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b_2-Metric spaces and some fixed point theorems

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Abstract

The aim of this paper is to establish the structure of b_2-metric spaces, as a generalization of 2-metric spaces. Some fixed point results for various contractive-type mappings in the context of ordered b_2-metric spaces are presented. We also provide examples to illustrate the results presented herein, as well as an application to integral equations.

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1 Introduction

The concept of metric spaces has been generalized in many directions.

The notion of a b-metric space was studied by Czerwik in [1, 2] and many fixed point results were obtained for single and multivalued mappings by Czerwik and many other authors.

On the other hand, the notion of a 2-metric was introduced by Gähler in [3], having the area of a triangle in $\mathbb{R}^2$ as the inspirative example. Similarly, several fixed point results were obtained for mappings in such spaces. Note that, unlike many other generalizations of metric spaces introduced recently, 2-metric spaces are not topologically equivalent to metric spaces and there is no easy relationship between the results obtained in 2-metric and in metric spaces.

In this paper, we introduce a new type of generalized metric spaces, which we call b_2-metric spaces, as a generalization of both 2-metric and b-metric spaces. Then we prove some fixed point theorems under various contractive conditions in partially ordered b_2-metric spaces. These include Geraghty-type conditions, conditions using comparison functions and almost generalized weakly contractive conditions. We illustrate these results by appropriate examples, as well as an application to integral equations.

2 Mathematical preliminaries

The notion of a b-metric space was studied by Czerwik in [1, 2].

Definition 1 [1] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric on X if, for all $x, y, z \in X$, the following conditions hold:

(b_1) $d(x, y) = 0$ if and only if $x = y$,

(b_2) $d(x,y) = d(y,x)$,
(b_3) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

In this case, the pair (X,d) is called a b-metric space.

Note that a b-metric is not always a continuous function of its variables (see, *e.g.*, [4, Example 2]), whereas an ordinary metric is.

On the other hand, the notion of a 2-metric was introduced by Gähler in [3].

Definition 2 [3] Let X be a nonempty set and let $d : X^3 \to \mathbb{R}$ be a map satisfying the following conditions:

1. For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x,y,z) \neq 0$.
2. If at least two of three points x, y, z are the same, then $d(x,y,z) = 0$.
3. The symmetry: $d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x)$ for all $x, y, z \in X$.
4. The rectangle inequality: $d(x,y,z) \leq d(x,y,t) + d(y,z,t) + d(z,x,t)$ for all $x, y, z, t \in X$.

Then d is called a 2-metric on X and (X,d) is called a 2-metric space.

Definition 3 [3] Let (X,d) be a 2-metric space, $a, b \in X$ and $r \geq 0$. The set $B(a,b,r) = \{x \in X : d(a,b,x) < r\}$ is called a 2-ball centered at a and b with radius r.

The topology generated by the collection of all 2-balls as a subbasis is called a 2-metric topology on X.

Note that a 2-metric is not always a continuous function of its variables, whereas an ordinary metric is.

Remark 1

1. [5] It is straightforward from Definition 2 that every 2-metric is non-negative and every 2-metric space contains at least three distinct points.
2. A 2-metric $d(x,y,z)$ is sequentially continuous in each argument. Moreover, if a 2-metric $d(x,y,z)$ is sequentially continuous in two arguments, then it is sequentially continuous in all three arguments; see [6].
3. A convergent sequence in a 2-metric space need not be a Cauchy sequence; see [6].
4. In a 2-metric space (X,d), every convergent sequence is a Cauchy sequence if d is continuous; see [6].
5. There exists a 2-metric space (X,d) such that every convergent sequence in it is a Cauchy sequence but d is not continuous; see [6].

For some fixed point results on 2-metric spaces, the readers may refer to [5–15].

Now, we introduce new generalized metric spaces, called b_2-metric spaces, as a generalization of both 2-metric and b-metric spaces.

Definition 4 Let X be a nonempty set, $s \geq 1$ be a real number and let $d : X^3 \to \mathbb{R}$ be a map satisfying the following conditions:

1. For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x,y,z) \neq 0$.
2. If at least two of three points x, y, z are the same, then $d(x,y,z) = 0$.

