On existence and uniqueness of best proximity points for proximal β-quasi contractive mappings on metric spaces

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Abstract. This paper focus on the study of the generalization of the best proximity point theorems for non-self contractions. In fact we propose two new theorems on the existence and the uniqueness of best proximity points for proximal β-quasi contractive mappings on metric spaces . The presented theorems extend and generalize the existence and the uniqueness of best proximity points for proximal contraction done by S.Basha and quasi-contraction mappings performed by Jleli and Samet.

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1. Introduction

The notion of best proximity point was proposed by Fan in [15] for non-self continuous mappings $T : A \to X$ where A is a nonempty compact convex subset of a Hausdorff locally convex topological vector space X. In fact, in metric space (X, d) he proved that there exists x such that $d(x, Tx) = d(Tx, A)$. Various extensions of Fan's theorem were established in the literature by Prolla [18], Reich [19], Singh and Sehgal [20].

In 2010, S. Bacha [6] extended the above definition to a pair of nonempty subsets (A, B) of a metric space (X, d) to introduce further extensions of Banach contraction principle by a best proximity theorem under the hypothesis that B is approximatively compact with respect to A.

Later on, several best proximity point results were derived (see eg. [7]-[9]). Best proximity point theorems for non-self set valued mappings have been obtained in [16] by M. Jleli and B. Samet, in the context of proximal orbital completeness condition which is weaker than the compactness condition. Best proximity point theorems serve for the generalization of fixed point theorems. In fact best proximity point becomes a fixed point in the self-mappings case.

Several extensions of non-self contractions for the existence of a best proximity points were studied in [13, 14, 1]. Moreover in [17, 2, 4] various best proximity theorems for some classes of non-self mappings were established.

Herein, we study the existence and the uniqueness of best proximity points for a novel class of non-self mappings. We show that the results obtained in [6, 16] are particular cases of our main result.

2. Preliminaries and definitions

Let (X, d) be a metric space and (A, B) be a pair of nonempty subsets of X. We consider the following notations:

$d(A, B) := \inf\{d(a, b) : a \in A, b \in B\}$;
$d(x, B) := \inf\{d(x, b) : b \in B\}$;
$\Delta(A) := \sup\{d(a, b) : a, b \in A\}$;
$A_0 := \{a \in A :$ there exists $b \in B$ such that $d(a, b) = d(A, B)\}$;
$B_0 := \{b \in B :$ there exists $a \in A$ such that $d(a, b) = d(A, B)\}$.

Definition 2.1 ([6]). *Let $T : A \to B$ be a mapping. An element $x_* \in A$ is said to be a best proximity point of T if $d(x_*, Tx_*) = d(A, B)$.*

Definition 2.2 ([3]). *Let $\beta \in (0, +\infty)$. A $\beta-$comparison function is a map $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying the following properties:*

(i) *φ is nondecreasing;*

(ii) $\lim_{n\to\infty} \varphi_\beta^n(t) = 0$ *for all $t > 0$, where φ_β^n denote the n^{th}-iterate of φ_β and $\varphi_\beta(t) = \varphi(\beta\, t)$;*

(*iii*) *there exists* $s \in (0, +\infty)$ *such that* $\sum_{n=1}^{\infty} \varphi_\beta^n(s) < \infty$.

(*iv*) $(id - \varphi_\beta) \circ \varphi_\beta(t) \le \varphi_\beta \circ (id - \varphi_\beta)(t)$ *for all* $t \ge 0$, *where* $id : [0, +\infty) \to [0, +\infty)$ *is the identity function.*

The set of all $\beta-$*comparison functions* φ *satisfying (i)–(iv) will be denoted by* Φ_β.

Remark 2.1. Let $\alpha, \beta \in (0, +\infty)$. If $\alpha < \beta$, then $\Phi_\beta \subset \Phi_\alpha$.

We recall some useful lemma concerning the comparison functions Φ_β.

Lemma 2.1 ([3]). *Let* $\beta \in (0, +\infty)$ *and* $\varphi \in \Phi_\beta$. *Then*

(*i*) φ_β *is nondecreasing;*

(*ii*) $\varphi_\beta(t) < t$ *for all* $t > 0$;

(*iii*) $\sum_{n=1}^{\infty} \varphi_\beta^n(t) < \infty$ *for all* $t > 0$;

(*iv*) $(id - \varphi_\beta) \circ \varphi_\beta^n(t) \le \varphi_\beta^n \circ (id - \varphi_\beta)(t)$ *for all* $t \ge 0$ *and* $n \in \mathbb{N}_0$.

