Some Fixed and Periodic Point Results for Generalized Contractions on Partial Ordered Metric Space with Applications in Metric Space

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ABSTRACT: In this paper, we introduce new type of Θ-contraction in term of a self-mapping on a metric space to obtain common fixed point results. Some examples are also provided to support the validity of our results and concepts presented herein. As an application of our results, periodic point results for these Θ-contractions in metric spaces are proved.

Key Words: Θ-contraction, Property P, Property Q, Orbit, Orbitally continuous, Fixed point.

Contents

1. Introduction and preliminaries

Let X be a nonempty set and $F, S : X \rightarrow X$ be a self-mapping. A point $x \in X$ is said to be a fixed point of F if $Fx = x$. A point $x \in X$ is a coincidence point (respectively common fixed point) of F and S if $Fx = Sx$ (respectively $Fx = Sx = x$). We denote $C(F, S)$ and $f(F, S)$ to be the set of all coincidence points and the set of all common fixed points of F and S, respectively.

In the theory of fixed point, Banach contraction mapping principle is a simple and powerful result with a wide range of applications, including iterative methods for solving linear, nonlinear, differential, integral, and difference equations (see [2]-[13] and [23,24]). There are several generalizations and extensions of the Banach contraction principle in the existing literature.

Definition 1.1. *Let X be a (nonempty) set. A function $d : X \times X \rightarrow \mathbb{R}^+$ is metric iff, for all $x, y, z \in X$, the following conditions are satisfied:*
(i) $d(x, y) = 0$ iff $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.
The pair (X, d) is called metric space.

Definition 1.2. [10] *Let (X, d) be a metric space and F, S be self-mappings on X. We say that the mapping F is an S-contraction if there is some $k \in (0, 1)$ such that*

$$d(Fx, Fy) \leq kd(Sx, Sy)$$

for all $x, y \in X$.

Definition 1.3. [11] *Let X be a nonempty set and F and S be self-mappings. Then F and S are said to be weakly compatible if they commute at their coincidence point, that is, if $Fx = Sx$, then $FSx = SFx$.*

Jungck [10] gave the concept of S-contraction and established an important generalization of the Banach contraction principle in 1976.

Theorem 1.4. [10] *Let (X, d) be a complete metric space and F, S be self-mappings on X with S also continuous. Then S has a fixed point in X if and only if there exists an S-contraction mapping $F : X \to X$ such that F commutes with S and $S(X) \subseteq F(X)$.*

Very recently, Jleli and Samet [9] introduced a new type of contraction and established some new fixed point theorems for such contraction in the context of generalized metric spaces.

Definition 1.5. *Let $\Theta : (0, \infty) \to (1, \infty)$ be a function satisfying:*

(Θ_1) Θ is nondecreasing;

(Θ_2) for each sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n \to \infty} \Theta(\alpha_n) = 1$ if and only if $\lim_{n \to \infty} (\alpha_n) = 0$;

(Θ_3) there exist $0 < k < 1$ and $l \in (0, \infty]$ such that $\lim_{a \to 0^+} \frac{\Theta(\alpha) - 1}{\alpha^k} = l$.

A mapping $F : X \to X$ is said to be a Θ-contraction if there exist a function Θ satisfying (Θ_1)-(Θ_3) and a constant $\alpha \in (0, 1)$ such that for all $x, y \in X$,

$$d(Fx, Fy) \neq 0 \Longrightarrow \Theta(d(Fx, Fy)) \leq [\Theta(d(x, y))]^\alpha.$$

Theorem 1.6. [9] *Let (X, d) be a complete metric space and $F : X \to X$ be a Θ-contraction. Then F has a unique fixed point.*

To be consistent with Jleli and Samet. [9], we denote by Ω the set of all functions $\Theta : (0, \infty) \to (1, \infty)$ satisfying the above conditions.

Hussain et al. [7] modified and extended the above result and proved the following fixed point theorem for Θ-contractive condition in the setting of complete metric spaces.

Theorem 1.7. [7] *Let (X, d) be a complete metric space and $F : X \to X$ be a self-mapping. If there exist a function $\Theta \in \Omega$ and positive real numbers $\alpha, \beta, \gamma, \delta$ with $0 \leq \alpha + \beta + \gamma + 2\delta < 1$ such that*

$$
\begin{aligned}
\Theta(d(Fx, Fy)) \quad \leq \quad & [\Theta(d(x, y))]^\alpha \cdot [\Theta(d(x, Fx))]^\beta \\
& \cdot [\Theta(d(y, Fy))]^\gamma \cdot [\Theta((d(x, Fy) + d(y, Fx))]^\delta
\end{aligned}
$$

for all $x, y \in X$, then F has a unique fixed point.

