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Extremal Number of Theta Graphs of Order 7

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abstract: For a set of graphs $\mathcal{F}$, let $\mathcal{H}(n;\mathcal{F})$ denote the class of non-bipartite Hamiltonian graphs on n vertices that does not contain any graph of $\mathcal{F}$ as a subgraph and $h(n;\mathcal{F}) = \max\{\mathcal{E}(G) : G \in \mathcal{H}(n;\mathcal{F})\}$ where $\mathcal{E}(G)$ is the number of edges in G. In this paper we determine $h(n;\{\theta_4,\theta_5,\theta_7\})$ and $h(n;\theta_7)$ for sufficiently odd large n. Our result confirms the conjecture made in [1] for $k = 3$.

Key Words: Tuŕan number, Theta graph, Extremal graph.

Contents

1 Introduction and preliminaries 21

2 Main results 23

1. Introduction and preliminaries

For our purposes a graph G is finite, undirected and has no loops or multiple edges. We denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. The cardinalities of these sets are denoted by $v(G)$ and $\mathcal{E}(G)$, respectively. The cycle on n vertices is denoted by C_n. A theta graph θ_n is defined to be a cycle C_n to which we add a new edge that joins two non-adjacent vertices. The neighbor set of a vertex u of G in a subgraph H of G, denoted by $N_H(u)$, consists of the vertices of H adjacent to u. The joint $G_1 \vee G_2$ of two vertex disjoint graphs G_1 and G_2 is the graph whose vertex set $V(G_1)\cup V(G_2)$ and edge set consists of $E(G_1)\cup E(G_2)$ together with all the edges joining $V(G_1)$ and $V(G_2)$. For vertex disjoint subgraphs H_1 and H_2 of G, we let $E(H_1,H_2) = \{xy \in E(G) : x \in V(H_1), y \in V(H_2)\}$ and $\mathcal{E}(H_1,H_2) = |E(H_1,H_2)|$.

For a proper subgraph H of G we write $G[V(H)]$ and $G-V(H)$ simply as $G[H]$ and $G - H$, respectively ($G[V(H)]$ is the induced subgraph). In this paper, we consider the Turán-type extremal problem with the θ-graph being the forbidden subgraph. Since a bipartite graph contains no odd θ-graph, we consider non-bipartite graphs. First, we recall some notation and terminology. For a positive integer n and a set of graphs $\mathcal{F}$, let $\mathcal{G}(n;\mathcal{F})$ (and $\mathcal{H}(n;\mathcal{F})$) denote the class of non-bipartite $\mathcal{F}$-free graphs (class of non-bipartite Hamiltonian $\mathcal{F}$-free graphs) on n vertices, and

$$f(n;\mathcal{F}) = \max\{\mathcal{E}(G) : G \in \mathcal{G}(n;\mathcal{F})\},$$
$$h(n;\mathcal{F}) = \max\{\mathcal{E}(G) : G \in \mathcal{H}(n;\mathcal{F})\}.$$

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An important problem in extremal graph theory is that of determining the values of the functions $f(n;\mathcal{F})$ and $h(n;\mathcal{F})$. Further, characterize the extremal graphs of $\mathcal{G}(n;\mathcal{F})$ and $\mathcal{H}(n;\mathcal{F})$ where $f(n;\mathcal{F})$ and $h(n;\mathcal{F})$ are attained. For a given C_r , the edge maximal graphs of $\mathcal{G}(n;C_r)$ have been studied by a number of authors see [6], [7], [8] and [10]. Bondy [5] proved that a Hamiltonian graph G on n vertices without a cycle of length r has at most $\frac{1}{2}n^2$ edges with equality holding if and only if n is even and r is odd.

Höggkvist, Faudree and Schelp [9] proved that $f(n;C_r) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1$ for all r. This result is sharp only for $r = 3$. Jia [12] proved that for $n \geq 9$, $f(n;C_5) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$ and he characterized the extremal graphs as well. In the same work, Jia conjectured that $f(n;C_{2k+1}) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$ for $n \geq 4k+2$. Bataineh [1] confirmed positively the above conjecture for $n \geq 36k$. Further, he showed that equality holds if and only if $G \in \mathcal{G}^*(n)$ where $\mathcal{G}^*(n)$ is the class of graphs obtained by adding a triangle, two vertices of which are new, to the complete bipartite graph $K_{\lfloor (n-2)/2 \rfloor, \lceil (n-2)/2 \rceil}$. Also, he proved the following result:

Theorem 1.1. *(Bataineh [1]) For positive integers $k \geq$ and $n > (4k+2)(4k^2+10k)$,*

$$h(n;C_{2k+1}) = \begin{cases} \frac{(n-2k+1)^2}{4} + 4k - 3, & \text{if } n \text{ is odd} \\ \frac{(n-2k)^2}{4} + 4k + 1, & \text{if } n \text{ is even.} \end{cases}$$

For θ_5-graph, Bataineh et al [2] proved that for $n \geq 5$

$$f(n;\theta_5) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1.$$

Later on, Bataineh et al [3], [4] and Jaradat et al [11] proved the following results

