(1-\frac{c\alpha}{\alpha+\beta}t+\frac{1}{2}\frac{c\alpha(\beta+1)}{\alpha+\beta+1}t\right)\\&+\frac{n^2}{2}r_0^2\left(\frac{2(1-b)}{2a+a^2-b^2}\right)^2\exp\left\{\left(2b+2a^2\sigma^2\right)t\right\}\left(1-\frac{2c\alpha}{\alpha+\beta}t+\frac{2c\alpha(\beta+1)}{\alpha+\beta+1}t\right)\end{aligned}$$

Where r_0 is the initial interest rate at time zero.

Note that when $\alpha=1$ and $\beta=1$, then $H(y)$ becomes

$$H(y)=\begin{cases}0, & y<0\\ y & 0\le y<1\\ 1, & y\ge 1\end{cases}$$

Therefore, $H(y)$ is the distribution function of the uniform random variable on $[0,1]$. Thus, substituting in equation (21), we obtain the $n-th$ moment approximation of the money return model $M_n(t)=E\left[B^n(t)\right]$ for all $n=1,2,3,\ldots,$ as follows

$$\begin{aligned}M_n(t)=E\left[B^n(t)\right]&\approx 1\\&+nr_0\left(\frac{2(1-b)}{2a+a^2-b^2}\right)\exp\left\{\left(b+\frac{1}{2}a^2\sigma^2\right)t\right\}\left(1-\frac{1}{6}ct\right)\\&+\frac{n^2}{2}r_0^2\left(\frac{2(1-b)}{2a+a^2-b^2}\right)^2\exp\left\{\left(2b+2a^2\sigma^2\right)t\right\}\left(1+\frac{1}{3}ct\right)\end{aligned} \qquad (22)$$

Where r_0 is the initial interest rate at time zero.

4.2 Exponential Distribution

Let $H(y)$ be the distribution function of exponential random variable with parameter λ, i.e.

$$H(y)=\begin{cases}0, & y<0\\ 1-e^{-\lambda y}, & y\ge 0\end{cases}$$

Note that

$$m=\frac{1}{\lambda}$$

And

$$\nu^2=\frac{1}{\lambda^2}$$

Now, substituting in equation (17), then the $n-th$ moment of the money return model $M_n(t)=E\left[B^n(t)\right]$ for all $n=1,2,3,\ldots,$ is given by

$$\begin{aligned}M_n(t)=E\left[B^n(t)\right]&\approx 1\\&+nr_0\left(\frac{2(1-b)}{2a+a^2-b^2}\right)\exp\left\{\left(b+\frac{1}{2}a^2\sigma^2\right)t\right\}\left(1-\frac{c}{\lambda}t+\frac{c}{\lambda^2}t\right)\\&+\frac{n^2}{2}r_0^2\left(\frac{2(1-b)}{2a+a^2-b^2}\right)^2\exp\left\{\left(2b+2a^2\sigma^2\right)t\right\}\left(1-\frac{2c}{\lambda}t+\frac{4c}{\lambda^2}t\right)\end{aligned} \qquad (23)$$

where r_0 is the initial interest rate at time zero.

5. Case of No External Effect

This case happens only when the external effect rate $c = 0$ and by substituting in equation (17), we obtain the $n-th$ moment approximation of the money return model $M_n(t) = E[B^n(t)]$ for all $n = 1,2,3,\ldots,$ is as follows

$$M_n(t) = E[B^n(t)] \approx 1 + nr_0\left(\frac{2(1-b)}{2a+a^2-b^2}\right)\exp\left\{\left(b+\frac{1}{2}a^2\sigma^2\right)t\right\} + \frac{n^2}{2}r_0^2\left(\frac{2(1-b)}{2a+a^2-b^2}\right)^2 \exp\{(2b+2a^2\sigma^2)t\} \quad (24)$$

where r_0 is the initial interest rate at time zero.

6. A Numerical Example

Consider as an example the following sample paths of the above money return model $B(t)$ that represents the annual interest rate of an economy in anonymous country when $r(0) = 0.02$, $b = 0.02$, $a = 2$, $n = 20$, and $c = 1$ for the following cases:

Case 1

In this case, we consider the sample path to the stochastic diffusion money return model $B(t)$ with a stochastic birth and death diffusion interest rate $r(t) = r$ (fixed value). Figure 1 and Figure 2 represent this case for $r(t) = 0.02$ and $B(t)$ respectively.