Mustafa and Sims [18] introduced the notion of $G-$metric spaces as a generalization of classical metric spaces and obtained some fixed point theorems for mappings satisfying different generalized contractive conditions. Thereafter, the concept of G-metric space has been studied and used to obtain various fixed point theorems by several mathematicians (see ([1,5], [14]-[19], [21] and [22])).

Definition 1.8. *[18]. Let X be a non empty and $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties*

$(G1)$ $G(a, b, c) = 0$ if $a = b = c$,
$(G2)$ $0 < G(a, a, b)$ for all $a, b \in X$ with $a \neq b$,
$(G3)$ $G(a, a, b) \leq G(a, b, c)$ for all $a, b, c \in X$ with $b \neq c$,
$(G4)$ $G(a, b, c) = G(a, c, b) = G(b, c, a) = \cdots$ (symmetry in all three variables),
$(G5)$ $G(a, b, c) \leq G(a, w, w) + G(w, b, c)$ for all $a, b, c, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or, a $G-$metric on X, and the pair (X, G) is called a $G-$metric space.

Thereafter, M.Jaradat et.al. generalized the above results in the setting of G-metric space as follows.

Definition 1.9. *Let (X, G) be a G-metric space, and let $g : X \to X$ be a self mapping. Then g is said to be a $JS-G$-contraction whenever there exist a function $\psi \in \Psi$ and positive real numbers r_1, r_2, r_3, r_4 with $0 \leq r_1 + 3r_2 + r_3 + 2r_4 < 1$ such that*

$$
\begin{aligned}
\psi(G(ga, gb, gc)) \leq \ & [\psi(G(a, b, c))]^{r_1} [\psi(G(a, ga, gc))]^{r_2} [\psi(G(b, gb, gc))]^{r_3} \\
& \times [\psi(G(a, gb, gb) + G(b, ga, ga))]^{r_4},
\end{aligned} \tag{1.1}
$$

for all $a, b, c \in X$.

Theorem 1.10. *[8] Let (X, G) be a complete G-metric space and $g : X \to X$ be a $JS - G$-contraction. Then g has a unique fixed point.*

The following definitions are needed in the proofs of our main results.

Definition 1.11. [2] *Let (X, d) be a metric space and $F : X \to X$ be a self-mapping. The set $O(x) = \{x, Fx, ..., F^n x, ...\}$ is called the orbit of x. A self-mapping F is called orbitally continuous at x^* if $\lim_{n \to \infty} F^n x = x^*$ implies that $\lim_{n \to \infty} F^{n+1} x = Fx^*$. A self-mapping F is orbitally continuous on X if F is orbitally continuous for all $x \in X$.*

Definition 1.12. [2] *Let $(X, \preceq)$ be a partially ordered set. A self-mapping $F : X \to X$ is said to be*

(D_1) a dominating mapping if $x \preceq Fx$;
(D_2) a dominated mapping if $Fx \preceq x$.

Definition 1.13. [4] *Let $(X, \preceq)$ be a partially ordered set. Self-mappings $F, S : X \to X$ are said to be weakly increasing if $Fx \preceq SFx$ and $Sx \preceq FSx$ for all $x \in X$.*

Definition 1.14. [2] *Let X be a nonempty set. Then $(X, d, \preceq)$ is called an ordered metric space if (X, d) is a metric space and $(X, \preceq)$ is a partially ordered set.*

Definition 1.15. [2] *Let $(X, \preceq)$ be a partially ordered set. Then $x, y \in X$ are said to be comparable elements if either $x \preceq y$ or $y \preceq x$ holds true. Moreover, we define $\Delta \subseteq X \times X$ by*

$$\Delta = \{(x, y) \in X \times X : x \preceq y \ or \ y \preceq x\}.$$

Definition 1.16. [2] *An ordered metric space $(X, d, \preceq)$ is said to have the sequential limit comparison property if for every non-decreasing sequence (non-increasing sequence) $\{x_n\}_{n \in \mathbb{N}}$ in X such that $x_n \to x$ implies that $x_n \preceq x$ $(x \preceq x_n)$.*

The aim of this paper is to present new type of generalized Θ-contractions in term of a self-mapping on a metric space to establish common fixed point results.

