

Research Article

Tawseef Rashid, Mohammed M. M. Jaradat*, Qamrul Haq Khan, Zoran D. Mitrović, Hassen Aydi, and Zead Mustafa

A new approach in the context of ordered incomplete partial b -metric spaces

<https://doi.org/10.1515/math-2020-0054>

received March 23, 2019; accepted October 28, 2019

Abstract: The main purpose of this paper is to find some fixed point results with a new approach, particularly in those cases where the existing literature remains silent. More precisely, we introduce partial completeness, \bar{f} -orbitally completeness, a new type of contractions and many other notions. We also ensure the existence of fixed points for non-contraction maps in the class of incomplete partial b -metric spaces. We have reported some examples in support of our results.

Keywords: partial b -metric space, \bar{f} -orbitally complete, partially complete, fixed point

MSC 2020: 47H10, 54H25, 46J10

1 Introduction

The Banach contraction principle plays a significant role in the development of fixed point theory. Banach [1] explained the uniqueness of a fixed point for a contraction map in complete metric spaces. The Banach contraction principle has been generalized by many researchers either by generalizing the distance space or modifying the contraction. In 1989, Bakhtin [2] generalized metric spaces by introducing the concept of b -metric spaces. In 1994, Matthews [3] introduced the concept of a partial metric space, in which a self-distance of any point may not be zero. Ran-Reurings [4] generalized the Banach contraction principle in setting of ordered metric spaces. The key feature in the Ran-Reurings theorem is that the contractive condition on the nonlinear map is only assumed to hold on the comparable elements instead of the whole space as in the Banach contraction principle. In 2005, Nieto and Rodríguez-López [5] proved a fixed point theorem by relaxing some conditions in Ran-Reurings [4]. In 2008, Suzuki [6] proved a fixed point theorem by assuming a contraction condition on those elements which satisfy the given condition. Recently in 2013, Shukla [7] generalized both b -metrics and partial metrics by introducing a partial b -metric. There is a bulk of literature on fixed point in all these spaces, see [8–35]. Almost all the existing

* **Corresponding author: Mohammed M. M. Jaradat**, Department of Mathematics, Statistics and Physics, Qatar University, Doha, 2713, Qatar, e-mail: mmjst4@qu.edu.qa

Tawseef Rashid, Qamrul Haq Khan: Department of Mathematics, Aligarh Muslim University, Aligarh, 202002, India, e-mail: tawseefrashid123@gmail.com, qhkhan.ssitm@gmail.com

Zoran D. Mitrović: University of Banja Luka, Faculty of Electrical Engineering, 78000, Banja Luka, Bosnia and Herzegovina, e-mail: zoran.mitrovic@etf.unibl.org

Hassen Aydi: Université de Sousse, Institut Supérieur d'Informatique et des Techniques de Communication, H. Sousse, 4000, Tunisia; China Medical University Hospital, China Medical University, Taichung, 40402, Taiwan, e-mail: hassn.aydi@isima.rnu.tn

Zead Mustafa: Department of Mathematics, Statistics and Physics, Qatar University, Doha, 2713, Qatar, e-mail: zead@qu.edu.qa

fixed point results involve the completeness of the underlying space. The main aim of this paper is to ensure the existence of fixed points either without assuming the completeness of the space or the contraction condition is not satisfied. In this direction, we give a brief introduction of our newly introduced concepts, especially partial completeness, a new contraction condition and many other concepts to tackle the problem as pointed out above. The contraction condition is not assumed to hold on the whole space, but we choose a subset with some given properties, on which the contraction condition is assumed to hold. Particularly, we prove fixed point results for non-contraction maps in the setting of ordered partial incomplete b -metric spaces. We give some examples in support of our obtained theorems. Before going to the main result, we recall some definitions.

Definition 1.1. [2] Let X be a nonempty set and $b : X \times X \rightarrow [0, \infty)$ be a function verifying:

- (1) $b(\rho, \xi) = 0$ iff $\rho = \xi$;
- (2) $b(\rho, \xi) = b(\xi, \rho)$;
- (3) There exists $s \geq 1$ such that $b(\rho, \xi) \leq s[b(\rho, \tau) + b(\tau, \xi)]$,
for all $\rho, \xi, \tau \in X$. Then, b is called a b -metric on X and (X, b) is called a b -metric space with a coefficient $s \geq 1$.

Definition 1.2. [3] Let X be a nonempty set and $p : X \times X \rightarrow [0, \infty)$ be a function such that:

- (1) $p(\rho, \rho) = p(\rho, \xi) = p(\xi, \xi)$ iff $\rho = \xi$;
- (2) $p(\rho, \rho) \leq p(\rho, \xi)$;
- (3) $p(\rho, \xi) = p(\xi, \rho)$;
- (4) $p(\rho, \xi) \leq p(\rho, \tau) + p(\tau, \xi) - p(\tau, \tau)$,
for all $\rho, \xi, \tau \in X$. Then, p is called a partial metric on X and (X, p) is called a partial metric space.

Definition 1.3. [7,36] Let X be a nonempty set and $d : X \times X \rightarrow [0, \infty)$ be a function such that:

- (1) $d(\rho, \rho) = d(\rho, \xi) = d(\xi, \xi)$ iff $\rho = \xi$;
- (2) $d(\rho, \rho) \leq d(\rho, \xi)$;
- (3) $d(\rho, \xi) = d(\xi, \rho)$;
- (4) There exists $s \geq 1$ such that $d(\rho, \xi) \leq s[d(\rho, \tau) + d(\tau, \xi)] - d(\tau, \tau)$,
for all $\rho, \xi, \tau \in X$. Then, d is called a partial b -metric on X and (X, d) is called a partial b -metric space with coefficient $s \geq 1$.