Definition 2.3 ([6]). *A mapping* $T : A \to B$ *is said to be a proximal contraction if there exists a nonnegative real number* $\alpha < 1$ *such that*

$$d(u, Tx) + d(Tx, Ty) + d(Ty, v) \le \alpha d(x, y)$$

whenever, x *and* y *are distinct elements in* A *satisfying the condition that* $d(u, Tx) = d(A, B)$ *and* $d(v, Ty) = d(A, B)$ *for some* $u, v \in A$.

Definition 2.4 ([16]). *A non self mapping* $T : A \to B$ *is said to be a proximal quasi-contraction if there exists a number* $q \in [0, 1)$ *such that*

$$d(u, v) \le q \max\{d(x, y), d(x, u), d(y, v), d(x, v), d(y, u)\}$$

whenever, $x, y, u, v \in A$ *satisfying the condition that* $d(u, Tx) = d(A, B)$ *and* $d(v, Ty) = d(A, B)$.

Lemma 2.2 ([16]). *Let* $T : A \to B$ *be a non self mapping. Suppose that the following conditions hold:*

(*i*) $A_0 \ne \emptyset$;

(*ii*) $T(A_0) \subseteq B_0$.

Then, for all $a \in A_0$, *there exists a sequence* $\{x_n\} \subset A_0$ *such that*

$$\begin{cases} x_0 = a \\ d(x_{n+1}, Tx_n) = d(A, B), \quad \forall n \in \mathbb{N}. \end{cases} \tag{1}$$

Definition 2.5 ([16]). *Under the assumptions of Lemma 2.2, any sequence* $\{x_n\} \subset A_0$ *satisfying* (1) *is called a proximal Picard sequence associated to* $a \in A_0$.

For every $a \in A_0$, we denote $PP(a)$ the set of all proximal Picard sequences associated to a. Let $a \in A_0$ and $\{x_n\} \in PP(a)$. For all $(n,k) \in \mathbb{N}^2$, we define the following sets $\mathcal{O}(x_n,k) := \{x_n,\dots,x_{n+k}\}$ and $\mathcal{O}(x_n,\infty) := \{x_k, k \geq n\}$

Definition 2.6 ([16]). *We say that* A_0 *is proximal* T*-orbitally complete if and only if every Cauchy sequence* $\{x_n\} \in PP(x_0)$ *for some* $x_0 \in A_0$*, converges to an element in* A_0.

Definition 2.7. *We say that* B *is approximatively compact with respect to* A *if and only if every sequence* $\{y_n\} \subset B$ *satisfying* $\lim_{n\to+\infty} d(x,y_n) = d(x,B)$ *for some* $x \in A$*, has a convergent subsequence.*

3. Main results and theorems

In [5] the following definition of Proximal β-quasi- contraction was introduced.

Definition 3.1 ([5]). *Let* $\beta \in (0,+\infty)$. *A non-self mapping* $T : A \to B$ *is said to be a proximal* β*-quasi-contraction if and only if there exist* $\varphi \in \Phi_\beta$ *and nonnegative numbers* $\alpha_0,\dots,\alpha_4$ *such that:*

$$d(u,v) \leq \varphi(\max\left\{\alpha_0 d(x,y), \alpha_1 d(x,u), \alpha_2 d(y,v), \alpha_3 d(x,v), \alpha_4 d(y,u)\right\}).$$

For all $x,y,u,v \in A$ *satisfying,* $d(u,Tx) = d(A,B)$ *and* $d(v,Ty) = d(A,B)$.

In the case of self mappings, a proximal β-quasi-contraction is exactly a β-quasi contractive mapping introduced first in [3].

Definition 3.2 ([3]). *Let* (X,d) *be a non empty complete space. A self mapping* $T : X \to X$ *is called* β*-quasi contractive if there exist* $\beta > 0$ *and* $\varphi \in \Phi_\beta$ *such that* $d(Tx,Ty) \leq \varphi(M_T(x,y))$, *where,*

$$M_T(x,y) = \max\left\{\alpha_0 d(x,y), \alpha_1 d(x,Tx), \alpha_2 d(y,Ty), \alpha_3 d(x,Ty), \alpha_4 d(y,Tx)\right\},$$

for all $x,y \in X$ *with* $\alpha_k \geq 0$ *for* $k = 0,1,\cdots,4$.