2. Main results

In this section, we define a Θ-contraction with respect to a self-mapping and establish a common fixed point theorem using the concept of dominating and dominated mappings.

Definition 2.1. *Let (X, d) be a metric space, $\Theta \in \Omega$ and $F, S : X \to X$ be self-mappings. Then F is said to be a Θ-contraction with respect to S if there exists some constant $k \in (0, 1)$ such that*

$$\Theta(d(Fx, Fy)) \leq [\Theta(d(Sx, Sy))]^k \tag{2.1}$$

for all $x, y \in X$ with $Fx \neq Fy$.

Example 2.2. *Let $\Theta : (0, \infty) \to (1, \infty)$ be a function given by $\Theta(t) = e^{\sqrt{t}}$. It is clear that $\Theta \in \Omega$. Suppose that $F : X \to X$ is a Θ-contraction with respect to a self-mapping S on X. From (2.1), we have*

$$e^{\sqrt{d(Fx, Fy)}} \leq [e^{\sqrt{d(Sx, Sy)}}]^k$$

which further implies that

$$d(Fx, Fy) \leq k^2 d(Sx, Sy).$$

Therefore a Θ-contraction F with respect to S reduces to an S-contraction.

Now we give an example of a Θ-contraction F with respect to a self-mapping S on X which is not an S-contraction on X. The idea of this example has been presented in [2].

Example 2.3. *Consider the sequence $\{\tau_n\}$ as follows:*
$\tau_n = 1 + 5 + 9 + ... + (4n - 3) = n(2n - 1)$.
Let $X = \{\tau_n : n \in \mathbb{N}\}$ and $d(x, y) = |x - y|$. Then (X, d) is a complete metric space. Define the mapping $F : X \to X$ by

$$F(\tau_1) = \tau_1, \quad F(\tau_n) = \tau_{n-1}, \quad for \ all \ n > 1$$

and $S : X \to X$ by

$$S(\tau_1) = \tau_1, \quad S(\tau_n) = \tau_{n+1}, \quad \text{for all } n > 1.$$

Now consider

$$\lim_{n \to \infty} \frac{d(F(\tau_n), F(\tau_1))}{d(S(\tau_n), S(\tau_1))} = \lim_{n \to \infty} \frac{\tau_{n-1} - 1}{\tau_{n+1} - 1} = \lim_{n \to \infty} \frac{(n-1)(2n-3) - 1}{(n+1)(2n+1) - 1} = 1.$$

So F is not an S-contraction. Consider the mapping $\Theta : (0, \infty) \to (1, \infty)$ defined by

$$\Theta(t) = e^{\sqrt{te^t}}.$$

We can easily show that $\Theta \in \Omega$. Now to show F is a Θ-contraction with respect to a mapping S we have to prove,

$$e^{\sqrt{d(F(\tau_n), F(\tau_m)) e^{d(F(\tau_n), F(\tau_m))}}} \leq e^{k\sqrt{d(S(\tau_n), S(\tau_m)) e^{d(S(\tau_n), S(\tau_m))}}}$$

for some $k \in (0, 1)$ which is equivalent to prove

$$d(F(\tau_n), F(\tau_m)) e^{d(F(\tau_n), F(\tau_m))} \leq k^2 d(S(\tau_n), S(\tau_m)) e^{d(S(\tau_n), S(\tau_m))}.$$

and so it is enough to show

$$\frac{d(F(\tau_n), F(\tau_m)) e^{d(F(\tau_n), F(\tau_m))}}{d(S(\tau_n), S(\tau_m)) e^{d(S(\tau_n), S(\tau_m))}} \leq k^2$$

for some $k \in (0, 1)$. We will consider two cases to show the above inequality:

Case 1. $n = 1$ and $m \geq 2$, we have

$$
\begin{aligned}
\frac{d(F(\tau_1), F(\tau_m)) e^{d(F(\tau_1), F(\tau_m)) - d(S(\tau_1), S(\tau_m))}}{d(S(\tau_1), S(\tau_m))} &= \frac{d(\tau_1, \tau_{m-1}) e^{d(\tau_1, \tau_{m-1}) - d(\tau_1, \tau_{m+1})}}{d(\tau_1, \tau_{m+1})} \\
&= \frac{2m^2 - 5m + 2}{2m^2 - m - 1} e^{-(4m-3)} \\
&\leq e^{-1}.
\end{aligned}
$$