3. The symmetry: $d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x)$ for all $x,y,z \in X$.
4. The rectangle inequality: $d(x,y,z) \leq s[d(x,y,t) + d(y,z,t) + d(z,x,t)]$ for all $x,y,z,t \in X$.

Then d is called a b_2-metric on X and (X,d) is called a b_2-metric space with parameter s.

Obviously, for $s = 1$, b_2-metric reduces to 2-metric.

Definition 5 Let $\{x_n\}$ be a sequence in a b_2-metric space (X,d).

1. $\{x_n\}$ is said to be b_2-convergent to $x \in X$, written as $\lim_n x_n = x$, if for all $a \in X$, $\lim_n d(x_n,x,a) = 0$.
2. $\{x_n\}$ is said to be a b_2-Cauchy sequence in X if for all $a \in X$, $\lim_n d(x_n,x_m,a) = 0$.
3. (X,d) is said to be b_2-complete if every b_2-Cauchy sequence is a b_2-convergent sequence.

The following are some easy examples of b_2-metric spaces.

Example 1 Let $X = [0,+\infty)$ and $d(x,y,z) = [xy + yz + zx]^p$ if $x \neq y \neq z \neq x$, and otherwise $d(x,y,z) = 0$, where $p \geq 1$ is a real number. Evidently, from convexity of function $f(x) = x^p$ for $x \geq 0$, then by Jensen inequality we have

$$(a + b + c)^p \leq 3^{p-1}\left(a^p + b^p + c^p\right).$$

So, one can obtain the result that (X,d) is a b_2-metric space with $s \leq 3^{p-1}$.

Example 2 Let a mapping $d : \mathbb{R}^3 \to [0,+\infty)$ be defined by

$$d(x,y,z) = \min\left\{|x-y|,|y-z|,|z-x|\right\}.$$

Then d is a 2-metric on $\mathbb{R}$, *i.e.*, the following inequality holds:

$$d(x,y,z) \leq d(x,y,t) + d(y,z,t) + d(z,x,t),$$

for arbitrary real numbers x, y, z, t. Using convexity of the function $f(x) = x^p$ on $[0,+\infty)$ for $p \geq 1$, we obtain that

$$d_p(x,y,z) = \left[\min\left\{|x-y|,|y-z|,|z-x|\right\}\right]^p$$

is a b_2-metric on $\mathbb{R}$ with $s < 3^{p-1}$.

Definition 6 Let (X,d) and (X',d') be two b_2-metric spaces and let $f : X \to X'$ be a mapping. Then f is said to be b_2-continuous at a point $z \in X$ if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in X$ and $d(z,x,a) < \delta$ for all $a \in X$ imply that $d'(fz,fx,a) < \varepsilon$. The mapping f is b_2-continuous on X if it is b_2-continuous at all $z \in X$.

Proposition 1 *Let (X,d) and (X',d') be two b_2-metric spaces. Then a mapping $f : X \to X'$ is b_2-continuous at a point $x \in X$ if and only if it is b_2-sequentially continuous at x; that is, whenever $\{x_n\}$ is b_2-convergent to x, $\{fx_n\}$ is b_2-convergent to $f(x)$.*

We will need the following simple lemma about the b_2-convergent sequences in the proof of our main results.

Lemma 1 *Let (X,d) be a b_2-metric space and suppose that $\{x_n\}$ and $\{y_n\}$ are b_2-convergent to x and y, respectively. Then we have*

$$\frac{1}{s^2}d(x,y,a) \le \liminf_{n\to\infty} d(x_n,y_n,a) \le \limsup_{n\to\infty} d(x_n,y_n,a) \le s^2 d(x,y,a),$$

for all a in X. In particular, if $y_n = y$ is constant, then

$$\frac{1}{s}d(x,y,a) \le \liminf_{n\to\infty} d(x_n,y,a) \le \limsup_{n\to\infty} d(x_n,y,a) \le sd(x,y,a),$$

for all a in X.