Remark 3.1. (i) Definition 2.3 [Basha, [6]] follows from definition 3.1 by taking $\beta = \alpha_0 = 1$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ and $\varphi(t) = \alpha t$ for some $0 < \alpha < 1$.

(ii) Definition 2.4 [Samet, [16]] follows from definition 3.1 by taking $\beta = \alpha_0 = \cdots = \alpha_4 = 1$ and $\varphi(t) = qt$ for some $0 < q < 1$.

Our main results are given by the following best proximity point theorem.

Theorem 3.1. *Let (A, B) be a pair of subsets of a metric space (X, d). Let $T : A \to B$ be a given non-self mapping. Suppose that the following conditions hold:*

(i) $A_0 \neq \emptyset$;

(ii) A_0 *is proximal* T*-orbitally complete;*

(iii) $T(A_0) \subseteq B_0$;

(iv) *there exists* $\beta \geq \max_{0 \leq k \leq 4}\{\alpha_k\}$ *such that* T *is a proximal* β*-quasi contraction.*

Furthermore, assume that one of the following conditions holds:

(a) φ *is continuous;*

(b) $\beta > \max\{\alpha_1, \alpha_4\}$;

Then, T *has a unique best proximity point* $x_* \in A_0$.

To prove Theorem 3.1, we need following preliminary lemmas.

Lemma 3.1. *Under the conditions of Theorem 3.1, let* $\{x_n\} \in PP(x_0)$. *Then for* $(s, n) \in \mathbb{N} \times \mathbb{N}^*$ *with* $1 \leq i \leq j \leq n$, *we have*

$$d(x_{s+i}, x_{s+j}) \leq \varphi_\beta(\Delta(\mathcal{O}(x_{s+i-1}, j-i+1))) \leq \varphi_\beta(\Delta(\mathcal{O}(x_s, n))) \tag{2}$$

Proof. Since T is a proximal β-quasi contraction and since $d(x_{s+i}, Tx_{s+i-1}) = d(x_{s+j}, Tx_{s+j-1}) = d(A, B)$ then

$$\begin{aligned} d(x_{s+i}, x_{s+j}) \quad \leq \quad & \varphi(\max\{\alpha_0 d(x_{s+i-1}, x_{s+j-1}), \alpha_1 d(x_{s+i-1}, x_{s+i}), \\ & \alpha_2 d(x_{s+j-1}, x_{s+j}), \alpha_3 d(x_{s+i-1}, x_{s+j}), \alpha_4 d(x_{s+j-1}, x_{s+i})\}). \end{aligned}$$

Using the fact that φ is nondecreasing, we get

$$\begin{aligned} d(x_{s+i}, x_{s+j}) \quad & \leq \quad \varphi(\beta \max\{d(x_{s+i-1}, x_{s+j-1}), d(x_{s+i-1}, x_{s+i}), \\ & \qquad d(x_{s+j-1}, x_{s+j}), d(x_{s+i-1}, x_{s+j}), d(x_{s+j-1}, x_{s+i})\}) \\ & \leq \quad \varphi(\beta\Delta(\mathcal{O}(x_{s+i-1}, j-i+1))) \\ & = \quad \varphi_\beta(\Delta(\mathcal{O}(x_{s+i-1}, j-i+1))) \\ & \leq \quad \varphi_\beta(\Delta(\mathcal{O}(x_s, n))). \end{aligned}$$

□

Lemma 3.2. *Under the conditions of Theorem 3.1, let* $\{x_n\} \in PP(x_0)$. *Then, we have the following two assertions*

(I) *For all* $(s, n) \in \mathbb{N} \times \mathbb{N}^*$, *there exists* $1 \leq j \leq n$ *such that*

$$\Delta(\mathcal{O}(x_s, n)) = d(x_s, x_{s+j}). \tag{3}$$

(II)