Case 2. $m > n > 1$, we have

$$
\begin{aligned}
&\frac{d(F(\tau_m), F(\tau_n)) e^{d(F(\tau_m), F(\tau_n)) - d(S(\tau_m), S(\tau_n))}}{d(S(\tau_m), S(\tau_n))} \\
&= \frac{d(\tau_{m-1}, \tau_{n-1}) e^{d(\tau_{m-1}, \tau_{n-1}) - d(\tau_{m+1}, \tau_{n+1})}}{d(\tau_{m+1}, \tau_{n+1})} \\
&= \frac{2m^2 - 5m - 2n^2 + 5n}{2m^2 + 3m - 2n^2 - 3n} e^{(4n-3) + (4n+1) - (4m-3) - (4m+1)} \\
&\qquad\qquad \leq e^{-1}
\end{aligned}
$$

with $k = e^{-\frac{1}{2}}$.

Now we state our main result which is a generalization of the result of [9].

Theorem 2.4. *Let $(X, d, \preceq)$ be a partially ordered metric space and let $F, S : X \to X$ be a self-mappings, F is Θ-contraction with respect to S on Δ, $F(X) \subseteq S(X)$, F is dominating and S is dominated mappings. Then*

(i) *F and S have a coincidence point in X provided that $S(X)$ is complete and has the sequential limit comparison property;*

(ii) *$C(F, S)$ is well-ordered if and only if $C(F, S)$ is a singleton;*

(iii) *F and S have a unique common fixed point if F and S are weakly compatible and $C(F, S)$ is well-ordered.*

Proof: (i) Let $x_0 \in X$ be an arbitrary point. Since the range of S contains the range of F, there exists a point $x_1 \in X$ such that $F(x_0) = S(x_1)$. Since F is a dominating mapping and S is a dominated mapping, we have

$$x_0 \preceq Fx_0 = S(x_1) \preceq x_1.$$

Hence $(x_0, x_1) \in \Delta$. Continuing in this way, for $x_n \in X$, we can obtain $x_{n+1} \in X$ such that

$$x_n \preceq Fx_n = S(x_{n+1}) \preceq x_{n+1}.$$

So we obtain $(x_n, x_{n+1}) \in \Delta$ for every $n \in \mathbb{N} \cup \{0\}$. If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0} = x_{n_0+1}$, then $Fx_{n_0} = Sx_{n_0+1}$ implies that $Fx_{n_0} = Sx_{n_0}$, that is, $x_{n_0} \in C(F, S)$. So we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Since F is a Θ-contraction with respect to S on Δ, it follows from the assumption that

$$
\begin{aligned}
1 \quad &< \quad \Theta(d(Sx_n, Sx_{n+1})) = \Theta(d(Fx_{n-1}, Fx_n)) \le [\Theta(d(Sx_{n-1}, Sx_n))]^k \\
&= \quad [\Theta(d(Fx_{n-2}, Fx_{n-1}))]^k \le [\Theta(d(Sx_{n-2}, Sx_{n-1}))]^{k^2} \\
&\vdots \\
&\le \quad [\Theta(d(Sx_1, Sx_2))]^{k^{n-1}}.
\end{aligned}
$$

Thus we have

$$1 < \Theta(d(Sx_n, Sx_{n+1})) \le [\Theta(d(Sx_1, Sx_2))]^{k^{n-1}}$$

for all $n \in \mathbb{N}$. So by taking limit as $n \to \infty$ in the above inequality, we have

$$\lim_{n \to \infty} \Theta(d(Sx_n, Sx_{n+1})) = 1$$

which implies that

$$\lim_{n \to \infty} d(Sx_n, Sx_{n+1}) = 0$$

by (Θ_2).

From the condition (Θ_3), there exist $0 < k < 1$ and $l \in (0, \infty]$ such that

$$\lim_{n \to \infty} \frac{\Theta(d(Sx_n, Sx_{n+1})) - 1}{d(Sx_n, Sx_{n+1})^k} = l.$$