Example 1.1. Set $p > 1$. The function $d : [0, \infty)^2 \rightarrow [0, \infty)^2$ given as

$$d(\rho, \xi) = (\max\{\rho, \xi\})^p + |\rho - \xi|^p,$$

for all $\rho, \xi \in X$, is a partial b -metric (here, $s = 2^{p-1} > 1$). It is neither a partial metric nor a b -metric.

Definition 1.4. [7] Given a sequence $\{\zeta_n\}$ in a partial b -metric space (X, d) .

- (1) $\{\zeta_n\}$ converges to $\zeta \in X$ (written as $\lim_{n \rightarrow \infty} \zeta_n = \zeta$) if
$$\lim_{n \rightarrow \infty} d(\zeta_n, \zeta) = d(\zeta, \zeta).$$
- (2) $\{\zeta_n\}$ is called Cauchy in X if $\lim_{n, m \rightarrow \infty} d(\zeta_n, \zeta_m)$ exists and is finite.
- (3) (X, d) is said complete if for every Cauchy sequence $\{\zeta_n\}$, there is $\zeta \in X$ so that

$$\lim_{n, m \rightarrow \infty} d(\zeta_n, \zeta_m) = \lim_{n \rightarrow \infty} d(\zeta_n, \zeta) = d(\zeta, \zeta).$$

Here, the limit of a convergent sequence may not be unique.

Example 1.2. [7] Take $d(\rho, \xi) = \max\{\rho, \xi\} + a$ for all $\rho, \xi \in [0, \infty)$, where $a > 0$. Then, (X, d) is a partial b -metric space with an arbitrary coefficient $s \geq 1$. If $\zeta_n = 1$ for each $n \in \mathbb{N}$, then $\lim_{n \rightarrow \infty} \zeta_n = \eta$, for each $\eta \geq 1$.

Definition 1.5. [36] Let (X, \preceq) be a partially ordered set and d be a partial b -metric on X . Then, the triplet (X, d, \preceq) is called an ordered partial b -metric space.

Definition 1.6. [37] Let f be a self-map on a metric space (X, d) . For $A \subset X$, define $\delta(A) = \sup\{d(\alpha, \beta) : \alpha, \beta \in A\}$. For each $\theta \in X$ and $n \geq 1$, take

$$O(\theta, n) = \{\theta, f\theta, f^2\theta, \dots, f^n\theta\}$$

and

$$O(\theta, \infty) = \{\theta, f\theta, f^2\theta, \dots\}.$$

The metric space (X, d) is called f -orbitally complete if each Cauchy sequence contained in $O(\theta, \infty)$ converges in X .

2 Results and discussions

First, we introduce some definitions.

Definition 2.1. Let f be a self-map on a metric space (X, d) . For a subset $A \subseteq X$ with $\theta \in A$, state

$$O_A(\theta, n) = \{\theta, f\theta, f^2\theta, \dots, f^n\theta\}$$

and

$$O_A(\theta, \infty) = \{\theta, f\theta, f^2\theta, \dots\}.$$

We say that (X, d) is f -orbitally complete with respect to A , if every Cauchy sequence contained in $O_A(\theta, \infty)$ converges in A .

Now, we give some examples to illustrate Definition 2.1.

Example 2.1. Let $X = (-1, 2)$ be endowed with the usual metric of \mathbb{R} . Define $f : X \rightarrow X$ by $f(\zeta) = \frac{\zeta}{2}$, for all $\zeta \in X$. Let $A = [0, 1)$. Then, (X, d) is f -orbitally complete with respect to A .

Example 2.2. Let $X = \mathbb{R}$ be endowed with the usual metric. Define $f : X \rightarrow X$ by $f(\zeta) = \zeta^2$, for all $\zeta \in X$. Let $A = [0, 1)$. Again (X, d) is f -orbitally complete with respect to A .

Definition 2.2. Let (X, d, \preceq) be an ordered partial b -metric space. Then, for any subset $A \subseteq X$, (X, d, \preceq) is said to be \bar{f} -orbitally complete with respect to A , if every strictly increasing Cauchy sequence contained in $O_A(\theta, \infty)$ for some $\theta \in A$ converges in A and the limit of the sequence is a strict upper bound of the sequence, i.e., if $\{\zeta_n\}$ is a strictly increasing Cauchy sequence contained in $O_A(\zeta, \infty)$, then there is $\zeta \in A$ such that $\zeta_n \rightarrow \zeta$ and $\zeta_n < \zeta$, for all $n \in \mathbb{N}$.

We present some examples in support of Definition 2.2.

Example 2.3. Let $X = (-2, 3)$ be endowed with usual metric and \preceq be defined as the natural ordering \leq . Consider $f(\zeta) = \frac{\zeta}{3}$ for all $\zeta \in X$. Take $A = (-1, 0]$. Note that (X, d, \preceq) is \bar{f} -orbitally complete with respect to A .

Example 2.4. Let $X = (-\infty, 1)$ be endowed with the usual metric and \preceq be the natural ordering \leq . Take $f(\zeta) = \zeta^3$ for all $\zeta \in X$. Let $A = (-1, 0]$. Then, (X, d, \preceq) is \bar{f} -orbitally complete with respect to A .

Before going to the main results of this section, we present some examples to support our claim of ensuring the existence of a fixed point in cases where known results are not applicable.