Proof Using the rectangle inequality in the given b_2-metric space, it is easy to see that

$$\begin{aligned} d(x,y,a) = d(x,a,y) &\le sd(x,a,x_n) + sd(a,y,x_n) + sd(y,x,x_n) \\ &\le sd(x,a,x_n) + s^2\bigl[d(a,y,y_n) + d(y,x_n,y_n) + d(x_n,a,y_n)\bigr] + sd(y,x,x_n) \end{aligned}$$

and

$$\begin{aligned} d(x_n,y_n,a) = d(x_n,a,y_n) &\le sd(x_n,a,x) + sd(a,y_n,x) + sd(y_n,x,x_n) \\ &\le sd(x_n,a,x) + s^2\bigl[d(a,y_n,y) + d(y_n,x,y) + d(x,a,y)\bigr] + sd(y_n,x,x_n). \end{aligned}$$

Taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality we obtain the desired result.

If $y_n = y$, then

$$d(x,y,a) \le sd(x,y,x_n) + sd(y,a,x_n) + sd(a,x,x_n)$$

and

$$d(x_n,y,a) \le sd(x_n,y,x) + sd(y,a,x) + sd(a,x_n,x). \qquad \square$$

3 Main results

3.1 Results under Geraghty-type conditions

In 1973, Geraghty [16] proved a fixed point result, generalizing the Banach contraction principle. Several authors proved later various results using Geraghty-type conditions. Fixed point results of this kind in b-metric spaces were obtained by Đukić *et al.* in [17].

Following [17], for a real number $s \ge 1$, let $\mathcal{F}_s$ denote the class of all functions $\beta : [0,\infty) \to [0,\frac{1}{s})$ satisfying the following condition:

$$\beta(t_n) \to \frac{1}{s} \quad \text{as } n \to \infty \quad \text{implies} \quad t_n \to 0 \quad \text{as } n \to \infty.$$

Theorem 1 *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b_2-metric d on X such that (X, d) is a b_2-complete b_2-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that*

$$sd(fx, fy, a) \le \beta\big(d(x, y, a)\big)M(x, y, a) \tag{3.1}$$

for all $a \in X$ and for all comparable elements $x, y \in X$, where

$$M(x, y, a) = \max\left\{d(x, y, a), \frac{d(x, fx, a)d(y, fy, a)}{1 + d(fx, fy, a)}\right\}.$$

If f is b_2-continuous, then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof Starting with the given x_0, put $x_n = f^n x_0$. Since $x_0 \preceq fx_0$ and f is an increasing function we obtain by induction that

$$x_0 \preceq fx_0 \preceq f^2 x_0 \preceq \cdots \preceq f^n x_0 \preceq f^{n+1} x_0 \preceq \cdots.$$

Step I: We will show that $\lim_n d(x_n, x_{n+1}, a) = 0$. Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, then by (3.1) we have

$$\begin{aligned} sd(x_n, x_{n+1}, a) = sd(fx_{n-1}, fx_n, a) &\le \beta\big(d(x_{n-1}, x_n, a)\big)M(x_{n-1}, x_n, a) \\ &\le \frac{1}{s}d(x_{n-1}, x_n, a) \le d(x_{n-1}, x_n, a), \end{aligned} \tag{3.2}$$

because

$$\begin{aligned} M(x_{n-1}, x_n, a) &= \max\left\{d(x_{n-1}, x_n, a), \frac{d(x_{n-1}, fx_{n-1}, a)d(x_n, fx_n, a)}{1 + d(fx_{n-1}, fx_n, a)}\right\} \\ &= \max\left\{d(x_{n-1}, x_n, a), \frac{d(x_{n-1}, x_n, a)d(x_n, x_{n+1}, a)}{1 + d(x_n, x_{n+1}, a)}\right\} \\ &= d(x_{n-1}, x_n, a). \end{aligned}$$