$$(Id - \varphi_\beta)(\Delta(\mathcal{O}(x_0, n))) \leq d(x_0, x_1) \tag{4}$$

Proof. Using Lemma 3.1 and the property $\phi_\beta(t) < t$, we obtain that for $(s,n) \in \mathbb{N} \times \mathbb{N}^*$ with $1 \leq i \leq j \leq n$, $d(x_{s+i}, x_{s+j}) \leq \varphi_\beta(\Delta(\mathcal{O}(x_s,n))) < \Delta(\mathcal{O}(x_s,n))$ Thus, $\Delta(\mathcal{O}(x_s,n)) \neq d(x_{s+i}, x_{s+j})$ when both $i,j \geq 1$. Therefore $\Delta(\mathcal{O}(x_s,n)) = d(x_{s+i}, x_{s+j})$ whenever one of i or j is equal to 0. Hence assertion (1) holds. Now, we prove the second assertion. Using the first assertion and Lemma 3.1, we have

$$\Delta(\mathcal{O}(x_0,n)) = d(x_0,x_j), \leq d(x_0,x_1) + d(x_1,x_j) \leq d(x_0,x_1) + \varphi_\beta(\Delta(\mathcal{O}(x_0,n))).$$

This proves the inequality (4). □

Lemma 3.3. *Under the conditions of Theorem 3.1, every sequence* $\{x_n\} \in PP(x_0)$ *is a Cauchy sequence.*

Proof. Let $(m,n) \in \mathbb{N}^2$ with $1 \leq n < m$. Using 2 we have

$$(5) \quad d(x_n,x_m) = d(x_{(n-1)+1}, x_{(n-1)+m-n+1}) \leq \varphi_\beta(\Delta(\mathcal{O}(x_{n-1}, m-n+1))).$$

On the other hand, from (3), we get $\Delta(\mathcal{O}(x_{n-1}, m-n+1)) = d(x_{n-1}, x_{n-1+j})$ for some $j \in \{1, \ldots, m-n+1\}$, and so equation (5) becomes

$$(6) \qquad d(x_n,x_m) \leq \varphi_\beta(d(x_{n-1}, x_{n-1+j})).$$

Using Lemma 3.1 we obtain

$$(7) \quad d(x_{n-1}, x_{n-1+j}) = d(x_{(n-2)+1}, x_{(n-2)+j+1}) \leq \varphi_\beta(\Delta(\mathcal{O}(x_{n-2}, m-n+2))).$$

From (6) and (7), we obtain $d(x_n,x_m) \leq \varphi_\beta^2(\Delta(\mathcal{O}(x_{n-2}, m-n+2)))$. Continuing this process, by induction, we obtain that

$$(8) \qquad d(x_n,x_m) \leq \varphi_\beta^n(\Delta(\mathcal{O}(x_0,m))).$$

Now by Lemma 3.2, we have $(Id - \varphi_\beta)(\Delta(\mathcal{O}(x_0,m))) \leq d(x_0,x_1)$. Applying the non-decreasing map $t \mapsto \sum_{\ell=0}^{p} \varphi_\beta^\ell(t)$ to both side of the previous inequality, we get

$$\sum_{\ell=0}^{p} \varphi_\beta^\ell \circ (id - \varphi_\beta)(\Delta(\mathcal{O}(x_0,m))) \leq \sum_{\ell=0}^{p} \varphi_\beta^\ell(d(x_0,x_1)).$$

Using property (iv) of φ_β, we obtain

$$\sum_{\ell=0}^{p} (id - \varphi_\beta) \circ \varphi_\beta^\ell(\Delta(\mathcal{O}(x_0,m))) \leq \sum_{\ell=0}^{p} \varphi_\beta^\ell(d(x_0,x_1)).$$

Which implies that

$$\Delta(\mathcal{O}(x_0,m)) - \varphi_\beta^{p+1}(\Delta(\mathcal{O}(x_0,m))) \leq \sum_{\ell=0}^{p} \varphi_\beta^\ell(d(x_0,x_1)).$$

Hence, by letting $p \to \infty$ in the above inequality and using (ii) of definition 2.2, we obtain

$$\Delta(\mathcal{O}(x_0, m)) \le \sum_{\ell=0}^{\infty} \varphi_\beta^\ell(d(x_0, x_1)), \quad \text{for all } m \in \mathbb{N}.$$

Taking φ_β^n for both sides of the above inequality and using (8) and the fact that φ_β^n is non-decreasing, we get

$$d(x_n, x_m) \le \varphi_\beta^n(\Delta(\mathcal{O}(x_0, m))) \le \varphi_\beta^n(\sum_{\ell=0}^{\infty} \varphi_\beta^\ell(d(x_0, x_1))).$$

Since, $\varphi_\beta^n(t) \to 0$ as $n \to +\infty$, we deduce that $\{x_n\}$ is a Cauchy sequence. □