Example 2.5. Let $X = (0, 2]$ be endowed with the usual metric and \leq be the natural ordering \leq . Define $f : X \rightarrow X$ by

$$f(\zeta) = \begin{cases} \frac{1}{\zeta^2}, & 0 < \zeta < 1, \\ \zeta, & 1 \leq \zeta \leq 2. \end{cases}$$

Clearly, X is not f -orbitally complete. It can be verified by taking $\zeta = \frac{2}{3}$ and the sequence $\{a_n = f^n(\zeta), n \text{ even}\}$ in $O(\zeta, \infty)$. But, if we take $A = [1, 2]$, then X is f -orbitally complete with respect to A .

Example 2.6. Let $X = (0, \infty)$ be endowed with the usual metric and \leq be the natural ordering \leq . Define $f : X \rightarrow X$ by

$$f(\zeta) = \begin{cases} \frac{\zeta}{3}, & 0 < \zeta < 1, \\ \zeta, & \text{otherwise.} \end{cases}$$

Clearly, X is not f -orbitally complete. If we take $A = [1, \infty)$, then X is f -orbitally complete with respect to A .

The following result is useful enough in the context of partial b -metric spaces.

Lemma 2.1. Let (X, d, \leq) be a partial b -metric space with a coefficient $s \geq 1$. For each sequence $\{\zeta_n\}$ in X , we have

$$d(\zeta_n, \zeta_m) \leq sd(\zeta_n, \zeta_{n+1}) + s^2d(\zeta_{n+1}, \zeta_{n+2}) + \dots + s^{m-n-1}d(\zeta_{m-2}, \zeta_{m-1}) + s^{m-n-1}d(\zeta_{m-1}, \zeta_m),$$

for all $n, m \in \mathbb{N}$ with $n < m$.

Proof. By using triangular inequality, one writes

$$\begin{aligned} d(\zeta_n, \zeta_m) &\leq s[d(\zeta_n, \zeta_{n+1}) + d(\zeta_{n+1}, \zeta_m)] - d(\zeta_{n+1}, \zeta_{n+1}) \\ &\leq s[d(\zeta_n, \zeta_{n+1}) + d(\zeta_{n+1}, \zeta_m)] \\ &\leq sd(\zeta_n, \zeta_{n+1}) + s[s(d(\zeta_{n+1}, \zeta_{n+2}) + d(\zeta_{n+2}, \zeta_m)) - d(\zeta_{n+2}, \zeta_{n+2})] \\ &\leq sd(\zeta_n, \zeta_{n+1}) + s[s(d(\zeta_{n+1}, \zeta_{n+2}) + d(\zeta_{n+2}, \zeta_m))] \\ &= sd(\zeta_n, \zeta_{n+1}) + s^2d(\zeta_{n+1}, \zeta_{n+2}) + s^2d(\zeta_{n+2}, \zeta_m). \end{aligned}$$

Continuing in this way, we get

$$d(\zeta_n, \zeta_m) \leq sd(\zeta_n, \zeta_{n+1}) + s^2d(\zeta_{n+1}, \zeta_{n+2}) + \dots + s^{m-n-1}d(\zeta_{m-2}, \zeta_{m-1}) + s^{m-n-1}d(\zeta_{m-1}, \zeta_m). \quad \square$$

Theorem 2.1. Let (X, d, \leq) be an ordered partial b -metric space with a coefficient $s \geq 1$. Suppose that the set $A = \{\theta \in X : \theta \leq f(\theta)\}$ is nonempty and that

$$d(\xi, f(\xi)) \leq \lambda d(\zeta, f(\zeta)), \tag{1}$$

for all $\zeta, \xi \in A$ with $\zeta < \xi$, where $\lambda \in \left[0, \frac{1}{s}\right)$. If further, $f(A) \subseteq A$ and (X, d, \leq) is \bar{F} -orbitally complete with respect to A , then there is a fixed point of f in A .

Proof. Since $A \neq \emptyset$, let $\zeta_0 \in A$. Then, $f(\zeta_0) \in A$. If $f(\zeta_0) = \zeta_0$, we are done. Otherwise, we choose $\zeta_1 = f(\zeta_0)$. Now, $f(\zeta_1) \in A$, and if $f(\zeta_1) = \zeta_1$, then we are through. Otherwise, choose $\zeta_2 = f(\zeta_1)$. Now, $f(\zeta_2) \in A$. As $\zeta_0, \zeta_1 \in A$ with $\zeta_0 < \zeta_1$, then by (1), we have

$$d(\zeta_1, f(\zeta_1)) \leq \lambda d(\zeta_0, f(\zeta_0)). \tag{2}$$

Again, as $\zeta_1, \zeta_2 \in A$ with $\zeta_1 < \zeta_2$, then by (1), we have

$$d(\zeta_2, f(\zeta_2)) \leq \lambda d(\zeta_1, f(\zeta_1)). \quad (3)$$

Using (2) in (3), we get

$$d(\zeta_2, f(\zeta_2)) \leq \lambda^2 d(\zeta_0, f(\zeta_0)).$$

Continuing similarly, we obtain

$$d(\zeta_n, f(\zeta_n)) \leq \lambda^n d(\zeta_0, f(\zeta_0)). \quad (4)$$

Continuing the process, we get a strictly increasing sequence $\{\zeta_n\}$ in A so that

$$\zeta_{n+1} = f(\zeta_n). \quad (5)$$

Now, we show that $\{\zeta_n\}$ is a Cauchy sequence in $O_A(\zeta_0, \infty)$. For this, take $n, m \in \mathbb{N}$ with $n < m$. From Lemma 2.1, one writes

$$d(\zeta_n, \zeta_m) \leq s d(\zeta_n, \zeta_{n+1}) + s^2 d(\zeta_{n+1}, \zeta_{n+2}) + \cdots + s^{m-n-1} d(\zeta_{m-2}, \zeta_{m-1}) + s^{m-n-1} d(\zeta_{m-1}, \zeta_m). \quad (6)$$