Figure 1: The fixed Annualized Interest Rate

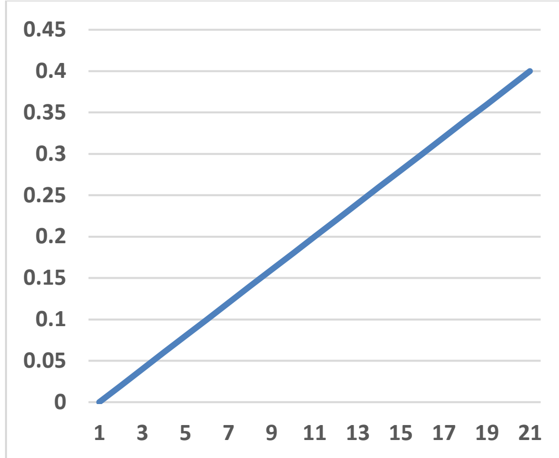
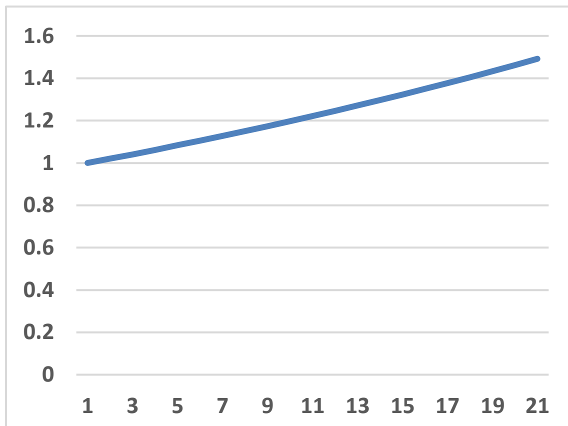


Figure 2: The Associated Annualized Money Return Model



One can note that both figures prove that the return is shaped as a positively increasing and linear function in the related plane.

Case 2

In this case, we consider the sample path to the stochastic diffusion money return model $B(t)$ with a stochastic birth and death diffusion interest rate $r(t)$ and no external jump processes. Note in this case the external jump rate $c = 0$. Figure 3 and Figure 4 represent this case for $r(t)$ and $B(t)$ respectively.

Figure 3: The Stochastic Interest Rate Process with no External Jump Process

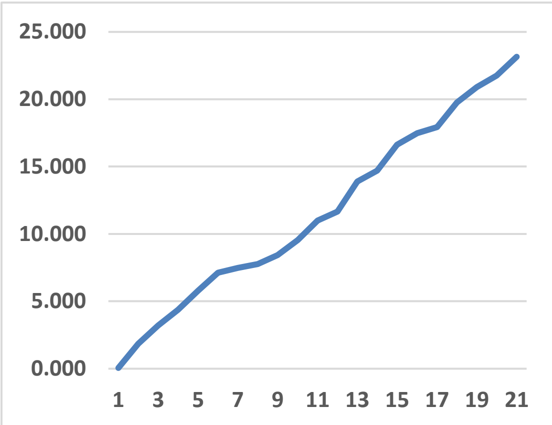
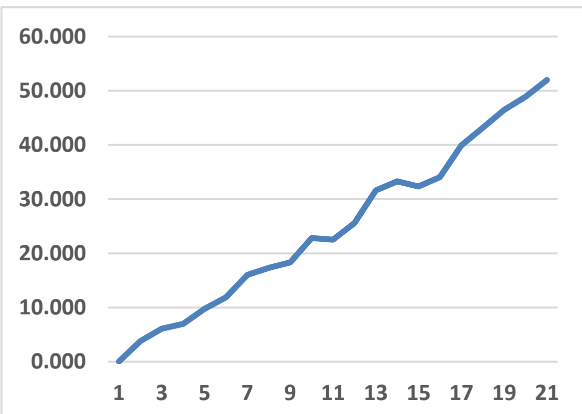


Figure 4: The Associated Stochastic Diffusion Money Return Model



In this example, it is clear that the return process is formed as a positively increasing but non-linear function in the related plane.

Case 3

In this case, we consider the sample path to the stochastic diffusion money return model $B(t)$ with a stochastic birth and death diffusion interest rate $r(t)$ with uniform external jump process. Note in this case the jump rate $c=1$. For simplicity, we take $H(\cdot)$ to be uniform on $[0,1]$. Thus

$$dH(y)=1, \quad 0 \le y \le 1 \qquad (13)$$

Note that $H(y)$ is independent of y with mean $\mu=\frac{1}{2}$, and variance $v^2=\frac{1}{12}$. Figure 5 and Figure 6 represent this case for $r(t)$ and $B(t)$ respectively.