Theorem 1.2. *(Jaradat et al [11]) For positive integers n and k, let G be a graph on $n \geq 6k+3$ vertices which contains no θ_{2k+1} as a subgraph, then*

$$\mathcal{E}(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Theorem 1.3. *(Jaradat et al [11] and Bataineh et al [4]) For sufficiently large integer n and for $k \geq 3$,*

$$f(n;\theta_{2k+1}) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3.$$

Caccetta and Jia [7] constructed the following class of graphs: The building blocks of this class are the path $P = u_1u_2\ldots u_{2k}$ and the complete bipartite graph $B = K_{\lceil \frac{1}{2}(n-2k) \rceil, \lfloor \frac{1}{2}(n-2k) \rfloor}$. For $1 \leq a \leq \lceil \frac{1}{2}(n-2k) \rceil - 1$, we let $\mathbb{B}(n,k,a)$ denote the class of graphs obtained by partitioning the $\lceil \frac{1}{2}(n-2k) \rceil$ vertices of the larger bipartitioning set of B into two sets V_1 and V_2 with $|V_1| = a$ and then joining each vertex of V_1 to u_1 and each vertex of V_2 to u_{2k}. Observe that for a graph $G \in \mathbb{B}(n, k, a)$

$$\mathcal{E}(G) = \lfloor \frac{1}{4}(n-2k+1)^2 \rfloor + 2k - 1.$$

Further, $G \in \mathcal{G}(n; C_3, C_5, \ldots, C_{2k+1})$. Caccetta and Jia [7] proved the following results:

Theorem 1.4. *(Caccetta and Jia [7]) Let $G \in \mathcal{G}(n; C_3, C_5, \ldots, C_{2k+1})$. Then*

$$\mathcal{E}(G) \leq \lfloor \frac{1}{4}(n-2k+1)^2 \rfloor + 2k - 1,$$

with equality possible if and only if $G \in \mathbb{B}(n, k, a)$.

Theorem 1.5. *(Caccetta and Jia [7]) Let $\mathcal{F}_k = \{C_3, C_5, C_7, \ldots, C_{2k+1}\}$. For even $n \geq 4k+4, k \geq 2$, we have*

$$h(n; F_k) = \frac{(n-4k-4)^2}{4} + 8k - 11.$$

Analoguely, In [1], Bataineh proved the following result concerning theta graphs:

Theorem 1.6. *(Bataineh [1])) Let $\Theta_k = \{\theta_4\} \cup \{\theta_5, \theta_7, \ldots, \theta_{2k+1}\}$, then for $k \geq 5$ and large odd n, we have*

$$h(n; \Theta_k) = \frac{(n-2k+3)^2}{4} + 2k - 3.$$

Bataineh [1] made the following conjecture

Conjecture 1. *Let $k \geq 3$ be a positive integer. For odd $n \geq 4k+4$, $h(n; \theta_{2k+1}) \leq \frac{(n-2k+3)^2}{4} + 2k - 3$.*

In this work, we prove the above conjecture for $k = 3$. In fact, we present exact values of $h(n; \mathcal{F})$ for sufficiently large odd n for $\mathcal{F} = \{\theta_4, \theta_5, \theta_7\}$ and $\mathcal{F} = \{\theta_7\}$.

2. Main results

We start this section by the following lemmas which will play a crucial role in proving our main results.

Lemma 2.1. *Let $H \in H(n, \{\theta_4, \theta_5, \theta_7\})$ and H contains a cycle C of length 7. If $u \in V(H-C)$, then $\mathcal{E}(u, C) \leq 3$. Moreover, if $B = \{u \in V(H-C) : \mathcal{E}(u, C) = 3\}$, then $|B| \leq 1$.*

Proof: Let $C = x_1x_2x_3 \dots x_7x_1$ be a cycle of length 7. Since H contains no θ_7 as a subgraph, so $H[C] = C$ and so $\mathcal{E}(H[C]) = 7$. If $u \in V(G-H)$ such that $\mathcal{E}(u, C) = 4$, then with out loss of generality one can easily check that $N_C(u) = \{x_1, x_2, x_3, x_4\}$ or $N_C(u) = \{x_1, x_2, x_3, x_5\}$ or $N_C(u) = \{x_1, x_2, x_4, x_5\}$ or $N_C(u) = \{x_1, x_2, x_4, x_6\}$ and each one of which produces a θ_7 as a subgraph in H. Thus, we conclude that $\mathcal{E}(u, C) \leq 3$ with equality holds only if $N_C(u) = \{x_i, x_{i+1}, x_{i+4}\}$ for some $i = 1, 2, \dots, 7$ ($x_j = x_{j-7}$ for $j > 7$). Suppose that $|B| \geq 2$. Let $x, y \in B$ with $x \neq y$. Without loss of generality, we may assume that $N_C(x) = \{x_1, x_2, x_5\}$. If $xy \in E(H)$ and y is adjacent to x_1, then the trail $xyx_1x_2xx_1$ would form a θ_4 as a subgraph in H, a contradiction. Similarly, one can show that y cannot be adjacent to x_2, x_4, x_5 or x_6 as otherwise a θ_4 or a θ_7 is produced as a subgraph. Thus, we assume that $xy \notin E(H)$. If $N_C(x) \cap N_C(y) = \varnothing$, then $N_C(y) = \{x_3, x_4, x_7\}$ or $\{x_3, x_6, x_7\}$. If $N_C(y) = \{x_3, x_4, x_7\}$, then the trail $xx_5x_4yx_7x_1x_2xx_1$ forms a θ_7 as a subgraph. Also if $N_C(y) = \{x_3, x_6, x_7\}$, then the trail $xx_5x_6yx_7x_1x_2xx_1$ forms a θ_7 as a subgraph. Therefore, $N_C(x) \cap N_C(y) \neq \varnothing$. We now consider the case that $x_1 \in N_C(y) \cap N_C(x)$. If y is adjacent to x_2, then the trail $x_1xx_2yx_1x_2$ forms a θ_4 as a subgraph, a contradiction. Similarly we can show that y cannot be adjacent to x_3, x_5 or x_7 as otherwise a θ_7 is produced as a subgraph. Thus y is adjacent to x_4 and x_6, but the trial $yx_6x_5x_4x_3x_2x_1yx_4$ forms a θ_7 as a subgraph, a contradiction. By using the same argument as a above one can show that if x_2 or x_5 belongs to $N_C(y) \cap N_C(x)$, then we get the same contradiction. Therefore, $|B| \leq 1$. This completes the proof. □