Now, by using (5) and (4), we get

$$\begin{aligned} d(\zeta_n, \zeta_m) &\leq s d(\zeta_n, f(\zeta_n)) + s^2 d(\zeta_{n+1}, f(\zeta_{n+1})) + \cdots + s^{m-n-1} d(\zeta_{m-2}, f(\zeta_{m-2})) + s^{m-n-1} d(\zeta_{m-1}, f(\zeta_{m-1})) \\ &\leq [s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-n-1}\lambda^{m-2} + s^{m-n-1}\lambda^{m-1}] d(\zeta_0, f(\zeta_0)) \\ &\leq s\lambda^n [1 + s\lambda + \cdots + (s\lambda)^{m-n-2} + s^{-1}(s\lambda)^{m-n-1}] d(\zeta_0, f(\zeta_0)) \\ &\leq s\lambda^n [1 + s\lambda + \cdots + (s\lambda)^{m-n-2} + (s\lambda)^{m-n-1}] d(\zeta_0, f(\zeta_0)) \\ &\leq \frac{s\lambda^n}{1 - s\lambda} \end{aligned} \quad (7)$$

$$\rightarrow 0, \quad \text{as } n, m \rightarrow \infty. \quad (8)$$

That is,

$$\lim_{n, m \rightarrow \infty} d(\zeta_n, \zeta_m) = 0.$$

Thus, the strictly increasing sequence $\{\zeta_n\}$ is Cauchy in $O_A(\zeta_0, \infty)$. But since (X, d, \leq) is \bar{f} -orbitally complete with respect to A , there is $u \in A$ so that $\zeta_n \rightarrow u$ and $\zeta_n < u$ for each $n \in \mathbb{N}$.

Since $\zeta_n, u \in A$ with $\zeta_n < u$ for all $n \in \mathbb{N}$, by (1) and (4), we have

$$d(u, f(u)) \leq \lambda d(\zeta_n, f(\zeta_n)) \leq \lambda^{n+1} d(\zeta_0, f(\zeta_0)).$$

Making $n \rightarrow \infty$ leads to $d(u, f(u)) = 0$. Hence, u is a fixed point of f in A . \square

Corollary 2.1. *It is clear from Eq. (4) and (7) that Theorem 2.1 still holds if Eq. (1) is replaced by*

$$d(\xi, f(\xi)) \leq \frac{1}{s + \delta} d(\rho, f(\rho)),$$

for $\rho, \xi \in A$ with $\rho < \xi$, where $\delta > 0$.

3 Fixed point theorems in partially complete spaces

Definition 3.1. Let $\{\zeta_n\}$ be a sequence in an ordered set (X, \leq) . An element $\zeta \in X$ is an upper bound of $\{\zeta_n\}$ if $\zeta_n \leq \zeta$, for each $n \in \mathbb{N}$. Such ζ is said to be a strict upper bound of $\{\zeta_n\}$ if $\zeta_n < \zeta$, for each $n \in \mathbb{N}$.

Still in the direction that the ordered partial b -metric space is not complete, we introduce the concept of partial completeness with respect to a subset A .

Definition 3.2. Let (X, d, \leq) be an ordered partial b -metric space ($s \geq 1$). For $A \subseteq X$, the triplet (X, d, \leq) is said partially complete with respect to A , if every strictly increasing Cauchy sequence in A has a strict upper bound in A , i.e., if $\{\zeta_n\}$ is a strictly increasing Cauchy sequence in A , then there exists $\zeta \in A$ such that $\zeta_n < \zeta$, for each $n \in \mathbb{N}$.

Example 3.1. Endow $X = \mathbb{Q}$ with the usual metric of \mathbb{R} and the natural order \leq . Note that (X, d, \leq) is an ordered partial b -metric space (with $s = 1$), but it is not complete. However, if we take $A = (1, 3] \cap \mathbb{Q}$, then (X, d, \leq) is partially complete with respect to A . Here, 3 is the strict upper bound for every strictly increasing Cauchy sequence in A .

The main objective of this concept is to present some fixed point results in noncomplete metric spaces. Here, we have dropped the contraction condition prescribed by Banach.

Remark 3.1. Let (X, \leq) be a totally ordered set. Then, every ordered complete metric space (X, d, \leq) is partially complete with respect to A , where A is any closed subset of X . The converse of the above statement is not true in general. It can be seen by taking $X = (0, 5)$ and $A = (1, 3]$. With the usual metric and the natural ordering \leq , X is partially complete with respect to A , but it is not complete. Also, if we take $A = [1, 3]$, then X is still not complete, but it is partially complete with respect to A .

Theorem 3.1. Let f be a self-map on an ordered partial b -metric space $(X, d, \leq, s \geq 1)$ such that the set $A = \{\zeta \in X : \zeta \leq f(\zeta)\}$ is nonempty and

$$d(\xi, f(\xi)) \leq \lambda d(\zeta, f(\zeta)), \tag{9}$$

for all $\zeta, \xi \in A$ with $\zeta < \xi$, where $\lambda \in \left[0, \frac{1}{s}\right)$. If further, $f(A) \subseteq A$ and (X, d, \leq) is partially complete with respect to A , then f has a fixed point in A .