Figure 5: The Stochastic Interest Rate Process with Uniform Jump Process

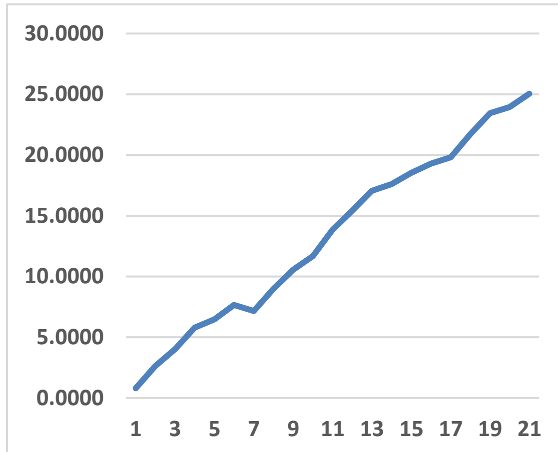
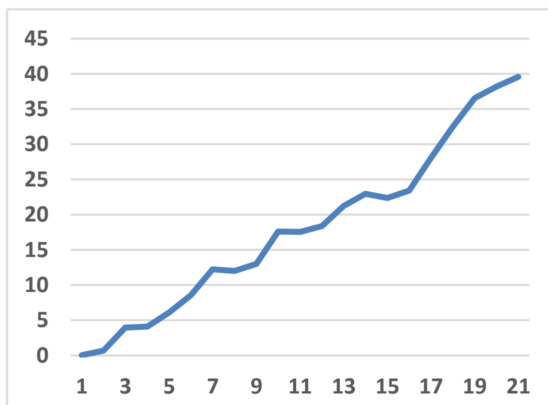


Figure 6: The Associated Stochastic Diffusion Money Return Model.



Case 4

In this case, we consider the sample path to the stochastic diffusion money return model $B(t)$ with a stochastic birth and death diffusion interest rate $r(t)$ with exponential external jump process. Note in this case the jump rate $c=1$. For simplicity, we consider $H(\cdot)$ to be exponential with mean 1. Thus

$$dH(y)=e^{-y}, \quad y>0 \tag{14}$$

Note that $H(y)$ depends on y with mean $\mu=1$, and variance $v^2=1$. Figure 7 and Figure 8 represent this case for $r(t)$ and $B(t)$ respectively.

Figure 7: The Stochastic Interest Rate Process with Exponential Jump Process

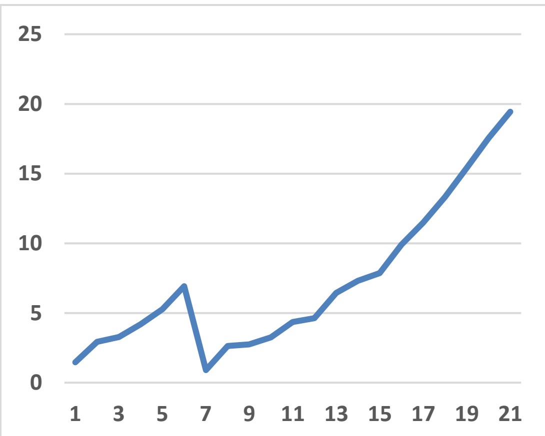
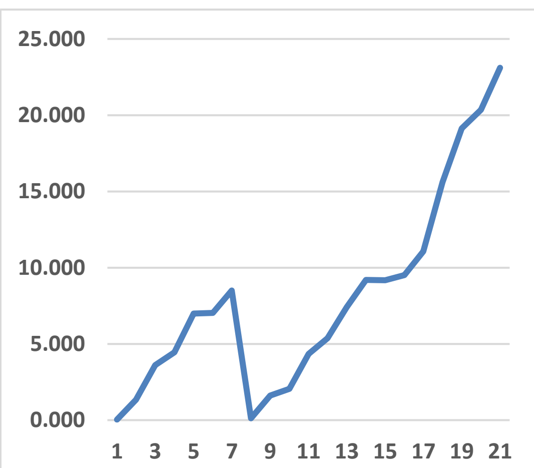


Figure 8: The Associated Stochastic Diffusion Money Return Model



In conclusion, the above figures show the difference between the uniform jump and the exponential jump, which is noted in the stochastic interest rate processes. The figures also show the difference among jump processes if these are dependent or independent of the interest rates. Indeed, these differences affect the parametrization and structure of the stochastic diffusion money return models.

7. Conclusion

In conclusion, this study efficiently provides a methodology for studying and analyzing the behavior of a money return model. More specifically, this study departs from the traditional regression techniques and the time series analysis, and develops a money return model using an interest rate process that explicitly accounts for the variations and volatilities in its values using an interest rate process that follows a birth and death diffusion with general downward external effect processes. The moment approximations of such a process are derived for some external effect distributions of Beta and Exponential, as well as for the case of no external effects. Numerical examples for a sample path of money return processes are also considered for the case of fixed annualized interest rate and for the case of no jumps as well as the case of the occurrence of jump process that follow a uniform and exponential distributions. The employed figures are all reasonable and suggested to be employed in modeling purposes for different money return models.