Suppose that $l < \infty$. In this case, let $B = \frac{l}{2} > 0$. From the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that

$$\left| \frac{\Theta(d(Sx_n, Sx_{n+1})) - 1}{d(Sx_n, Sx_{n+1})^k} - l \right| \leq B$$

for all $n > n_1$. This implies that

$$\frac{\Theta(d(Sx_n, Sx_{n+1})) - 1}{d(Sx_n, Sx_{n+1})^k} \geq l - B = \frac{l}{2} = B$$

for all $n > n_1$. Then

$$nd(Sx_n, Sx_{n+1})^k \leq An[\Theta(d(Sx_n, Sx_{n+1})) - 1]$$

for all $n > n_1$, where $A = \frac{1}{B}$. Now we suppose that $l = \infty$. Let $B > 0$ be an arbitrary positive real number. From the definition of the limit, there exists $n_2 \in \mathbb{N}$ such that

$$B \leq \frac{\Theta(d(Sx_n, Sx_{n+1})) - 1}{d(Sx_n, Sx_{n+1})^k}$$

for all $n > n_2$. This implies that

$$nd(Sx_n, Sx_{n+1})^k \leq An[\Theta(d(Sx_n, Sx_{n+1})) - 1]$$

for all $n > n_2$, where $A = \frac{1}{B}$. Thus, in all cases, there exist $A > 0$ and $n_3 \in \mathbb{N}$ such that

$$nd(Sx_n, Sx_{n+1})^k \leq An[\Theta(d(Sx_n, Sx_{n+1})) - 1] \tag{2.2}$$

for all $n > n_3$. Thus by (2.2) and being $\Theta(d(Sx_n, Sx_{n+1})) \leq [\Theta(d(Sx_1, Sx_2))]^{k^{n-1}}$, we obtain

$$nd(Sx_n, Sx_{n+1})^k \leq An([(\Theta d(Sx_1, Sx_2))]^{k^{n-1}} - 1).$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n \to \infty} nd(Sx_n, Sx_{n+1})^k = 0.$$

Thus there exists $n_4 \in \mathbb{N}$ such that

$$d(Sx_n, Sx_{n+1}) \leq \frac{1}{n^{1/k}}$$

for all $n > n_4$.

Now we prove that $\{Sx_n\}$ is a Cauchy sequence. For $m > n > n_4$, we have

$$d(Sx_n, Sx_m) \leq \sum_{i=n}^{m-1} d(Sx_i, Sx_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/k}}.$$

Since $0 < k < 1$, $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ converges. Therefore, $d(Sx_n, Sx_m) \to 0$ as $m, n \to \infty$. Thus $\{Sx_n\}$ is a Cauchy sequence in $S(X)$. The completeness of $S(X)$ ensures that there exists $v \in S(X)$ such that $\lim_{n\to\infty} Sx_n \to v$. Let $u \in X$ be such that $Su = v$. The sequential limit comparison property implies that $Sx_{n+1} \preceq v$. Since

$$x_n \preceq Fx_n = Sx_{n+1} \preceq v = Su \preceq u,$$

then $(x_n, u) \in \Delta$. So from (2.1), we get

$$1 < \Theta(d(Sx_n, Fu)) = \Theta(d(Fx_{n-1}, Fu)) \leq [\Theta(d(Sx_{n-1}, Su))]^k.$$

Since $\lim_{n\to\infty} d(Sx_{n-1}, Su) = 0$, by (Θ_2), we have $\lim_{n\to\infty} \Theta(d(Sx_{n-1}, Su)) = 1$. This implies that

$$\lim_{n\to\infty} \Theta(d(Sx_n, Fu)) = 1,$$

which further implies that $\lim_{n\to\infty} d(Sx_n, Fu) = 0$. Hence $\lim_{n\to\infty} Sx_n = Fu$. The uniqueness of limit implies that $Fu = Su$, that is, $u \in C(F, S)$.

(ii) Now we suppose that $C(F, S)$ is well-ordered. We prove that $C(F, S)$ is a singleton. Assume on the contrary that there exists another point $w \in X$ such that $Fw = Sw$. Since $C(F, S)$ is well-ordered, $(u, w) \in \Delta$. Now from (2.1) we have

$$1 < \Theta(d(Su, Sw)) = \Theta(d(Fu, Fw)) \leq [\Theta(d(Su, Sw))]^k < \Theta(d(Su, Sw)).$$

which is a contradiction to the fact that $k \in (0, 1)$. Therefore $u = w$. Hence F and S have a unique coincidence point u in X. The converse follows immediately.