Proof of Theorem 3.1. By Lemma 2.2 there is a sequence $\{x_n\} \subset A_0$ such that $d(x_{n+1}, Tx_n) = d(A, B)$ and so by Lemma 3.3 the sequence is Cauchy. Since A_0 is proximal $T-$orbitally complete the sequence $\{x_n\}$ converge to $x^* \in A_0$. Since $T(A_0) \subseteq B_0$ there exists $u \in A_0$ such that $d(u, Tx^*) = d(A, B)$. Using (1) and the fact that T is β-proximal quasi-contraction we get

$$d(u, x_{n+1}) \le \varphi\Big(\max\left\{\begin{array}{c} \alpha_0 d(x^*, x_n), \alpha_1 d(x^*, u), \alpha_2 d(x_n, x_{n+1}), \\ \alpha_3 d(x^*, x_{n+1}), \alpha_4 d(x_n, u) \end{array}\right\}\Big). \tag{9}$$

For simplicity, denote $\rho = d(u, x^*)$ and

$$X_n = \max\left\{\alpha_0 d(x^*, x_n), \alpha_1 d(x^*, u), \alpha_2 d(x_n, x_{n+1}), \alpha_3 d(x^*, x_{n+1}), \alpha_4 d(x_n, u)\right\}.$$

Thus,

$$\lim_{n\to\infty} X_n = \max\{\alpha_1\rho, \alpha_4\rho\} = \max\{\alpha_1, \alpha_4\}\rho. \tag{10}$$

Now, we will prove that $\rho = 0$. Assume that $\rho > 0$. If φ is continuous, then by taking limit of (9) as $n \to \infty$ and using (10) we get

$$\rho \le \varphi(\max\{\alpha_1, \alpha_4\}\rho) \le \varphi(\beta\rho) = \varphi_\beta(\rho) < \rho$$

which is a contradiction. If $\beta > \max\{\alpha_1, \alpha_4\}$, then there exists $\varepsilon > 0$ and $N > 0$ such that for all $n > N$, we have

$$X_n < (\max\{\alpha_1, \alpha_4\} + \varepsilon)\rho \ \text{ and } \ \beta > \max\{\alpha_1, \alpha_4\} + \varepsilon.$$

Therefore,

$$\begin{aligned} d(u, x_{n+1}) &\le \varphi(X_n) \\ &\le \varphi((\max\{\alpha_1, \alpha_4\} + \varepsilon)\rho) = \varphi_\beta(\frac{\max\{\alpha_1, \alpha_4\} + \varepsilon}{\beta}\rho) \\ &< \frac{\max\{\alpha_1, \alpha_4\} + \varepsilon}{\beta}\rho < \rho. \end{aligned}$$

Thus, by letting $n \to \infty$, we get

$$\rho < \frac{\max\{\alpha_1, \alpha_4\} + \varepsilon}{\beta}\rho < \rho,$$

which is a contradiction as well. Therefore, our claim holds. Assume that x^* and y^* are two distinct best proximity points of T on A_0 that is $d(x^*, Tx^*) = d(A, B) = d(y^*, Ty^*)$. Since T is a proximal β-quasi contraction, we obtain the following inequality $d(x^*, y^*) \leq \varphi(\max\{\alpha_0, \alpha_3, \alpha_4\}d(x^*, y^*)) \leq \varphi(\beta d(x^*, y^*)) = \varphi_\beta(d(x^*, y^*)) < d(x^*, y^*)$, which is a contradiction.

Theorem 3.2. *Let (A, B) be a pair of non empty closed subsets of a complete metric space (X, d). Further, suppose that A_0 and B_0 are non empty. Let $T : A \to B$ be a single valued mapping satisfying the following conditions:*

(i) B is approximatively compact with respect to A;

(ii) $T(A_0) \subset B_0$;

(iii) there exists $\beta \geq \max_{0\leq k\leq 4}\{\alpha_k, 2\alpha_4\}$ such that T is a proximal β-quasi contraction.

Moreover, assume that one of the following conditions holds:

(a) φ is continuous;

(b) $\beta > \max\{\alpha_2, \alpha_3\}$;

Then, T has a unique best proximity point $x_* \in A$ such that $d(x_*, Tx_*) = d(A, B)$.