Therefore, the sequence $\{d(x_n, x_{n+1}, a)\}$ is decreasing. Then there exists $r \ge 0$ such that $\lim_n d(x_n, x_{n+1}, a) = r$. Suppose that $r > 0$. Then, letting $n \to \infty$, from (3.2) we have

$$\frac{1}{s}r \le sr \le \lim_n \beta\big(d(x_{n-1}, x_n, a)\big)r \le r.$$

So, we have $\lim_n \beta(d(x_{n-1}, x_n, a)) \ge \frac{1}{s}$ and since $\beta \in \mathcal{F}_s$ we deduce that $\lim_n d(x_{n-1}, x_n, a) = 0$ which is a contradiction. Hence, $r = 0$, that is,

$$\lim_n d(x_n, x_{n+1}, a) = 0. \tag{3.3}$$

Step II: As $\{d(x_n, x_{n+1}, a)\}$ is decreasing, if $d(x_{n-1}, x_n, a) = 0$, then $d(x_n, x_{n+1}, a) = 0$. Since from part 2 of Definition 4, $d(x_0, x_1, x_0) = 0$, we have $d(x_n, x_{n+1}, x_0) = 0$ for all $n \in \mathbb{N}$. Since

$d(x_{m-1}, x_m, x_m) = 0$, we have

$$d(x_n, x_{n+1}, x_m) = 0 \tag{3.4}$$

for all $n \geq m-1$. For $0 \leq n < m-1$, we have $m-1 \geq n+1$, and from (3.4) we have

$$d(x_{m-1}, x_m, x_{n+1}) = d(x_{m-1}, x_m, x_n) = 0. \tag{3.5}$$

It implies that

$$\begin{aligned} d(x_n, x_{n+1}, x_m) &\leq sd(x_n, x_{n+1}, x_{m-1}) + sd(x_{n+1}, x_m, x_{m-1}) + sd(x_m, x_n, x_{m-1}) \\ &= sd(x_n, x_{n+1}, x_{m-1}). \end{aligned}$$

Since $d(x_n, x_{n+1}, x_{n+1}) = 0$, from the above inequality, we have

$$d(x_n, x_{n+1}, x_m) \leq s^{m-n-1} d(x_n, x_{n+1}, x_{n+1}) = 0 \tag{3.6}$$

for all $0 \leq n < m-1$. From (3.4) and (3.6), we have

$$d(x_n, x_{n+1}, x_m) = 0 \tag{3.7}$$

for all $n, m \in \mathbb{N}$.

Now, for all $i, j, k \in \mathbb{N}$ with $i < j$, we have

$$d(x_{j-1}, x_j, x_i) = d(x_{j-1}, x_j, x_k) = 0. \tag{3.8}$$

Therefore, from (3.8) and triangular inequality

$$\begin{aligned} d(x_i, x_j, x_k) &\leq s\bigl[d(x_i, x_j, x_{j-1}) + d(x_j, x_k, x_{j-1}) + d(x_k, x_i, x_{j-1})\bigr] \\ &= sd(x_i, x_{j-1}, x_k) \leq \cdots \leq s^{j-i} d(x_i, x_i, x_k) = 0. \end{aligned}$$

This proves that for all $i, j, k \in \mathbb{N}$

$$d(x_i, x_j, x_k) = 0. \tag{3.9}$$

Step III: Now, we prove that the sequence $\{x_n\}$ is a b_2-Cauchy sequence. Using the rectangle inequality and by (3.1) we have

$$\begin{aligned} d(x_n, x_m, a) &\leq sd(x_n, x_m, x_{n+1}) + sd(x_m, a, x_{n+1}) + sd(a, x_n, x_{n+1}) \\ &\leq sd(x_n, x_{n+1}, x_m) + s^2\bigl[d(x_m, x_{m+1}, a) + d(x_{n+1}, x_{m+1}, a) \\ &\quad + d(x_m, x_{m+1}, x_{n+1})\bigr] + sd(x_n, x_{n+1}, a) \\ &\leq sd(x_n, x_{n+1}, x_m) + s^2 d(x_m, x_{m+1}, a) + s\beta\bigl(d(x_n, x_m, a)\bigr)M(x_n, x_m, a) \\ &\quad + s^2 d(x_m, x_{m+1}, x_{n+1}) + sd(x_n, x_{n+1}, a). \end{aligned}$$

Letting $m,n\to\infty$ in the above inequality and applying (3.3) and (3.7) we have

$$\lim_{m,n\to\infty} d(x_n,x_m,a) \le s \lim_{m,n\to\infty} \beta\bigl(d(x_n,x_m,a)\bigr) \lim_{m,n\to\infty} M(x_n,x_m,a). \tag{3.10}$$