Lemma 2.2. *Let $H \in \mathcal{H}(n, \{\theta_4, \theta_5, \theta_7\})$ and H contains a cycle C of length 7. If $|B| = 1$ and uv is an edge in the subgraph $H - C - B$, then $\mathcal{E}(\{u, v\}, C) \leq 3$ where B is as defined in Lemma 2.1.*

Proof: Let uv be an edge in $H - C - B$. Then by Lemma 2.1., $\mathcal{E}(u, C), \mathcal{E}(v, C) \leq 2$. Now we shall prove by contradiction that the case $\mathcal{E}(u, C) = \mathcal{E}(v, C) = 2$ is impossible. Suppose $\mathcal{E}(u, C) = \mathcal{E}(v, C) = 2$, then one can see that each of $N_C(u)$ and $N_C(v)$ is of the form $\{x_i, x_{i+2}\}$ or $\{x_i, x_{i+3}\}$ or $\{x_i, x_{i+4}\}$ as otherwise at least one of θ_4, θ_5, and θ_7 is produced as a subgraph. Let $B = \{x\}$ and with out loss of generality assume x is adjacent to x_1, x_2 and x_5. Note that if $N_C(u)$ or $N_C(v)$ is of the form $\{x_i, x_{i+2}\}$, then the only possibilities for that are $\{x_2, x_4\}, \{x_3, x_5\}, \{x_5, x_7\}$ and $\{x_1, x_6\}$ as otherwise at least one of θ_4, θ_5 and θ_7 is produced as a subgraph. Further, if $N_C(u)$ or $N_C(v)$ is of the form $\{x_i, x_{i+3}\}$ or $\{x_i, x_{i+4}\}$, then the only possibilities for that are $\{x_1, x_4\}, \{x_2, x_6\}$ and $\{x_3, x_7\}$ as otherwise at least one of θ_4, θ_5 and θ_7 is produced as a subgraph. Note that, $|N_C(u) \cap N_C(v)| = 0$ or 1 as otherwise a θ_4 is produced as a subgraph. To this end we consider two cases:

Case 1: $|N_C(u) \cap N_C(v)| = 0$. Then, without loss of generality, we list all the possibilities as follows:

1) $N_C(u) = \{x_2, x_4\}$ and $N_C(v) = \{x_3, x_5\}$. Then the trail $uvx_3x_4x_5xx_2ux_4$ is a θ_7 subgraph, a contradiction.

2) $N_C(u) = \{x_2, x_4\}$ and $N_C(v) = \{x_5, x_7\}$. Then the trail $ux_4x_3x_2xx_5vux_2$ is a θ_7 subgraph, a contradiction.

3) $N_C(u) = \{x_2, x_4\}$ and $N_C(v) = \{x_1, x_4\}$ or $\{x_1, x_6\}$. Then the trail $x_2xx_1vux_2x_1$ is a θ_5 subgraph, a contradiction.

4) $N_C(u) = \{x_2, x_4\}$ and $N_C(v) = \{x_2, x_6\}$. Then the trail $ux_4x_3x_2vux_2$ is a θ_5 subgraph, a contradiction.

5) $N_C(u) = \{x_2, x_4\}$ and $N_C(v) = \{x_3, x_7\}$. Then the trail $x_3vux_2xx_5x_4x_3x_2$ is a θ_7 subgraph, a contradiction.

6) $N_C(u) = \{x_3, x_5\}$ and $N_C(v) = \{x_1, x_6\}$ or $\{x_2, x_6\}$. Then the trail $x_5x_6vux_3x_2xx_5u$ is a θ_7 subgraph, a contradiction.

7) $N_C(u) = \{x_3, x_5\}$ and $N_C(v) = \{x_1, x_4\}$. Then the trail $x_3x_2xx_5uvxx_4x_3u$ is a θ_7 subgraph, a contradiction.

8) $N_C(u) = \{x_5, x_7\}$ and $N_C(v) = \{x_6, x_1\}$. Then by symmetry we get the same contradiction as in (1).

9) $N_C(u) = \{x_5, x_7\}$ and $N_C(v) = \{x_1, x_4\}$. Then the trail $ux_7x_1xx_5x_4vux_5$ is a θ_7 subgraph, a contradiction.

10) $N_C(u) = \{x_5, x_7\}$ and $N_C(v) = \{x_2, x_6\}$. Then the trail $x_1xx_2vux_1x_2$ is a θ_5 subgraph, a contradiction.

11) $N_C(u) = \{x_1, x_6\}$ and $N_C(v) = \{x_3, x_7\}$. Then the trail $x_1x_7x_6uvx_3x_2x_1u$ is a θ_7 subgraph, a contradiction.