Let $x_0 \in A_0$. As before, we can find $x_{n+1} \in A_0$ such that

$$d(x_{n+1}, Tx_n) = d(A, B).$$

for every positive integer n. If $x_n = x_{n+1}$ are equals for some non negative integer n, then nothing to prove. Thus, we assume that $x_n \neq x_{n+1}$ for every nonnegative integer n. Now we have the following claim

Claim. The sequence $\{x_n\}$ is Cauchy.

Proof of the claim. Since $d(x_{n+1}, Tx_n) = d(A, B)$ and $d(x_n, Tx_{n-1}) = d(A, B)$ and T is a proximal β-quasi contraction, we get

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \varphi(\max\{\alpha_0 d(x_n, x_{n-1}), \alpha_1 d(x_n, x_{n+1}), \alpha_2 d(x_{n-1}, x_n), \\ &\quad \alpha_4 d(x_{n-1}, x_{n+1})\}) \\ &\leq \varphi(\max\{\alpha_0 d(x_n, x_{n-1}), \alpha_1 d(x_n, x_{n+1}), \alpha_2 d(x_{n-1}, x_n), \\ &\quad \alpha_4 d(x_{n-1}, x_n) + \alpha_4 d(x_n, x_{n+1})\}) \\ &\leq \varphi(\max\{\alpha_0 d(x_n, x_{n-1}), \alpha_1 d(x_n, x_{n+1}), \alpha_2 d(x_{n-1}, x_n), \\ &\quad 2\alpha_4 \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}\}) \\ &\leq \varphi(\beta \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}). \end{aligned}$$

If $d(x_n, x_{n-1}) \leq d(x_n, x_{n+1})$, then by Lemma 2.1,

$$d(x_{n+1}, x_n) \leq \varphi(\beta d(x_{n+1}, x_n)) = \varphi_\beta(d(x_{n+1}, x_n)) < d(x_{n+1}, x_n)$$

which is a contradiction. So, for each $n \geq 1$ we have $d(x_{n-1}, x_n) > d(x_{n+1}, x_n)$ and so,

$$d(x_{n+1}, x_n) \quad \leq \quad \varphi_\beta(d(x_n, x_{n-1})), \quad \forall n \geq 1.$$

Then, by induction we obtain that

$$d(x_{n+1}, x_n) \quad \leq \quad \varphi_\beta^n(d(x_0, x_1)), \quad \forall n \geq 1.$$

Now, for $n < m$ and using triangle inequalities, we obtain

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \varphi_\beta^k(d(x_0, x_1)) \leq \sum_{k=1}^{\infty} \varphi_\beta^k(d(x_0, x_1)).$$

By Lemma 2.1, $\sum_{k=1}^{\infty} \varphi_\beta^k(d(x_0, x_1)) < \infty$, so for every $\epsilon > 0$ there exists $N > 0$ such that

$$\sum_{k=n}^{m-1} \varphi_\beta^k(d(x_0, x_1)) < \epsilon \quad \text{for all } m > n > N.$$

Thus, $d(x_n, x_m) < \epsilon$. This imply that $\{x_n\}$ is a Cauchy sequence in X.

To this end, as the space X is complete, the sequence $\{x_n\} \in A$ and converges to some element $x \in A$, (since A is closed). Furthermore,

$$\begin{aligned} d(A,B) \leq d(x,B) &\leq d(x,Tx_n) \leq d(x,x_{n+1}) + d(x_{n+1},Tx_n) \\ &= d(x,x_{n+1}) + d(A,B) \\ &\leq d(x,x_{n+1}) + d(x,B). \end{aligned} \tag{11}$$

Therefore, as $n \to +\infty$ in the above inequalities we get $d(x, Tx_n) \to d(x, B)$ and $d(x, B) = d(A, B)$ which implies that

$$d(x, Tx_n) \to d(x, B) = d(A, B) \tag{12}$$

Using B is approximatively compact with respect to A, it follows that there is a subsequence Tx_{n_k} which converges to an element $y \in B$. Thus, by (12)

$$d(x, y) = \lim_{k\to\infty} d(x, Tx_{n_k}) = d(x, B) = d(A, B) \tag{13}$$

So, $x \in A_0$. Since $T(A_0) \subset B_0$, there is $u \in A$ such that$d(u, Tx) = d(A, B)$. Since, $d(x_{n+1}, Tx_n) = d(A, B) = d(u, Tx)$, then by definition 3.1 we have

$$d(x_{n+1}, u) \leq \varphi\Big(\max\Big\{ \begin{array}{c} \alpha_0 d(x, x_n), \alpha_1 d(x_n, x_{n+1}), \alpha_2 d(x, u), \\ \alpha_3 d(x_n, u), \alpha_4 d(x, x_{n+1}) \end{array} \Big\}\Big) \tag{14}$$