Here

$$\begin{aligned} d(x_n,x_m,a) &\le M(x_n,x_m,a) \\ &= \max\left\{d(x_n,x_m,a), \frac{d(x_n,fx_n,a)d(x_m,fx_m,a)}{1+d(fx_n,fx_m,a)}\right\} \\ &= \max\left\{d(x_n,x_m,a), \frac{d(x_n,x_{n+1},a)d(x_m,x_{m+1},a)}{1+d(x_{n+1},x_{m+1},a)}\right\}. \end{aligned}$$

Letting $m,n\to\infty$ in the above inequality we get

$$\lim_{m,n\to\infty} M(x_n,x_m,a) = \lim_{m,n\to\infty} d(x_n,x_m,a). \tag{3.11}$$

Hence, from (3.10) and (3.11), we obtain

$$\lim_{m,n\to\infty} d(x_n,x_m,a) \le s \lim_{m,n\to\infty} \beta\bigl(d(x_n,x_m,a)\bigr) \lim_{m,n\to\infty} d(x_n,x_m,a). \tag{3.12}$$

Now we claim that $\lim_{m,n\to\infty} d(x_n,x_m,a) = 0$. If, to the contrary, $\lim_{m,n\to\infty} d(x_n,x_m,a) \neq 0$, then we get

$$\frac{1}{s} \le \lim_{m,n\to\infty} \beta\bigl(d(x_n,x_m,a)\bigr).$$

Since $\beta \in \mathcal{F}_s$ we deduce that

$$\lim_{m,n\to\infty} d(x_n,x_m,a) = 0, \tag{3.13}$$

which is a contradiction. Consequently, $\{x_n\}$ is a b_2-Cauchy sequence in X. Since (X,d) is b_2-complete, the sequence $\{x_n\}$ b_2-converges to some $z\in X$, that is, $\lim_n d(x_n,z,a) = 0$.

Step IV: Now, we show that z is a fixed point of f.

Using the rectangle inequality, we get

$$d(fz,z,a) \le sd(fz,fx_n,z) + sd(z,a,fx_n) + sd(a,fz,fx_n).$$

Letting $n\to\infty$ and using the continuity of f, we have $fz = z$. Thus, z is a fixed point of f.

Step V: Finally, suppose that the set of fixed point of f is well ordered. Assume, to the contrary, that u and v are two distinct fixed points of f. Then by (3.1), we have

$$\begin{aligned} sd(u,v,a) = sd(fu,fv,a) &\le \beta\bigl(d(u,v,a)\bigr)M(u,v,a) \\ &= \beta\bigl(d(u,v,a)\bigr)d(u,v,a) < \frac{1}{s}d(u,v,a), \end{aligned} \tag{3.14}$$

because

$$\begin{aligned}M(u,v,a) &= \max\left\{d(u,v,a), \frac{d(u,fu,a)d(v,fv,a)}{1+d(fu,fv,a)}\right\}\\ &= \max\{d(u,v,a),0\} = d(u,v,a).\end{aligned}$$

Thus, we get $sd(u,v,a) < \frac{1}{s}d(u,v,a)$, a contradiction. Hence, f has a unique fixed point. The converse is trivial. □

Note that the continuity of f in Theorem 1 can be replaced by certain property of the space itself.

Theorem 2 *Under the hypotheses of Theorem* 1, *without the b_2-continuity assumption on f, assume that whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$. Then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.*

Proof Repeating the proof of Theorem 1, we construct an increasing sequence $\{x_n\}$ in X such that $x_n \to z \in X$. Using the assumption on X we have $x_n \preceq z$. Now, we show that $z = fz$. By (3.1) and Lemma 1,

$$\begin{aligned}s\left[\frac{1}{s}d(z,fz,a)\right] &\le s\limsup_{n\to\infty} d(x_{n+1},fz,a)\\ &\le \limsup_{n\to\infty}\beta\bigl(d(x_n,z,a)\bigr)\limsup_{n\to\infty}M(x_n,z,a),\end{aligned}$$

where

$$\begin{aligned}\lim_{n\to\infty}M(x_n,z,a) &= \lim_n \max\left\{d(x_n,z,a), \frac{d(x_n,fx_n,a)d(z,fz,a)}{1+d(fx_n,fz,a)}\right\}\\ &= \lim_n \max\left\{d(x_n,z,a), \frac{d(x_n,x_{n+1},a)d(z,fz,a)}{1+d(x_{n+1},fz,a)}\right\} = 0 \quad \bigl(\text{see (3.3)}\bigr).\end{aligned}$$

Therefore, we deduce that $d(z,fz,a) \le 0$. As a is arbitrary, hence, we have $z = fz$.