Proof. Going through the same lines of proof of Theorem 2.1, we get a strictly increasing Cauchy sequence $\{\zeta_n\}$ in A such that

$$\zeta_{n+1} = f(\zeta_n), \tag{10}$$

$$d(\zeta_n, f(\zeta_n)) \leq \lambda^n d(\zeta_0, f(\zeta_0)). \tag{11}$$

Since (X, d, \leq) is partially complete with respect to A , there is $\zeta \in A$ such that $\zeta_n < \zeta$, for each $n \in \mathbb{N}$. Thus, from (9) and (11), we get

$$d(\zeta, f(\zeta)) \leq \lambda d(\zeta_n, f(\zeta_n)) \leq \lambda^{n+1} d(\zeta_0, f(\zeta_0)).$$

Since $s \geq 1$, we have that $d(\zeta, f(\zeta)) = 0$, i.e., ζ is the fixed point of f in A . □

Example 3.2. Let $X = \mathbb{Q}$ be endowed with $d(\zeta, \xi) = |\zeta - \xi| - \min\{\zeta, \xi\}$. Let \leq be the natural order \leq . Note that (X, d, \leq) is an ordered partial b -metric space (with $s = 2$), which is not complete. Consider $f : X \rightarrow X$ by

$$f(\zeta) = \begin{cases} \frac{\zeta+1}{2}, & 0 \leq \zeta \leq 1, \\ \zeta^2, & \text{otherwise.} \end{cases}$$

Since X is not a complete partial b -metric space, we cannot apply the previous results. Take $A = \left\{\frac{2^n - 1}{2^n} : n \geq 1\right\} \cup \{1\}$. Here, $f(A) \subseteq A$ and $\zeta \leq f(\zeta)$ for each $\zeta \in A$. Also, (X, d, \leq) is partially complete with respect to A . Therefore, it remains to prove that (9) is satisfied. Let $\zeta, \xi \in A$ be such that $\zeta < \xi$. We have $d(\zeta, f(\zeta)) = \frac{1-3\zeta}{2}$, $d(\xi, f(\xi)) = \frac{1-3\xi}{2}$ and $2\xi - \zeta \geq \frac{1}{3}$. For $\lambda = \frac{1}{2}$, one writes

$$\lambda d(\zeta, f(\zeta)) - d(\xi, f(\xi)) = \frac{1 - 3\zeta}{4} - \frac{1 - 3\xi}{2} = \frac{6\xi - 3\zeta - 1}{4} \geq 0.$$

Hence, all the conditions of Theorem 3.1 hold and f has a fixed point in A , which is, $u = 1$.

Example 3.3. Let $X = (-\infty, 0] \cap \mathbb{Q}$ be endowed with $d(\zeta, \xi) = |\zeta - \xi| - \min\{\zeta, \xi\}$ and \leq be the natural order \leq . Note that (X, d, \leq) is an ordered partial b -metric space (with $s = 2$), which is not complete. Take

$$f(\zeta) = \begin{cases} \frac{\zeta}{5}, & -1 < \zeta \leq 0, \\ \zeta^3, & \zeta \leq -1. \end{cases}$$

Take $A = \left\{ a_n : a_{n+1} = \frac{a_n}{5}, n \geq 0 \text{ with } a_0 = -\frac{1}{2} \right\} \cup \{0\}$. Then, $A = \left\{ -\frac{1}{2}, -\frac{1}{10}, -\frac{1}{50}, \dots \right\} \cup \{0\}$. Here, $f(A) \subseteq A$ and $\zeta \leq f(\zeta)$ for each $\zeta \in A$. Also, (X, d, \leq) is partially complete with respect to A . Therefore, it remains to prove that (9) is satisfied. Let $\zeta, \xi \in A$ such that $\zeta < \xi$. We have $d(\zeta, f(\zeta)) = -\frac{11\zeta}{5}$, $d(\xi, f(\xi)) = -\frac{11\xi}{5}$ and $4\xi - \zeta \geq 0$. For $\lambda = \frac{1}{4}$, one writes

$$\lambda d(\zeta, f(\zeta)) - d(\xi, f(\xi)) = -\frac{11\zeta}{20} + \frac{11\xi}{5} = \frac{11}{20}[4\xi - \zeta] \geq 0.$$

Thus, $d(\xi, f(\xi)) \leq \lambda d(\zeta, f(\zeta))$. Hence, all the conditions of Theorem 3.1 hold and f has a fixed point in A .

Corollary 3.1. Let f be a self-map on an ordered partial b -metric space $(X, d, \leq, s \geq 1)$. Assume that $f(X)$ is nonempty with $\zeta \leq f(\zeta)$, for each $\zeta \in f(X)$, and

$$d(\xi, f(\xi)) \leq \lambda d(\zeta, f(\zeta)), \tag{12}$$

for all $\zeta, \xi \in f(X)$ with $\zeta < \xi$, $\zeta \neq f(\zeta)$ and $\xi \neq f(\xi)$, where $\lambda \in \left[0, \frac{1}{s} \right)$. Furthermore, if (X, d, \leq) is partially complete with respect to $f(X)$, then f has a fixed point in $f(X)$.