(iii) Now if F and S are weakly compatible mappings, then we have $Fv = FSu = SFu = Sv$, that is, v is the coincidence point of F and S. But v is the only point of coincidence of F and S and so $Fv = Sv = v$. Hence v is the unique common fixed point of F and S. $\qquad\square$

Example 2.5. *Let $X = [0, 6]$ be endowed with usual metric and usual order. Define the mappings $F, S : X \to X$ by*

$$F(x) = \begin{cases} 4 & \text{if } x \in [0, 4) \\ 6 & \text{if } x \in [4, 6] \end{cases}$$

and

$$S(x) = \begin{cases} 0 & \text{if } x \in [0, 4) \\ 4 & \text{if } x \in [4, 6) \\ 6 & \text{if } x = 6. \end{cases}$$

Clearly, F is a dominating mapping and S is a dominated mapping. Define the mapping $\Theta : (0, \infty) \to (1, \infty)$ by

$$\Theta(t) = e^{\sqrt{t}}.$$

If $x \in [0, 4)$ and $y \in [4, 6)$, then

$$\begin{aligned} \Theta(d(Fx, Fy)) &= \Theta(d(4, 6)) = e^{\sqrt{d(4,6)}} = e^{\sqrt{2}} = e^{\sqrt[4]{4}} \\ &= [\Theta(d(Sx, Sy))]^k \end{aligned}$$

with $k = \frac{1}{2}$.

Similarly, for $x \in [0, 4)$ and $y = 6$, we have

$$\begin{aligned} \Theta(d(Fx, Fy)) &= \Theta(d(4, 6)) = e^{\sqrt{d(4,6)}} = e^{\sqrt{2}} < e^{\sqrt[4]{6}} \\ &= [\Theta(d(Sx, Sy))]^k \end{aligned}$$

with $k = \frac{1}{2}$. Hence all the conditions of Theorem 2.4 are satisfied. Moreover, $x = 6$ is the coincidence point of F and S. Also F and S are weakly compatible and $x = 6$ is the common fixed point of F and S as well.

Now we prove a common fixed point theorem without imposing any type of commutativity condition for self-mappings $F, S : X \to X$. Moreover, we relax the dominance conditions on F and S as well.

Theorem 2.6. *Let $(X, d, \preceq)$ be a partially ordered complete metric space and $F, S : X \to X$ be weakly increasing mappings satisfying*

$$\Theta(d(Fx, Sy)) \leq [\Theta(d(x, y))]^k \tag{2.3}$$

for all $(x, y) \in \Delta$. Then the set of common fixed points of F and S, $f(F, S)$, is nonempty provided that X has the sequential limit comparison property. Furthermore, F and S have a unique common fixed point if and only if the set of fixed points of F and S is well-ordered.

Proof: Let $x_0 \in X$ be an arbitrary point. Define the sequence $\{x_n\}_{n \in \mathbb{N}}$ by $x_{2n+1} = Fx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$. Since F and S are weakly increasing, we have

$$x_{2n+1} = Fx_{2n} \preceq SFx_{2n} = Sx_{2n+1} = x_{2n+2}$$

and

$$x_{2n+2} = Sx_{2n+1} \preceq FSx_{2n+1} = Fx_{2n+2} = x_{2n+3}.$$

Hence $(x_{2n+1}, x_{2n+2}) \in \Delta$ and $(x_{2n+2}, x_{2n+3}) \in \Delta$ for every $n \in \mathbb{N} \cup \{0\}$. It follows from the assumption that

$$1 < \Theta(d(x_{2n+1}, x_{2n+2})) = \Theta(d(Fx_{2n}, Sx_{2n+1})) \leq [\Theta(d(x_{2n}, x_{2n+1}))]^k.$$

Similarly,

$$\begin{aligned} 1 &< \Theta(d(x_{2n+2}, x_{2n+3})) = \Theta(d(Sx_{2n+1}, Fx_{2n+2})) = \Theta(d(Fx_{2n+2}, Sx_{2n+1})) \\ &\leq [\Theta(d(x_{2n+2}, x_{2n+1}))]^k = [\Theta(d(x_{2n+1}, x_{2n+2}))]^k. \end{aligned}$$

Thus for all $n \in \mathbb{N} \cup \{0\}$, we have

$$1 < \Theta(d(x_n, x_{n+1})) \leq [\Theta(d(x_{n-1}, x_n))]^k.$$

which yield

$$1 < \Theta(d(x_n, x_{n+1})) \leq [\Theta(d(x_1, x_2))]^{k^n}. \tag{2.4}$$

So by taking limit as $n \to \infty$ in (2.4), we have

$$\lim_{n \to \infty} \Theta(d(x_n, x_{n+1})) = 1,$$

which implies by the property of (Θ_2) that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0$$

.