The proof of uniqueness is the same as in Theorem 1. □

If in the above theorems we take $\beta(t) = r$, where $0 \le r < \frac{1}{s}$, then we have the following corollary.

Corollary 1 *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a b_2-metric d on X such that (X,d) is a b_2-complete b_2-metric space. Let $f: X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that for some r, with $0 \le r < \frac{1}{s}$,*

$$sd(fx,fy,a) \le rM(x,y,a)$$

holds for each $a \in X$ and all comparable elements $x, y \in X$, where

$$M(x,y,a) = \max\left\{d(x,y,a), \frac{d(x,fx,a)d(y,fy,a)}{1+d(fx,fy,a)}\right\}.$$

If f is continuous, or, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$ one has $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point. Additionally, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Corollary 2 *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b_2-metric d on X such that (X, d) is a b_2-complete b_2-metric space. Let $f: X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that*

$$d(fx, fy, a) \leq \alpha d(x, y, a) + \beta \frac{d(x, fx, a)d(y, fy, a)}{1 + d(fx, fy, a)}$$

for each $a \in X$ and all comparable elements $x, y \in X$, where $\alpha, \beta \geq 0$ and $\alpha + \beta \leq \frac{1}{s}$.

If f is continuous, or, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$ one has $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof Since

$$\begin{aligned} &\alpha d(x, y, a) + \beta \frac{d(x, fx, a)d(y, fy, a)}{1 + d(fx, fy, a)} \\ &\quad \leq (\alpha + \beta) \max\left\{d(x, y, a), \frac{d(x, fx, a)d(y, fy, a)}{1 + d(fx, fy, a)}\right\}. \end{aligned}$$

Putting $r = \alpha + \beta$, the conditions of Corollary 1 are satisfied and f has a fixed point. □

Example 3 Let $X = \{(\alpha, 0) : \alpha \in [0, +\infty)\} \cup \{(0, 2)\} \subset \mathbb{R}^2$ and let $d(x, y, z)$ denote the square of the area of triangle with vertices $x, y, z \in X$, *e.g.*,

$$d\big((\alpha, 0), (\beta, 0), (0, 2)\big) = (\alpha - \beta)^2.$$

It is easy to check that d is a b_2-metric with parameter $s = 2$. Introduce an order $\preceq$ in X by

$$(\alpha, 0) \preceq (\beta, 0) \quad \iff \quad \alpha \geq \beta,$$

with all other pairs of distinct points in X incomparable.

Consider the mapping $f: X \to X$ given by

$$f(\alpha, 0) = \left(\frac{\alpha}{3}, 0\right) \quad \text{for } \alpha \in [0, +\infty) \text{ and } f(0, 2) = (0, 2),$$

and the function $\beta \in \mathcal{F}_2$ given as

$$\beta(t) = \frac{1 + t}{2 + 4t} \quad \text{for } t \in [0, +\infty).$$

Then f is an increasing mapping with $(\alpha, 0) \preceq f(\alpha, 0)$ for each $\alpha \geq 0$. If $\{x_n\} = \{(\alpha_n, 0)\}$ is a nondecreasing sequence in X, converging to some $z = (\gamma, 0)$, then $(\alpha_n, 0) \preceq (\gamma, 0)$ for all $n \in \mathbb{N}$. Finally, in order to check the contractive condition (3.1), only the case when $x = (\alpha, 0)$, $y = (\beta, 0)$, $a = (0, 2)$ is nontrivial. But then $d(x, y, a) = (\alpha - \beta)^2$ and

$$\begin{aligned} sd(fx, fy, a) &= 2d\left(\left(\tfrac{1}{3}\alpha, 0\right), \left(\tfrac{1}{3}\beta, 0\right), (0, 2)\right) = 2 \cdot \frac{1}{9}(\alpha - \beta)^2 \leq \frac{1}{4}(\alpha - \beta)^2 \\ &\leq \beta\big(d(x, y, a)\big)d(x, y, a) \leq \beta\big(d(x, y, a)\big)M(x, y, a). \end{aligned}$$

All the conditions of Theorem 2 are satisfied and f has two fixed points, $(0, 0)$ and $(0, 2)$. Note that the condition (stated in Theorem 1 and Theorem 2) for the uniqueness of a fixed point is here not satisfied.