Proof. As $f(X) \neq \emptyset$, let $\zeta_0 \in f(X)$. Note that $\zeta_0 \leq f(\zeta_0)$. If $\zeta_0 = f(\zeta_0)$, then we have nothing to prove. Otherwise, choose $\zeta_1 = f(\zeta_0) \in f(X)$. Then, again $\zeta_1 \leq f(\zeta_1)$. If $\zeta_1 = f(\zeta_1)$, then we are through. Otherwise, choose $\zeta_2 = f(\zeta_1) \in f(X)$. Continuing the same process, we get a strictly increasing sequence $\{\zeta_n\}$ in $f(X)$ such that

$$\zeta_{n+1} = f(\zeta_n). \tag{13}$$

Now, repeating the process as in Theorem 2.1, we get for each $n \in \mathbb{N}$,

$$d(\zeta_n, f(\zeta_n)) \leq \lambda^n d(\zeta_0, f(\zeta_0)). \tag{14}$$

Let $n, m \in \mathbb{N}$ with $n < m$. Using Lemma 2.1, (14) and the fact that $s \geq 1$, we get

$$d(\zeta_n, \zeta_m) \rightarrow 0, \text{ as } n, m \rightarrow \infty.$$

Thus, $\{\zeta_n\}$ is a strictly increasing Cauchy sequence in $f(X)$. But, (X, d, \leq) is partially complete with respect to $f(X)$, then $\{\zeta_n\}$ has a strict upper bound in $f(X)$, say ζ .

Since $\zeta_n, \zeta \in f(X)$, for each n with $\zeta_n < \zeta$, by (12) and (14), we have

$$d(\zeta, f(\zeta)) \leq \lambda^{n+1} d(\zeta_0, f(\zeta_0)).$$

As $s \geq 1$, so we have $d(\zeta, f(\zeta)) = 0$, so ζ is a fixed point of f in $f(X)$. □

Now, we present an example where the given partial b -metric space is not complete. In such a case, Corollary 3.1 can be applied and a fixed point result is obtained.

Example 3.4. Let $A = \left\{ a_n : a_{n+1} = \frac{a_n}{3}, n \geq 0 \text{ with } a_0 = -2 \right\}$ and $B = (0, 3] \cap \mathbb{Q}$. Then, $A = \left\{ -2, -\frac{2}{3}, -\frac{2}{9}, -\frac{2}{27}, -\frac{2}{81}, \dots \right\}$. Consider $X = A \cup B$. If we take \leq as the natural ordering \leq and $d(\zeta, \xi) = |\zeta - \xi|$, then (X, d, \leq) is an ordered partial b -metric space with $s = 1$. Choose

$$f(\zeta) = \begin{cases} \frac{\zeta}{3}, & \zeta \in A \\ \zeta, & \zeta \in A^c. \end{cases} \tag{15}$$

Here, $f(X) = X$. Also, every strictly increasing Cauchy sequence has a strict upper bound in $f(X)$. Thus, (X, d, \leq) is partially complete with respect to $f(X)$. Note that $\zeta \leq f(\zeta)$, for each $\zeta \in f(X)$. Let $\zeta, \xi \in f(X)$ with $\zeta < \xi$, $\zeta \neq f(\zeta)$ and $\xi \neq f(\xi)$. We have $\zeta, \xi \in A$, where $\zeta < \xi \Rightarrow 3\xi - \zeta \geq 0$. Furthermore, for such $\zeta, \xi \in A$, $d(\zeta, f(\zeta)) = -\frac{2\zeta}{3}$ and $d(\xi, f(\xi)) = -\frac{2\xi}{3}$. For $\lambda = \frac{1}{3}$, one writes

$$\lambda d(\zeta, f(\zeta)) - d(\xi, f(\xi)) = -\frac{2\zeta}{9} + \frac{2\xi}{3} = \frac{2}{9}(3\xi - \zeta) \geq 0.$$

Hence, $d(\xi, f(\xi)) \leq \lambda d(\zeta, f(\zeta))$ for all $\zeta, \xi \in f(X)$ with $\zeta < \xi$, $\zeta \neq f(\zeta)$ and $\xi \neq f(\xi)$. Thus, all the assumptions of Corollary 3.1 hold and there is at least one fixed point of f in $f(X)$.

4 A fixed point result with a continuous partial b -metric

In this section, we assume that the partial b -metric is continuous. Following [38], we state the following.

Theorem 4.1. Let (S_1, p) and (S_2, p) be two partial b -metric spaces, $A \subseteq S_1$ and $c \in A$. Given a function $f : A \rightarrow S_2$. Then, if f is continuous at c , then for each convergent sequence $\{\zeta_n\}$ to some $c \in A$, we have $\{f(\zeta_n)\}$ is convergent to $f(c)$.

Theorem 4.2. Let (X, d, \leq) be a complete ordered partial b -metric space ($s \geq 1$) and f be a continuous self-mapping on X . Let $A = \{\zeta \in X : \zeta \leq f(\zeta)\} \neq \emptyset$ such that $f(A) \subseteq A$. Assume that

$$d(\xi, f(\xi)) \leq \lambda d(\zeta, f(\zeta)), \tag{16}$$

for all $\zeta, \xi \in A$ with $\zeta < \xi$, where $\lambda \in \left[0, \frac{1}{s}\right)$. Suppose in addition that the partial b -metric is continuous, then f has a fixed point in X .