12) $N_C(u) = \{x_1, x_4\}$ and $N_C(v) = \{x_2, x_6\}$. Then the trail $x_1xx_2uvx_1x_2$ is a θ_5 subgraph, a contradiction.

13) $N_C(u) = \{x_1, x_4\}$ and $N_C(v) = \{x_3, x_7\}$. Then the trail $uvx_1xx_5x_4x_3vx_3$ is a θ_7 subgraph, a contradiction.

14) $N_C(u) = \{x_2, x_6\}$ and $N_C(v) = \{x_3, x_7\}$. Then the trail $uvx_7x_1xx_5x_6uv$ is a θ_7 subgraph, a contradiction.

Case 2: $|N_C(u) \cap N_C(v)| = 1$. Then, without loss of generality, we list all of the possibilities as follows:

1) $N_C(u) = \{x_1, x_6\}$ and $N_C(v) = \{x_1, x_4\}$. Then the trail $uvx_1x_7x_6x_5x_4vx_1$ is a θ_7 subgraph, a contradiction.

2) $N_C(u) = \{x_2, x_4\}$ and $N_C(v) = \{x_2, x_6\}$. Then the trail $x_2x_3x_4uvx_2u$ is a θ_5 subgraph, a contradiction.

3) $N_C(u) = \{x_3, x_5\}$ and $N_C(v) = \{x_3, x_7\}$. Then the trail $ux_3x_4x_5x_6x_7vux_5$ is a θ_7 subgraph, a contradiction.

4) $N_C(u) = \{x_1, x_4\}$ and $N_C(v) = \{x_2, x_4\}$. Then the trail $ux_1xx_2x_3x_4vux_4$ is a θ_7 subgraph, a contradiction.

5) $N_C(u) = \{x_3, x_5\}$ and $N_C(v) = \{x_5, x_7\}$. Then the trail $x_5x_6x_7vux_5v$ is a θ_5 subgraph, a contradiction.

6) $N_C(u) = \{x_1, x_6\}$ and $N_C(v) = \{x_2, x_6\}$. Then the trail $x_6vux_1x_7x_6u$ is a θ_5 subgraph, a contradiction.

7) $N_C(u) = \{x_3, x_7\}$ and $N_C(v) = \{x_5, x_7\}$. Then the trail $x_7uvx_5x_6x_7v$ is a θ_5 subgraph, a contradiction. $\square$

The following remark follows from the fact that if $H \in \mathcal{H}(n, \{\theta_4, \theta_5, \theta_7\})$, C is a cycle of length 7 in H, $u \in V(H - C)$ and $\mathcal{E}(u, C) = 3$, then $N_C(u) = \{x_i, x_{i+1}, x_{i+4}\}$.

Remark 2.3. *For $H \in \mathcal{H}(n, \{C_3, \theta_4, \theta_5, \theta_7\})$, if H contains a cycle C of length 7, then $B = \varnothing$ where B is as defined in Lemma 2.1.*

We now establish the following result which will be used in the rest of this section. We begin with the following construction. For odd n, let $\mathcal{H}_1$ be the class of graphs obtained from $\overline{K}_{\frac{n-3}{2}} \vee \overline{K}_{\frac{n-3}{2}}$ by replacing one edge, say $y_1y_2 \in \overline{K}_{\frac{n-3}{2}} \vee \overline{K}_{\frac{n-3}{2}}$, by the path $y_1w_2w_3w_4y_2$ with the vertices w_2, w_3, w_4, being all new vertices. Note that $\mathcal{H}_1$ is a class of non-bipartite Hamiltonian graphs containing none of θ_4, θ_5 and θ_7 as a subgraphs. Also $\mathcal{E}(H) = \left\lfloor \frac{(n-3)^2}{4} \right\rfloor + 3$ for any $H \in \mathcal{H}_1$. Thus

$$h(n; \{\theta_4, \theta_5, \theta_7\}) \geq \frac{(n-3)^2}{4} + 3 \text{ for odd } n. \tag{2.1}$$

Theorem 2.4. *For sufficiently large odd n, we have*

$$h(n; \{\theta_4, \theta_5, \theta_7\}) = \frac{(n-3)^2}{4} + 3.$$

Proof: Let $H \in \mathcal{H}(n, \{\theta_4, \theta_5, \theta_7\})$. By 2.1 it is enough to show that $\mathcal{E}(H) \leq \frac{(n-3)^2}{4} + 3$. If H contains no cycle of length 7, then by Theorem 1.1, we have

$$\mathcal{E}(H) \leq \frac{(n-5)^2}{4} + 9 \leq \frac{(n-3)^2}{4} + 3,$$

for sufficiently large odd n, as required. Suppose H contains a cycle C of length 7. Define the set $B = \{u \in V(H-C) : \mathcal{E}(u, C) = 3\}$. Then from Lemma 2.1, $|B| \leq 1$. If $|B| = 0$, then again from Lemma 2.1 $\mathcal{E}(u, C) \leq 2$ for all $u \in V(H-C)$ and so $\mathcal{E}(H-C, C) \leq 2(n-7)$. Now, suppose $|B| = 1$. Since H is Hamiltonian, the graph $H - C - B$ must have an edge uv. By Lemma 2.2, we obtain $\mathcal{E}(\{u, v\}, C) \leq 3$, thus

$$\begin{aligned} \mathcal{E}(H-C, C) &= \mathcal{E}(H - B - \{u, v\}, C) + \mathcal{E}(B, C) + \mathcal{E}(\{u, v\}, C) \\ &\leq 2(n-10) + 3 + 3 = 2(n-7). \end{aligned}$$