By the same reasoning as in the proof of Theorem 2.4, we obtain that $\{x_n\}$ is a Cauchy sequence in X. The completeness of X ensures that there exists $v \in X$ such that $\lim_{n \to \infty} x_n = x^*$. Since X has the sequential limit comparison property, $(x_n, x^*), (x_{2n}, x^*), (x_{2n+1}, x^*) \in \Delta$. Therefore from (2.3), we get

$$1 < \Theta(d(x_{2n+1}, Sx^*)) = \Theta(d(Fx_{2n}, Sx^*)) \leq [\Theta(d(x_{2n}, x^*))]^k.$$

Since $\lim_{n \to \infty} d(x_{2n}, x^*) = 0$, by (Θ_2), we have $\lim_{n \to \infty} \Theta(d(x_{2n}, x^*)) = 1$. This implies that

$$\lim_{n \to \infty} \Theta(d(x_{2n+1}, Sx^*)) = 1,$$

which further implies that

$$\lim_{n \to \infty} d(x_{2n+1}, Sx^*) = 0.$$

Hence $x^* = \lim_{n \to \infty} x_{2n+1} = Sx^*$.

Similarly, from (2.3), we get

$$\begin{aligned}
1 < \Theta(d(x_{2n+2}, Fx^*) \quad) = & \quad \Theta(d(Sx_{2n+1}, Fx^*)) = \Theta(d(Fx^*, Sx_{2n+1})) \\
\leq & \quad [\Theta(d(x^*, x_{2n+1}))]^k.
\end{aligned}$$

Since $\lim_{n \to \infty} d(x^*, x_{2n+1}) = 0$, by (Θ_2), we have $\lim_{n \to \infty} \Theta(d(x^*, x_{2n+1})) = 1$. This implies that

$$\lim_{n \to \infty} \Theta(d(x_{2n+2}, Fx^*)) = 1,$$

which further implies that $d(x_{2n+2}, Fx^*) = 0$. Hence $d(x^*, Fx^*) = 0$ and $x^* = Fx^*$. Thus x^* is a common fixed point of F and S.

Now we suppose that the set of common fixed points of F and S is well-ordered. We prove that this set is a singleton. Assume on the contrary that there exists another point x' in X such that $Fx' = x' = Sx'$ with $x' \neq x^*$. From the assumption we have, $(x^*, x') \in \Delta$. So, from (2.3), we have

$$1 < \Theta(d(x^*, x')) = \Theta(d(Fx^*, Sx')) \leq [\Theta(d(x^*, x'))]^k,$$

which is a contradiction since $k < 1$. Thus $x' = x^*$. Hence F and S have a unique common fixed point x^* in X.

The converse follows immediately. $\square$

3. Periodic point theorems

The aim of this section is to use the properties P and Q of self-mappings to prove some periodic point results in the conext of metric spaces.

If x is a fixed point of a self-mapping F, then x is a fixed point of F^n for every $n \in \mathbb{N}$, but the converse is not true. In the sequel, we denote by $f(F)$ the set of all fixed points of F.

Definition 3.1. *A self-mapping* $F : X \to X$ *is said to have the property* P *if* $f(F^n) = f(F)$ *for every* $n \in \mathbb{N}$. *A pair* (F, S) *of self-mappings is said to have the property* Q *if*

$$f(F) \cap f(S) = f(F^n) \cap f(S^n)$$

for every $n \in \mathbb{N}$.

Theorem 3.2. *Let* $(X, d, \preceq)$ *be a partially ordered complete metric space and* $F : X \to X$ *be a self-mapping satisfying*

$$\Theta(d(Fx, F^2 x)) \leq [\Theta(d(x, Fx))]^k \tag{3.1}$$

for all $x \in X$ *for some* $k \in (0, 1)$ *such that* $d(Fx, F^2 x) > 0$. *Then* F *has fixed point, further* F *has the property* P *provided that* F *is orbitally continuous on* X.

Proof:

Step I: we will prove that F has fixed point. Let $x_0 \in X$ be an arbitrary point. Define the sequence $\{x_n\}_{n \in \mathbb{N}}$ by $x_{n+1} = Fx_n$ for every $n \in \mathbb{N} \cup \{0\}$. If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0} = x_{n_0+1}$, then $x_{n_0} = Fx_{n_0}$ and the proof is finished. So we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. From (3.1), we obtain

$$
\begin{aligned}
1 \quad < \quad & \Theta(d(x_n, x_{n+1})) = \Theta(d(Fx_{n-1}, F^2 x_{n-1})) \leq [\Theta(d(x_{n-1}, Fx_{n-1}))]^k \\
= \quad & [\Theta(d(Fx_{n-2}, F^2 x_{n-2}))]^k \leq [\Theta(d(x_{n-2}, Fx_{n-2}))]^{k^2} \\
\vdots \quad & \\
\leq \quad & [\Theta(d(x_0, Fx_0))]^{k^n}
\end{aligned}
\tag{3.2}
$$

for all $n \in \mathbb{N} \cup \{0\}$. By taking the limit as $n \to \infty$ in the above inequality, we obtain that

$$\lim_{n \to \infty} \Theta(d(x_n, x_{n+1})) = 1,$$

which implies by property of (Θ_2) that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0$$

.