In addition, some inference problems could be carried out for the employed models. The results should be useful in further empirical tests of estimating moments in diffusion return models as different diffusions impose different constraints on interest rate processes.

In terms of future research, the employed methodology could be applied not only to money return models, but also on all aspects of finance and operations research problems.

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References

Al-Eideh, B. M. (2001). Moment approximations of life table survival diffusion process with general external effect. *Intern. J. of Appl. Math.*, 5 (1), 107-114.

Al-Eideh, B. M. and Al-Hussainan, A. A. (2002). A quasi-stochastic diffusion process of the Lorenz curve. *Intern. Math. J*., 1 (4), 377-383.

Brownen-Trinh, R., (2019). Effects of winsorization: The cases of forecasting non-GAAP and GAAP earnings. *Journal of Business Finance & Accounting*, 46, 105-135.

Carpinteyro, M.; Venegas-Martinez, F. and Aali-Bujari, A. (2021). Article Modeling Precious Metal Returns through Fractional Jump-Diffusion Processes Combined with Markov Regime-Switching Stochastic Volatility. *Mathematics* 407 (9): 1-17.

Daykin, C. D., Pentikainen, T. and Pesonen, M. (1996). *Practical Risk Theory for Actuaries*, Chapman and Hall, USA. Geoghegan, T.J., and, Clarkson, R. S., Feldman, K.S., et al. (1992). *Report on the Wilkie Stochastic Model*. *JIA*, 119. 173-228.

De Simone, L., (2016). Does a common set of accounting standards affect tax-motivated income shifting for multinational firms? *Journal of Accounting and Economics*, 61 (1), 145-165.

Easton, P., Harris, T., (1991). Earnings as an explanatory variable for returns. *Journal of Accounting Research* 29, 19-36

Geoghegan, P., Matison, M. T., Reichle, J. J. and Keppel, R. J. (1992). Influence of salt front position on the occurrence of uncommon marine fishes in the Hudson River estuary. *Estuaries* 15 (2): 251-254

Gihman, I. and Skorohod, A. V. (1974). *The Theory Of Stochastic Processes*. Springer-Verlag, Berlin and New York.

Goh, B. W., Li, D. Jeffrey, Ng and Yong, K. O. (2015). Market Pricing of Banks′ Fair Value Assets Reported Under SFAS 157 Since the 2008 Financial Crisis. *Journal of Accounting and Public Policy*, Vol. 34, No. 2, 129-145.

Gu, S., Kelly, B., and Xiu, D. (2020). Empirical Asset Pricing via Machine Learning. *Review of Financial Studies*, 33(5), 2223 – 2273

Hasan, M. H. and Al-Eideh, B. M. (2002). Modeling Default Rick Using a Stochastic Process approach. *International Mathematical Journal.* Vol.1, No.6, 591-599.

Ibbotson, R.G. and Sinquefield, RV. (1977). Stocks, Bonds, Bills and Inflation: The Past (1926-1976) and Future (1977-2000), Financial Analysts Research Foundation, Charlottesville.

Kothari, S. P. and Zimmerman, J. L. (1995). Price and Return Models. *Journal of Accounting and Economics*, Vol. 20, No. 2, 155-192.

Naeem, M., Tiwari, K., Mubashra, S. and Shahbaz, M. (2019). Modeling Volatility of Precious Metals Markets by Using Regime-Switching GARCH Models. *Resources Policy* 64, 101497.

Simon, S., (2021). International Stock Return Predictability. *International Review of Financial Analysis*, 78. 1057 – 101963

Taylor, H. M. and Karlin, S. (1981). A Second Course in Stochastic Processes. *Academic Press*, USA.

Vallejo-Jiménez, B.; Venegas-Martinez, F. (2017). Optimal Consumption and Portfolio Rules when the Asset Price is driven by a Time-Inhomogeneous Markov Modulated Fractional Brownian Motion with Multiple Poisson Jumps. *Economics Bulletin,* 37, 1 314–326.

Wilkie, A.D. (1984) Steps towards a Comprehensive Stochastic Investment Model. *Occasional Actuarial Research Discussion Paper No. 36*. Institute of Actuaries, London.

Wilkie, A. D. (1986). Stochastic investment model for Actuarial use. *Transactions of the Faculty of Actuaries,* Montreal, 39, 341.

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