Proof. Since $A \neq \emptyset$, let $\zeta_0 \in A$. By definition of A , we have $\zeta_0 \leq f(\zeta_0)$. As $f(A) \subseteq A$, we have $\zeta_0, f(\zeta_0) \in A$. If $f(\zeta_0) = \zeta_0$, then we are done. Otherwise, choose $\zeta_1 \in A$ such that $\zeta_0 < f(\zeta_0) = \zeta_1$. Again, $\zeta_1 \leq f(\zeta_1)$. If $\zeta_1 = f(\zeta_1)$, then nothing to prove. Otherwise, we choose $\zeta_2 \in A$ such that $\zeta_1 < f(\zeta_1) = \zeta_2$. Continuing in this process, we get an increasing sequence $\{\zeta_n\}$ in A such that

$$\zeta_{n+1} = f(\zeta_n). \tag{17}$$

Also, $\{\zeta_n\}$ is a Cauchy sequence in X (see the proof of Theorem 2.1). Hence, there is at least one $\zeta \in X$ such that $\zeta_n \rightarrow \zeta$. Since f is continuous, we get $f(\zeta_n) \rightarrow f(\zeta)$. By taking $n \rightarrow \infty$, we obtain that

$$\lim_{n \rightarrow \infty} d(\zeta_n, f(\zeta_n)) = 0$$

As the partial b -metric is assumed to be continuous, we get that

$$d\left(\lim_{n \rightarrow \infty} \zeta_n, \lim_{n \rightarrow \infty} f(\zeta_n)\right) = 0,$$

so $d(\zeta, f(\zeta)) = 0$. Hence, $f(\zeta) = \zeta$. □

Example 4.1. Let $X = \mathbb{R}$ be endowed with $d(\zeta, \xi) = |\zeta - \xi|$. Let \leq be the natural order \leq . Note that (X, d, \leq) is an ordered partial b -metric space (with $s = 1$). Define $f : X \rightarrow X$ by

$$f(\zeta) = \begin{cases} \frac{\zeta}{3}, & \zeta \leq 0, \\ \zeta^2, & \zeta > 0. \end{cases}$$

Clearly, for $\zeta = 1$ and $\xi = 3$, we have $d(f\zeta, f\xi) > kd(\zeta, \xi)$ for each $k \in (0, 1)$. Consider $A = \{a_n : a_{n+1} = \frac{a_n}{3}, n \geq 0 \text{ with } a_0 = -\frac{1}{2}\}$. Note that $A = \{-\frac{1}{2}, -\frac{1}{6}, -\frac{1}{18}, \dots\}$. Here, $f(A) \subseteq A$ and $\zeta \leq f(\zeta)$ for each $\zeta \in A$. Then,

it remains to prove that (16) is satisfied. Let $\zeta, \xi \in A$ be such that $\zeta < \xi$. For $\lambda = \frac{1}{2} \in [0, 1)$, we have

$$\lambda d(\zeta, f(\zeta)) - d(\xi, f(\xi)) = -\frac{\zeta}{3} + \frac{2\xi}{3} = \frac{1}{3}[2\xi - \zeta] \geq 0.$$

Thus, $d(\xi, f(\xi)) \leq \lambda d(\zeta, f(\zeta))$. Hence, all the conditions of Theorem 4.2 hold and f has a fixed point in X .

5 Conclusion

We proved some fixed point theorems with a new approach. In this direction, we introduced partial completeness, \bar{f} -orbitally completeness, a new contraction type and other notions. The aim was to find the existence of a fixed point in the cases where many known results cannot work. The proof for uniqueness of a fixed point needs further explorations. It would be very interesting to analyze the existing literature in light of the results discussed in this paper. Particularly, if we replace Banach contraction by other newly introduced contractions, we may get further interesting results.

Acknowledgments: The publication of this article was funded by the Qatar National Library. The authors highly appreciate the efforts of referees and editor of this paper who helped us to improve it in several places.

References

- [1] S. Banach, *Sur operations dans les ensembles abstraits et leur application aux equations integrals*, Fund. Math. **3** (1922), 133–181.
- [2] I. A. Bakhtin, *The contraction principle in quasimetric spaces*, It. Funct. Anal. **30** (1989), 26–37.
- [3] S. G. Matthews, *Partial metric topology*, in: S. Andima et al. (Eds), Proc. 8th Summer Conference on General Topology and Applications, Annals of the New York Academy of Sciences, vol. 728, pp. 183–197.
- [4] A. C. M. Ran and M. C. B. Reurings, *A fixed point theorem in partial ordered sets and some applications to matrix equations*, Proc. Am. Math. Soc. **132** (2004), no. 5, 1435–1443.
- [5] J. J. Nieto and R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order **22** (2005), 223–239.
- [6] T. Suzuki, *A generalized Banach contraction principle that characterizes metric completeness*, Proc. Am. Math. Soc. **136** (2008), 1861–1869.
- [7] S. Shukla, *Partial b-metric spaces and fixed point theorems*, Mediterr. J. Math. **11** (2014), 703–711.
- [8] S. Aleksić, H. Huang, Z. D. Mitrović, and S. Radenović, *Remarks on some fixed point results in b-metric spaces*, J. Fixed Point Theory Appl. **20** (2018), no. 4, 2153–2163.
- [9] S. Aleksić, Z. D. Mitrović, and S. Radenović, *On some recent fixed point results for single and multi-valued mappings in b-metric spaces*, Fasc. Mathematici **61** (2018), 5–16.
- [10] S. Aleksić, T. Došenović, Z. D. Mitrović, and S. Radenović, *Remarks on common fixed point results for generalized α - ψ -contraction multivalued mappings in b-metric spaces*, Adv. Fixed Point Theory **9** (2019), no. 1, 1–16.