3.2 Results using comparison functions

Let Ψ denote the family of all nondecreasing and continuous functions $\psi : [0, \infty) \to [0, \infty)$ such that $\lim_n \psi^n(t) = 0$ for all $t > 0$, where ψ^n denotes the nth iterate of ψ. It is easy to show that, for each $\psi \in \Psi$, the following are satisfied:

(a) $\psi(t) < t$ for all $t > 0$;
(b) $\psi(0) = 0$.

Theorem 3 *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b_2-metric d on X such that (X, d) is a b_2-complete b_2-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that*

$$sd(fx, fy, a) \leq \psi\big(M(x, y, a)\big), \tag{3.15}$$

where

$$M(x, y, a) = \max\left\{d(x, y, a), \frac{d(x, fx, a)d(y, fy, a)}{1 + d(fx, fy, a)}\right\},$$

for some $\psi \in \Psi$ and for all elements $x, y, a \in X$, with x, y comparable. If f is b_2-continuous, then f has a fixed point. In addition, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof Since $x_0 \preceq fx_0$ and f is an increasing function, we obtain by induction that

$$x_0 \preceq fx_0 \preceq f^2x_0 \preceq \cdots \preceq f^nx_0 \preceq f^{n+1}x_0 \preceq \cdots.$$

By letting $x_n = f^nx_0$, we have

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots.$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} = fx_{n_0}$ and so we have nothing to prove. Hence, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Step I. We will prove that $\lim_n d(x_n, x_{n+1}, a) = 0$. Using condition (3.15), we obtain

$$d(x_{n+1}, x_n, a) \leq sd(x_{n+1}, x_n, a) = sd(fx_n, fx_{n-1}, a) \leq \psi\big(M(x_n, x_{n-1}, a)\big).$$

Here

$$\begin{aligned} M(x_{n-1},x_n,a) &= \max\left\{d(x_{n-1},x_n,a), \frac{d(x_{n-1},fx_{n-1},a)d(x_n,fx_n,a)}{1+d(fx_{n-1},fx_n,a)}\right\} \\ &= d(x_{n-1},x_n,a). \end{aligned}$$

Hence,

$$d(x_n,x_{n+1},a) \leq sd(x_n,x_{n+1},a) \leq \psi\bigl(d(x_{n-1},x_n,a)\bigr) < d(x_{n-1},x_n,a). \tag{3.16}$$

By induction, we get

$$d(a,x_{n+1},x_n) \leq \psi\bigl(d(a,x_n,x_{n-1})\bigr) \leq \psi^2\bigl(d(a,x_{n-1},x_{n-2})\bigr) \leq \cdots \leq \psi^n\bigl(d(a,x_1,x_0)\bigr).$$

As $\psi \in \Psi$, we conclude that

$$\lim_n d(x_n,x_{n+1},a) = 0. \tag{3.17}$$

From similar arguments as in Theorem 1, since $\{d(x_n,x_{n+1},a)\}$ is decreasing, we can conclude that

$$d(x_i,x_j,x_k) = 0 \tag{3.18}$$

for all $i,j,k \in \mathbb{N}$.