For simplicity, denote $\rho = d(u, x)$. We will prove that $\rho = 0$. Assume, by contradiction that $\rho > 0$. Now, If φ is continuous, then by letting $n \to +\infty$ on the above inequality, we get $\rho \leq \varphi(\max\{\alpha_2, \alpha_3\}\rho) \leq \varphi(\beta\rho) = \varphi_\beta(\rho) < \rho$ which is a contradiction. Now consider $\beta > \max\{\alpha_2, \alpha_3\}$. Let

$$X_n = \max\{\alpha_0 d(x_n, x), \alpha_1 d(x_n, x_{n+1}), \alpha_2 d(x, u), \alpha_3 d(x_n, u), \alpha_4 d(x, x_{n+1}))\}.$$

As $n \to +\infty$ in the above equality, then $X_n \to \max\{\alpha_2, \alpha_3\}\rho$. We show that $\rho = 0$. Suppose $\rho > 0$. Since $\beta > \max\{\alpha_2, \alpha_3\}$, then there exists $\epsilon > 0$ and $N > 0$ such that for all $n > N$

$$X_n < (\max\{\alpha_2, \alpha_3\} + \epsilon)\rho \text{ and } \beta > \max\{\alpha_2, \alpha_3\} + \epsilon.$$

Therefore,

$$\begin{aligned} d(u, x_{n+1}) &\leq \varphi(X_n) \\ &\leq \varphi((\max\{\alpha_2, \alpha_3\} + \epsilon)\rho) = \varphi_\beta(\tfrac{\max\{\alpha_2,\alpha_3\}+\epsilon}{\beta}\rho) \\ &\leq \tfrac{\max\{\alpha_2,\alpha_3\}+\epsilon}{\beta}\rho < \rho. \end{aligned}$$

Thus, by letting $n \to \infty$ we get

$$\rho \leq \frac{\max\{\alpha_2, \alpha_3\} + \epsilon}{\beta}\rho < \rho,$$

which is a contradiction as well. Hence, $u = x$ and therefore $d(x, Tx) = d(A, B)$, which implies that x is a best proximity point for the mapping T. For uniqueness of best proximity point, we proceed in a similar fashion as in the proof of the uniqueness in Theorem 3.1.

4. Consequences

The following result of [16] is a direct consequence of Theorem 3.1 by taking $\beta = \alpha_i = 1, i \in \{0, 1, 2, 3, 4\}$ and $\varphi(t) = qt$.

Corollary 4.1 ([16]). *Let (A, B) a pair subsets of a metric space (X, d). Let $T : A \to B$ be a giving mapping. Suppose that the following conditions hold:*

- *(i) $A_0 \neq \emptyset$;*
- *(ii) A_0 is proximal T-orbitally complete;*
- *(iii) $T(A_0) \subseteq B_0$;*
- *(iv) T is a proximal quasi contraction.*

Then, T has a unique best proximity point $x_* \in A_0$. Moreover, for any $x_0 \in A_0$, any sequence $\{x_n\} \in PP(x_0)$ converges to x_*.

Specializing $\beta = \alpha_i = 1, i \in \{0, 1, 2, 3, 4\}$ and $\varphi(t) = qt$ in Theorem 3.2, we obtain the following result of [16].

Corollary 4.2 ([16]). *Let (A, B) a pair of closed subsets of a complete metric space (X, d). Let $T : A \to B$ be a giving mapping. Suppose that the following conditions hold:*

(i) $A_0 \neq \emptyset$;

(ii) *B is approximatively compact with respect to A;*

(iii) $T(A_0) \subseteq B_0$;

(iv) *T is a proximal quasi contraction.*

Then, T has a unique best proximity point $x_* \in A_0$. Moreover, for any $x_0 \in A_0$, any sequence $\{x_n\} \in PP(x_0)$ converges to x_*.

Taking $\beta = \alpha_0 = 1$, $\alpha_i = 0, i \in \{1, 2, 3, 4\}$ and $\varphi(t) = \alpha t$, in Theorem 3.2 we obtain the following result of [6].