By Theorem 1.2, we have

$$\mathcal{E}(H - C) \leq \frac{(n-7)^2}{4}.$$

Therefore

$$\begin{aligned} \mathcal{E}(H) &= \mathcal{E}(H-C) + \mathcal{E}(H-C, C) + \mathcal{E}(C) \\ &\leq \frac{(n-7)^2}{4} + 2(n-7) + 7 \\ &= \frac{(n-3)^2}{4} + 3. \end{aligned}$$

This completes the proof. □

We now determine $h(n;\theta_7)$ for sufficiently large odd n. Note that the class $\mathcal{H}_1$ consists of non-bipartite Hamiltonian graphs containing no θ_7 as a subgraph. Further, $\mathcal{E}(H) = \frac{(n-3)^2}{4} + 3$ for any $H \in \mathcal{H}_1$. Thus we establish that

$$h(n;\theta_7) \geq \frac{(n-3)^2}{4} + 3 \tag{2.2}$$

for sufficiently large odd n.

Theorem 2.5. *For sufficiently large odd n, let $H \in \mathcal{H}(n;\theta_7)$ with $\delta(H) \geq 20$. Then*

$$\mathcal{E}(H) \leq \frac{(n-3)^2}{4} + 3.$$

Proof: To prove the theorem, we split the proof into two cases, according to the existence of θ_5 in H as a subgraph:
Case 1: H contains θ_5 as a subgraph, namely let $x_1x_2x_3x_4x_5x_1x_4$ be a θ_5-graph in H. Since $\delta(H) \geq 20$, we can define the sets A_i for $i = 1, 2, 3$, that consist of 5 neighbors of x_i in $H - \{x_1, x_2, x_3, x_4, x_5\}$ so that $A_i \cap A_j = \varnothing$ for $i \neq j$. Let $T = H[x_1, x_2, x_3, x_4, x_5, A_1, A_2, A_3]$ and $B = H - T$. Let $u \in V(B)$, if u is adjacent to a vertex in one of the sets A_1, A_2 or A_3, then u cannot be adjacent to a vertex in the other two sets, as otherwise H would have a θ_7-graph as a subgraph. Also, if u is adjacent to a vertex in A_i for some $i = 1, 2, 3$, then u cannot be adjacent to any of x_{i+1} and x_{i-1}, as otherwise H would have a θ_7-graph as a subgraph. Thus,

$$\mathcal{E}(u, T) \leq 8,$$

which implies that

$$\mathcal{E}(B, T) \leq 8(n - 20).$$

Also, by Theorem 1.2, we have

$$\mathcal{E}(B) \leq \left\lfloor \frac{(n-20)^2}{4} \right\rfloor \quad \text{and} \quad \mathcal{E}(T) \leq \left\lfloor \frac{(20)^2}{4} \right\rfloor.$$

Consequently

$$\begin{aligned}
\mathcal{E}(H) &= \mathcal{E}(B) + \mathcal{E}(B, T) + \mathcal{E}(T) \\
&\leq \frac{(n-20)^2}{4} + 8(n-20) + \frac{(20)^2}{4} \\
&\leq \frac{n^2 - 8n + 160}{4} \\
&= \frac{(n-4)^2}{4} + 36 \\
&< \frac{(n-3)^2}{4} + 3,
\end{aligned}$$

for sufficiently large odd n, as required.

Case 2: H contains no θ_5-graph as a subgraph. If H contains no θ_4 as a subgraph, then the result is immediate from Theorem 2.4. So, assume H contains a θ_4-graph, namely let $x_1x_2x_3x_4x_1x_3$ be a θ_4-graph in H. Since $\delta(H) \geq 20$, we can define the sets A_i $(i = 1, 2, 4)$ that consist of 5 neighbors of x_i in $H - \{x_1x_2x_3x_4\}$ selected so that $A_i \cap A_j = \varnothing$ for $i \neq j$. Let $T = H[x_1, x_2, x_3, x_4, A_1, A_2, A_4]$ and $B = H - T$. Then, the rest of the proof is rather similar to that of Case 1. □

Now we are ready to establish our main result. In the following theorem we determine $h(n; \theta_7)$ for odd large n and $\delta(H) \geq 7$.

Theorem 2.6. *For sufficiently large odd n, let $H \in \mathcal{H}(n; \theta_7)$ with $\delta(H) \geq 7$. Then*

$$\mathcal{E}(H) \leq \frac{(n-3)^2}{4} + 3.$$

Proof: Let $H \in \mathcal{H}(n; \theta_7)$ with $\delta(H) \geq 7$. Let A be the set of vertices in H with degree less than or equal to 19. Let $|A| = m$. Observe that,

$$\mathcal{E}(H - A, A) + \mathcal{E}(A) \leq 19m.$$

By Theorem 1.2,

$$\begin{aligned} \mathcal{E}(H) &= \mathcal{E}(H - A) + \mathcal{E}(H - A, A) + \mathcal{E}(A) \\ &\leq \left\lfloor \frac{(n-m)^2}{4} \right\rfloor + 19m. \end{aligned}$$

If $m \geq 4$, then by remembering that n is sufficiently large, we have that the right hand side of the last inequality is maximum when $m = 4$. Thus,

$$\begin{aligned} \mathcal{E}(H) &\leq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 76 \\ &< \left\lfloor \frac{(n-3)^2}{4} \right\rfloor + 3. \end{aligned}$$

If $m = 0$, then by Theorem 2.5, we have

$$\mathcal{E}(H) \leq \left\lfloor \frac{(n-3)^2}{4} \right\rfloor + 3,$$

as required. Now, for $m = 1, 2, 3$, we consider two cases according to the graph $H - A$.