By the same reasoning as in the proof of Theorem 2.4, we obtain that $\{F^n x_0\}$ is a Cauchy sequence in X. Since $\{F^n x_0 : n \in \mathbb{N}\} \subseteq O(x_0) \subseteq X$ and X is complete, there exists x^* in X such that $\lim_{n \to \infty} F^n x_0 = x^*$. Since F is orbitally continuous

at x^*, $x^* = \lim_{n \to \infty} F^n x_0 = F(\lim_{n \to \infty} F^{n-1} x_0) = Fx^*$. Hence F has a fixed point.

Step II: Now we prove $f(F^n) = f(F)$ by induction. For $n = 1$ clearly it is true, now we assume that $n > 1$. Suppose on the contrary that $x^{'} \in f(F^n)$ but $x^{'} \notin f(F)$. Then $d(x^{'}, Fx^{'}) > 0$. From (3.1), we have

$$
\begin{aligned}
1 \quad &< \quad \Theta(d(x^{'}, Fx^{'})) = \Theta(d(F^n x^{'}, F^{n+1} x^{'})) = \Theta(d(F(F^{n-1} x^{'}), F^2(F^{n-1} x^{'}))) \\
&\leq \quad [\Theta(d(F^{n-1} x^{'}, F(F^{n-1} x^{'})))]^k \\
&\leq \quad [\Theta(d(F^{n-2} x^{'}, F(F^{n-2} x^{'})))]^{k^2} \\
&\vdots \\
&\leq \quad [\Theta(d(x^{'}, Fx^{'}))]^{k^n}.
\end{aligned}
$$

By taking the limit as $n \to \infty$ in the above inequality, we obtain that

$$
\lim_{n \to \infty} \Theta(d(x^{'}, Fx^{'})) = 1,
$$

which implies that

$$
\lim_{n \to \infty} d(x^{'}, Fx^{'}) = 0
$$

by (Θ_2). That is, $d(x^{'}, Fx^{'}) = 0$, which is a contradiction. Hence $x^{'} \in f(F)$. $\square$

Theorem 3.3. *Let $(X, d, \preceq)$ be a partially ordered complete metric space and $F, S : X \to X$ be weakly increasing mappings satisfying*

$$
\Theta(d(Fx, Sy)) \leq [\Theta(d(x, y))]^k
$$

for all $x, y \in X$. Then F and S have the property Q provided that X has the sequential limit comparison property.

Proof: By Theorem (2.6), F and S have a common fixed point. Suppose on the contrary that $x^* \in f(F^n) \cap f(S^n)$ but $x^* \notin f(F) \cap f(S)$. Then we have the following three possibilites:
(i) $x^* \in f(F) \backslash f(S)$;
(ii) $x^* \in f(S) \backslash f(F)$;
(iii) $x^* \notin f(F)$ and $x^* \notin f(S)$.

Now we will show case (i) and (iii), while case (ii) is similar argument. Let $x^* \notin f(S)$, that is, $d(x^*, Sx^*) > 0$. So we have

$$
\begin{aligned}
1 \quad &< \quad \Theta(d(x^*, Sx^*)) = \Theta(d(F(F^{n-1} x^*), S(S^n x^*))) \\
&\leq \quad [\Theta(d(F^{n-1} x^*, S^n x^*))]^k \\
&\leq \quad [\Theta(d(F^{n-2} x^*, S^{n-1} x^*))]^{k^2} \\
&\vdots \\
&\leq \quad [\Theta(d(x^*, Sx^*))]^{k^n}.
\end{aligned}
$$

By taking the limit as $n \to \infty$ in the above inequality, we obtain that

$$\lim_{n \to \infty} \Theta(d(x^*, Sx^*)) = 1,$$

which implies that

$$\lim_{n \to \infty} d(x^*, Sx^*) = 0$$

by (Θ_2). That is, $d(x^*, Sx^*) = 0$, which is a contradiction. Hence $x^* \in f(F) \cap f(S)$.

$\square$

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