Corollary 4.3 ([6]). *Let X be a complete metric space. Let A and B be nonempty, closed subsets of X such that is approximatively compact with respect to A. Further, suppose that A_0 and B_0 are nonempty. Let $T : A \to B$ be a single valued map satisfying the following conditions:*

(i) *$T(A_0)$ is contained in B_0;*

(ii) *T is a proximal contraction.*

Then, there exists a unique element x in A such that $d(x, Tx) = d(A, B)$.

The preceding results subsume the following fixed point theorem. First, as a consequence of Theorem 3.1, we obtain the following result:

Corollary 4.4. *Let A be a nonempty subsets of a metric space (X, d); Let $T : A \to A$ be a giving mapping. Assume that the following conditions hold:*

(i) *A is proximal T-orbitally complete;*

(ii) *there exists $\beta \geq \max_{0 \leq k \leq 4}\{\alpha_k\}$ such that T is a β-quasi contraction.*

Moreover, assume that one of the following conditions holds:

(a) *φ is continuous;*

(b) $\beta > \max\{\alpha_1, \alpha_4\}$.

Then, T has a unique fixed point in A.

Since, every set is approximatively compact with it self. As a consequence of Theorem 3.2, we obtain the following fixed point theorem

Corollary 4.5. *Let (X, d) be a complete metric space. Let $T : X \to X$ be a giving mapping. Suppose there exists $\beta \geq \max_{0 \leq k \leq 4}\{\alpha_k, 2\alpha_4\}$ such that T is a β-quasi contraction.*

Moreover, assume that one of the following conditions holds:

(a) *φ is continuous;*

(b) *$\beta > \max\{\alpha_2, \alpha_3\}$.*

Then, T has a unique fixed point in X.

Example 4.1. Consider the complete metric space $X = \mathbb{R}$ with the metric $d(x, y) = |x - y|$. Let $A = [0, 2]$ and $B = [3, 5]$. Also, let $T : A \to B$ be defined by $T(x) = 5 - x$. Then, it is easy to see that $d(A, B) = 1$ and $A_0 = \{2\}$, $B_0 = \{3\}$. Thus, $T(A_0) = T(\{2\}) = \{3\} = B_0$. It is clear that A_0 is proximal T-orbitally complete since the only sequence can be formed in A_0 is the constant sequence $x_n = \{2\}$ which is Cauchy sequence and converge to $2 \in A_0$.
Now we shall show that T is proximal β-quasi-contraction mapping with $\phi(t) = \frac{1}{10}t, \beta = 2$ and $\alpha_i = \frac{1}{3}$ for $i = 0, 1, 2, 3, 4$. Note that $\phi(t) = \frac{1}{10}t \in \Phi_2$ since $\phi_\beta t = \phi_2 t = \frac{2}{10}t = \frac{1}{5}t$.

As above the only $x, y, u, v \in A$ such that $d(u, Tx) = d(A, B) = 1 = d(v, Ty)$ is $x = y = u = v = 2 \in A$. Now

$$\begin{aligned} 0 = d(u,v) &= d(2,2) \\ &\leq \frac{2}{10}\max\{\frac{1}{3}d(x,y), \frac{1}{3}d(x,u), \frac{1}{3}d(y,v), \frac{1}{3}d(x,v), \frac{1}{3}d(y,u)\} \\ &= \phi(\max\{\frac{1}{3}d(1,1), \frac{1}{3}d(1,1), \frac{1}{3}d(1,1), \frac{1}{3}d(1,1), \frac{1}{3}d(1,1)\}) \\ &= \phi(\max\{0,0,0,0,0\}) \\ &= 0 \end{aligned}$$

So, T is a proximal β-quasi-contraction mapping with $\phi(t) = \frac{1}{10}t$. We deduce using our Theorem 3.1, that T has a unique best proximity point which is $x_* = 2$ in this example.

Finally, $\phi(t)$ is continuous mappings as well as $\beta > \max_{0 \leq i \leq 4}\{\alpha_i\}$.

$$d(x_*, Tx_*) = d(2, 3) = 1 = d(A, B).$$

5. Conclusion

A novel class of non-self mappings is given in this paper. Under the proximal orbital completeness condition and compactness condition, we established the existence and uniqueness of best proximity points for such mappings. As a consequence of these theorems, we obtained some fixed point results for the self mapping case.

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