Case 1: If $H - A$ is a non-bipartite graph. Then Theorem 1.3 implies that

$$\mathcal{E}(H - A) \leq \left\lfloor \frac{(n-m-2)^2}{4} \right\rfloor + 3.$$

And so,

$$\begin{aligned}\mathcal{E}(H) &= \mathcal{E}(H-A)+\mathcal{E}(H-A,A)+\mathcal{E}(A)\\ &\leq \left\lfloor\frac{(n-m-2)^2}{4}\right\rfloor+3+19m.\end{aligned}$$

For $m=2$ and $m=3$, the above inequality has it is maximum at $m=2$, so

$$\begin{aligned}\mathcal{E}(H) &\leq \left\lfloor\frac{(n-4)^2}{4}\right\rfloor+41\\ &< \left\lfloor\frac{(n-3)^2}{4}\right\rfloor+3,\end{aligned}$$

for odd large n, as required. Therefore, we now consider only the case when $m=1$. Assume $A=\{x_0\}$, then according to the existance of θ_4 and θ_5 in H, we consider the following three cases:

Subcase 1.1: H contains niether θ_5-graph as a subgraph nor θ_4-graph as subgraph. Then as a above, the result follows from Theorem 2.4.

Subcase 1.2: H contains θ_5-graph as a subgraph. Assume $x_0 \notin V(\theta_5)$ and let $x_1x_2x_3x_4x_5x_1x_4$ be a θ_5-graph. Consider the same construction as in Case 1 of Theorem 2.5 and define $R=H-A-T$, then we have

$$\mathcal{E}(R,T)\leq 8(n-21).$$

Observe that $\mathcal{E}(R,A)+\mathcal{E}(T,A)+\mathcal{E}(A)\leq 19$. Also, by Theorem 1.2 we have

$$\mathcal{E}(R)\leq\frac{(n-21)^2}{4}\qquad\text{and}\quad\mathcal{E}(T)\leq\frac{(20)^2}{4}.$$

Consequently

$$\begin{aligned}\mathcal{E}(H) &= \mathcal{E}(R)+\mathcal{E}(R,T)+\mathcal{E}(T)+\mathcal{E}(R,A)+\mathcal{E}(T,A)+\mathcal{E}(A)\\ &\leq \frac{(n-21)^2}{4}+8(n-21)+\frac{(20)^2}{4}+19\\ &\leq \frac{n^2-10n+245}{4}\\ &= \frac{(n-5)^2}{4}+55\\ &< \frac{(n-3)^2}{4}+3,\end{aligned}$$

for odd large n, as required.

Now we consider $x_0\in V(\theta_5)$. Assume that $x_0=x_5$ that is $x_1x_2x_3x_4x_0x_1x_4$ be a θ_5-graph in H. Let $T=H[x_1,x_2,x_3,x_4,x_0,A_1,A_2,A_3]$ and $R=H-T$ where A_i is as defined in Theorem2.5, then as in Case 1 of Theorem 2.5, $\mathcal{E}(x,T)\leq 8$ for each $x\in R$, and so

$$\mathcal{E}(R,T)\leq 8(n-20).$$

Also, by Theorem 1.2 we get

$$\mathcal{E}(R) \leq \frac{(n-20)^2}{4} \qquad \text{and} \quad \mathcal{E}(T) \leq \frac{(20)^2}{4}.$$

As a consequence

$$\begin{aligned}\mathcal{E}(H) &= \mathcal{E}(R) + \mathcal{E}(R,T) + \mathcal{E}(T)\\ &\leq \frac{(n-20)^2}{4} + 8(n-20) + \frac{(20)^2}{4}\\ &\leq \frac{n^2-8n+144}{4}\\ &= \frac{(n-4)^2}{4} + 36\\ &< \frac{(n-3)^2}{4} + 3.\end{aligned}$$

Similarly, if $x_0 = x_1$ or x_2 or x_3 or x_4 in θ_5, then we can choose $i's$ so that $A_{i's}$ satisfied the required properties as in above and then word by word we use the above technique.

Subcase 1.3: H contains no θ_5-graph as a subgraph but it contains θ_4-graph as a subgraph. Assume that $x_0 \notin V(\theta_4)$. By Consideing the same construction as in Theorem 2.5 and define $R = H - A - T$, we obtain that

$$\mathcal{E}(R,T) \leq 6(n-17).$$

Recall that $\mathcal{E}(R,A) + \mathcal{E}(T,A) + \mathcal{E}(A) \leq 19$. Also, by Theorem 1.2 we have

$$\mathcal{E}(R) \leq \frac{(n-17)^2}{4} \qquad \text{and} \quad \mathcal{E}(T) \leq \frac{(16)^2}{4}.$$

Therefore,

$$\begin{aligned}\mathcal{E}(H) &= \mathcal{E}(R) + \mathcal{E}(R,T) + \mathcal{E}(T) + \mathcal{E}(R,A) + \mathcal{E}(T,A) + \mathcal{E}(A)\\ &\leq \frac{(n-17)^2}{4} + 6(n-17) + \frac{(16)^2}{4} + 19\\ &\leq \frac{n^2-10n+213}{4}\\ &= \frac{(n-5)^2}{4} + 47\\ &< \frac{(n-3)^2}{4} + 3,\end{aligned}$$

for odd large n as required.