- [11] N. Tahat, H. Aydi, E. Karapinar, and W. Shatanawi, *Common fixed points for single-valued and multi-valued maps satisfying a generalized contraction in G -metric spaces*, Fixed Point Theory Appl. **2012** (2012), 48, DOI: 10.1186/1687-1812-2012-48.
- [12] H. Aydi, M. Abbas, and C. Vetro, *Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces*, Topology Appl. **159** (2012), 3234–3242.
- [13] H. Aydi, E. Karapinar, and P. Kumam, *A note on modified proof of Caristi's fixed point theorem on partial metric spaces*, J. Inequal. Appl. **2013** (2013), 355, DOI: 10.1186/1029-242X-2013-355.
- [14] H. Aydi, E. Karapinar, and W. Shatanawi, *Coupled fixed point results for (ψ, ϕ) -weakly contractive condition in ordered partial metric spaces*, Comput. Math. Appl. **62** (2011), 4449–4460.
- [15] H. Aydi, A. Felhi, E. Karapinar, and S. Sahmim, *A Nadler-type fixed point theorem in dislocated spaces and applications*, Miscolc Math. Notes **19** (2018), no. 1, 111–124.
- [16] S. Czerwik, *Contraction mappings in b -metric spaces*, Acta. Math. Univ. Ostrav. **1** (1993), no. 1, 5–11.
- [17] S. Chandok, V. Osturk, and S. Radenović, *On fixed point in context of b -metric spaces*, Mat. Vesnik **71** (2019), no. 1–2, 21–30.
- [18] Z. Mustafa, M. M. Jaradat, A. Ansari, B. Popović, and H. Jaradat, *C -class functions with new approach on coincidence point results for generalized (ψ, ϕ) -weakly contractions in ordered b -metric spaces*, SpringerPlus **5** (2016), no. 1, 802, DOI: 10.1186/s40064-016-2481-1.
- [19] Z. Mustafa, M. M. Jaradada, and H. Jaradat, *Some common fixed point results of graphs on b -metric space*, J. Nonlinear Sci. Appl. **9** (2016), no. 6, 4838–4851.
- [20] N. V. Dung and V. T. L. Hang, *Remarks on partial b -metric spaces and fixed point theorems*, Mat. Vesnik **69** (2017), no. 4, 231–240.
- [21] Z. Mustafa, H. Aydi, and E. Karapinar, *On common fixed points in G -metric spaces using $(E.A)$ property*, Comput. Mat. Appl. **6** (2012), 1944–1956.
- [22] N. Hussain and Z. D. Mitrović, *On multi-valued weak quasi-contractions in b -metric spaces*, J. Nonlinear Sci. Appl. **10** (2017), 3815–3823.
- [23] Z. Mustafa, J. Roshan, V. Parvaneh, and Z. Kadelburg, *Some common fixed point results in ordered partial b -metric space*, J. Inequal. Appl. **2013** (2013), 562, DOI: 10.1186/1029-242X-2013-562.
- [24] Z. Mustafa, J. Roshan, V. Parvaneh, and Z. Kadelburg, *Fixed point theorems for Weakly T -Chatterjea and weakly T -Kannan contractions in b -metric spaces*, J. Inequal. Appl. **2014** (2014), 46, DOI: 10.1186/1029-242X-2014-46.
- [25] E. Karapinar, R. P. Agarwal, and H. Aydi, *Interpolative Reich-Rus-Ciric type contractions on partial metric spaces*, Mathematics **6** (2018), 256, DOI: 10.3390/math6110256.
- [26] Q. H. Khan and T. Rashid, *Coupled coincidence point of ϕ -contraction type T -coupling in partial metric spaces*, J. Math. Anal. **9** (2018), no. 1, 136–149.
- [27] W. Kirk and N. Shahzad, *Fixed Point Theory in Distance Spaces*, Springer International Publishing, Switzerland, 2014.
- [28] G. S. Lin, *A note on partial b -metric spaces*, Mediterr. J. Math. **13** (2016), 1273–1276.
- [29] R. Miculescu and A. Mihail, *New fixed point theorems for set-valued contractions in b -metric spaces*, J. Fixed Point Theory Appl. **1** (2017), 2153–2163.
- [30] Z. D. Mitrović, S. Radenović, F. Vetro, and J. Vujaković, *Some remark on TAC-contractive mappings in b -metric spaces*, Mat. Vesnik **70** (2018), no. 2, 167–175.
- [31] Z. D. Mitrović, *A note on a Banach's fixed point theorem in b -rectangular metric space and b -metric space*, Math. Slovaca **68** (2018), no. 5, 1113–1116.
- [32] T. Rashid, Q. H. Khan, H. Aydi, H. Alsamir, and M. S. Noorani, *t -property of metric spaces and fixed point theorems*, Ital. J. Pure Appl. Math. **41** (2019), no. 1, 422–433.
- [33] T. Rashid, Q. H. Khan, and H. Aydi, *On strong coupled coincidence points of g -Couplings and an application*, J. Funct. Spaces **2018** (2018), 4034535, DOI: 10.1155/2018/4034535.
- [34] T. Rashid, N. Alharbi, Q. H. Khan, H. Aydi, and C. Özel, *Order-theoretic metrical coincidence theorems involving (ϕ, ψ) -contractions*, J. Math. Anal. **9** (2018), no. 5, 119–135.
- [35] J. Vujaković, H. Aydi, and S. Radenović, *Some remarks and new results in ordered partial b -metric spaces*, Mathematics **7** (2019), no. 4, 334, DOI: 10.3390/math7040334.
- [36] Z. Mustafa, V. Parvaneh, M. M. Jaradat, and Z. Kadelburg, *Extended rectangular b -metric spaces and some fixed point theorems for contractive mappings*, Symmetry **11** (2019), 594, DOI: 10.3390/sym11040594.
- [37] O. Popescu, *A partial generalization of Ćirić theorem*, J. Nonlinear Sci. Appl. **6** (2015), 152–161.
- [38] H. Aydi, W. Shatanawi, and C. Vetro, *On generalized weakly G -contraction mapping in G -metric spaces*, Comput. Math. Appl. **62** (2011), 4222–4229.