Now, we consider $x_0 \in V(\theta_4)$, then assume that $x_0 = x_4$ that is $x_1x_2x_3x_0x_1x_3$ forms θ_4-graph is in H. Since $\delta(H) \geq 7$, so for $i = 0, 1, 2$, let A_i be the set that consist of 4 neighbors of x_i in H selected so that $A_i \cap A_j = \varnothing$ for $i \neq j$. Let $T = H[x_0, x_1, x_2, x_3, A_0, A_1, A_2]$ and $R = H - T$. Observe that

$$\mathcal{E}(R, T) \leq 6(n - 16).$$

Also, by Theorem 1.2, we have

$$\mathcal{E}(R) \leq \left\lfloor \frac{(n-16)^2}{4} \right\rfloor \quad \text{ and } \quad \mathcal{E}(T) \leq \left\lfloor \frac{(16)^2}{4} \right\rfloor.$$

Consequently

$$\begin{aligned} \mathcal{E}(H) &= \mathcal{E}(R) + \mathcal{E}(R, T) + \mathcal{E}(T) \\ &\leq \left\lfloor \frac{(n-4)^2}{4} \right\rfloor + 28 \\ &< \frac{(n-3)^2}{4} + 3, \end{aligned}$$

for odd large n as required. Similarly, we can do the same construction and get the same result if $x_0 = x_1$ or x_2 or x_3.

Case 2: $H - A$ is a bipartite graph with the partitioning sets X and Y. Recall that A is the set of vertices in H with degree less than or equal to 19 and we have proved the theorem for the case when $m \geq 4$ or $m = 0$ where $|A| = m$. Sine H is a non-bipartite graph, then it contains an odd cycle, in fact any odd cycle in H must involve vertices of A. If H contains no cycles of length 3 and 5, then the result follows from Theorem 1.5. So, we have to study two cases according to the length of the odd cycles in H.

Subcase 2.1: H contains an odd cycle of length 5. Let $C = x_1x_2x_3x_4x_5x_1$ be a cycle of length 5 with minimum vertices of A and n_1, n_2 be the cardinalities of $X - V(C) - A, Y - V(C) - A$, respectively. According to the possibilities of m we consider the following three cases:

Subsubcase 2.1.1. $m = 1$. Let $A = \{x_5\}$ and $x_1, x_2, x_3, x_4 \in H - A$. Observe that, $N_{H-C}(x_i) \cap N_{H-C}(x_{i+1}) = \varnothing$ for $i = 1, 2, 3, 4$, otherwise $H - A$ would have an odd cycle of length 3. Also, $E(N_{H-C}(x_i), N_{H-C}(x_{i+1})) = \varnothing$ for $i = 1$ and 3, otherwise H would have a θ_7-graph as subgraph of H. Let $|N_{H-C}(x_i)| = k_i$, for $i = 1, ..., 4$. Note that $H - C$ is a bipartite graph with the above observations, we have

$$\mathcal{E}(H - C) \leq n_1n_2 - k_1k_2 - k_3k_4,$$

where $n_1 + n_2 = n - 5$. Now

$$\begin{aligned} \mathcal{E}(H) &= \mathcal{E}(H - C) + \mathcal{E}(H - C, C) + \mathcal{E}(C) \\ &\leq n_1n_2 - k_1k_2 - k_3k_4 + k_1 + k_2 + k_3 + k_4 + 27. \end{aligned}$$

Note that $k_i \geq 18$ and the right hand side of the above inequality is maximum when $k_i = 18$ and $n_1 = n_2 = \frac{n-5}{2}$, thus

$$\mathcal{E}(H) \leq \frac{(n-5)^2}{4} - 549 < \frac{(n-3)^2}{4} + 3,$$

as required.

Subsubcase 2.1.2. $m = 2$. It is easy to see that there is an edge of C non of its end points in A, say $x_1, x_2 \notin A$. Then by the same argument as above we have $N_{H-C}(x_1) \cap N_{H-C}(x_2) = \varnothing$ and $E(N_{H-C}(x_1), N_{H-C}(x_2)) = \varnothing$. If $A \subseteq V(C)$, then $H - C^{'}$ is a bipartite graph with the above observations, we have

$$\mathcal{E}(H - C) \leq n_1 n_2 - k_1 k_2,$$

where $N_{H-C}(x_2)| = k_1$ and $|N_{H-C}(x_4)| = k_2$. Thus,

$$\begin{aligned} \mathcal{E}(H) &= E(H - C) + E(H - C, C) + E(C) \\ &\leq n_1 n_2 - k_1 k_2 + k_1 + k_2 + \max\{n_1, n_2\} + 44. \end{aligned}$$

Recall that $n_1 + n_2 = n - 5$ and the right hand side of the above inequality is maximum when $n_1 = n_2 = \frac{n-5}{2}$. Thus

$$\mathcal{E}(H) \leq \frac{(n-4)^2}{4} - k_1 k_2 + k_1 + k_2 + 43.$$

Note that $k_i \geq 18$ and the right hand side of the above is maximum when $k_i = 18$, thus

$$\mathcal{E}(H) \leq \frac{(n-4)^2}{4} - 245 < \frac{(n-3)^2}{4} + 3,$$

as required.

If $A \nsubseteq V(C)$, then C contains only one vertex of A, say x_5. As in Subsubcase 2.1.1 we have $N_{H-C}(x_i) \cap N_{H-C}(x_{i+1}) = \varnothing$ for $i = 1, 2, 3, 4$ and

$$E(N_{H-C}(x_i), N_{H-C}(x_{i+1})) = \varnothing$$

for $i = 1, 3$. Note that $H - C - \{x_5\}$ is a bipartite graph with the above observations, we have

$$\mathcal{E}(H - C - x_5) \leq n_1 n_2 - k_1 k_2 - k_3 k_4.$$

where $k_i = |N_{H-C-x_5}(x_i)|$. Thus,

$$\begin{aligned} \mathcal{E}(H) &= \mathcal{E}(H - C - x_5) + \mathcal{E}(H - C - x_5, C) + \mathcal{E}(C) + \mathcal{E}(x_5) \\ &\quad + \mathcal{E}(H - C - x_5, x_5) + \mathcal{E}(C, x_5) \\ &\leq n_1 n_2 - k_1 k_2 - k_3 k_4 + k_1 + k_2 + k_3 + k_4 + 44 \end{aligned}$$

Recall that $n_1 + n_2 = n - 6$ and the right hand side of the above is maximum when $n_1 = n_2 = \frac{n-6}{2}$. Thus,

$$\mathcal{E}(H) \leq \frac{(n-6)^2}{4} - k_1 k_2 - k_3 k_4 + k_1 + k_2 + k_3 + k_4 + 46.$$

Note that $k_i \geq 18$. The right hand side of the above is maximum when $k_i = 18$, thus

$$\mathcal{E}(H) \leq \frac{(n-6)^2}{4} - 622 < \frac{(n-3)^2}{4} + 3,$$

as required.

Subsubcase 2.1.3. $m = 3$. If C has an edge none of its end points belongs to A, then by applying a similar argument as above, we get the result. So, without loss of generality, assume that x_1, x_3, x_5 are in A and x_2, x_4 are in $H - A$. Observe that $N_{H-C}(x_2) \cap N_{H-C}(x_4) = \varnothing$ and $E(N_{H-C}(x_2), N_{H-C}(x_4)) = \varnothing$, otherwise a new cycle of length 5 with minimum vertices of A is produced. If x_2 and x_4 are not in the same partition of the bipartite graph $H - A$, then the result holds as above. If x_2 and x_4 are in the same partition, then

$$\begin{aligned} E(H) &= E(H-C) + E(H-C,C) + E(C) \\ &\leq n_1 n_2 + k_1 + k_2 + 61, \end{aligned}$$

where $|N_{H-C}(x_2)| = k_1, |N_{H-C}(x_4)| = k_2$and $n_1 + n_2 = n - 5$. Note that $k_1 + k_2 \leq \max\{n_1, n_2\}$. Thus

$$\mathcal{E}(H) \leq n_1 n_2 + \max\{n_1, n_2\} + 61 < \frac{(n-3)^2}{4} + 3,$$

as required.

Subcase 2.2: H contains no cycle of length 5 but it contains cycles of length 3. Let $C = x_1x_2x_3$ be a cycle of length 3 with minimum vertices of A. As above we consider three cases according to the value of m.

Subsubcase 2.2.1. $m = 1$. Let $x_1, x_2 \in H - A$ and $x_3 \in A$. Then, $N_{H-C}(x_1) \cap N_{H-C}(x_2) = \varnothing$ as otherwise $H - A$ would have an odd cycle. Also $E(N_{H-C}(x_1), N_{H-C}(x_2)) = \varnothing$, as otherwise H would have a cycle of length 5. Using the same arguments as above, we get the result.

Subsubcase 2.2.2. $m = 2$. If only one vertex of A belongs to $V(C)$, then we use the same argument as in Subsubcases 1.2.2 and 2.2.1. So, we assume that $x_1 \in H - A$ and $x_2, x_3 \in A$. Since H is Hamiltonian, then there is a vertex $z \notin \{x_1, x_2, x_3\}$ such that $x_2 z \in E(H)$. Define $C^* = H[x_1, x_2, x_3, z]$, then $N_{H-C^*}(x_1) \cap N_{H-C^*}(z) = \varnothing$, as otherwise H would have a cycle of length 5. Also, $E(N_{H-C^*}(x_1), N_{H-C^*}(z)) = \varnothing$, as otherwise a cycle of length 5 is produced. Apply the same argument as in above, we get the result.

Subsubcase 2.2.3. $m = 3$. If $|A \cap V(C)| = 1$ or 2, then we use the same argument as in Subsubcases 2.2.2 and 1.2.3. Thus, we assume that $x_1, x_2, x_3 \in A$. Since H is Hamiltonian, then there are two different vertices w, z with $w, z \notin \{x_1, x_2, x_3\}$, $wx_1 \in E(H)$ and $zx_2 \in E(H)$. Define $C^* = H[x_1, x_2, x_3, w, z]$, then $N_{H-C^*}(w) \cap N_{H-C^*}(z) = \varnothing$, as otherwise we have a cycle of length 5 in H. Also, $E(N_{H-C^*}(w), N_{H-C^*}(z)) = \varnothing$, as otherwise a θ_7 is produced. Apply the same argument as in above, we get the result. This completes the proof. □

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