Step II. We will prove that $\{x_n\}$ is a b_2-Cauchy sequence. Suppose the contrary. Then there exist $a \in X$ and $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \quad \text{and} \quad d(x_{m_i},x_{n_i},a) \geq \varepsilon. \tag{3.19}$$

This means that

$$d(x_{m_i},x_{n_i-1},a) < \varepsilon. \tag{3.20}$$

From (3.19) and using the rectangle inequality, we get

$$\varepsilon \leq d(x_{m_i},x_{n_i},a) \leq sd(x_{m_i},x_{n_i},x_{m_i+1}) + sd(x_{m_i+1},x_{n_i},a) + sd(x_{m_i+1},x_{m_i},a).$$

Taking the upper limit as $i \to \infty$, from (3.17) and (3.18) we get

$$\frac{\varepsilon}{s} \leq \limsup_{i\to\infty} d(x_{m_i+1},x_{n_i},a). \tag{3.21}$$

From the definition of $M(x,y,a)$ we have

$$\begin{aligned} M(x_{m_i},x_{n_i-1},a) &= \max\left\{d(x_{m_i},x_{n_i-1},a), \frac{d(x_{m_i},fx_{m_i},a)d(x_{n_i-1},fx_{n_i-1},a)}{1+d(fx_{m_i},fx_{n_i-1},a)}\right\} \\ &= \max\left\{d(x_{m_i},x_{n_i-1},a), \frac{d(x_{m_i},a,x_{m_i+1})d(x_{n_i-1},a,x_{n_i})}{1+d(x_{m_i+1},x_{n_i},a)}\right\} \end{aligned}$$

and if $i \to \infty$, by (3.17) and (3.20) we have

$$\limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}, a) \leq \varepsilon.$$

Now, from (3.15) we have

$$sd(x_{m_i+1}, x_{n_i}, a) = sd(fx_{m_i}, fx_{n_i-1}, a) \leq \psi\bigl(M(x_{m_i}, x_{n_i-1}, a)\bigr).$$

Again, if $i \to \infty$ by (3.21) we obtain

$$\varepsilon = s \cdot \frac{\varepsilon}{s} \leq s \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i}, a) \leq \psi(\varepsilon) < \varepsilon,$$

which is a contradiction. Consequently, $\{x_n\}$ is a b_2-Cauchy sequence in X. Therefore, the sequence $\{x_n\}$ b_2-converges to some $z \in X$, that is, $\lim_n d(x_n, z, a) = 0$ for all $a \in X$.

Step III. Now we show that z is a fixed point of f.

Using the rectangle inequality, we get

$$d(z, fz, a) \leq sd(z, fz, fx_n) + sd(fx_n, fz, a) + sd(fx_n, z, a).$$

Letting $n \to \infty$ and using the continuity of f, we get

$$d(z, fz, a) \leq 0.$$

Hence, we have $fz = z$. Thus, z is a fixed point of f.

The uniqueness of the fixed point can be proved in the same manner as in Theorem 1. □

Theorem 4 *Under the hypotheses of Theorem* 3, *without the b_2-continuity assumption on f, assume that whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$. Then f has a fixed point. In addition, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.*

Proof Following the proof of Theorem 3, we construct an increasing sequence $\{x_n\}$ in X such that $x_n \to z \in X$. Using the given assumption on X we have $x_n \preceq z$. Now, we show that $z = fz$. By (3.15) we have

$$sd(fz, x_n, a) = sd(fz, fx_{n-1}, a) \leq \psi\bigl(M(z, x_{n-1}, a)\bigr), \tag{3.22}$$

where

$$M(z, x_{n-1}, a) = \max\left\{d(z, x_{n-1}, a), \frac{d(z, fz, a)d(x_{n-1}, fx_{n-1}, a)}{1 + d(fz, fx_{n-1}, a)}\right\}.$$

Letting $n \to \infty$ in the above relation, we get

$$\limsup_{n\to\infty} M(z, x_{n-1}, a) = 0. \tag{3.23}$$

Again, taking the upper limit as $n \to \infty$ in (3.22) and using Lemma 1 and (3.23) we get

$$\begin{aligned} s\left[\frac{1}{s}d(z,fz,a)\right] &\le s\limsup_{n\to\infty} d(x_n,fz,a) \\ &\le \limsup_{n\to\infty} \psi\bigl(M(z,x_{n-1},a)\bigr) = 0. \end{aligned}$$

So we get $d(z,fz,a) = 0$, *i.e.*, $fz = z$. □

Corollary 3 *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a b_2-metric d on X such that (X,d) is a b_2-complete b_2-metric space. Let $f: X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that*

$$sd(fx,fy,a) \le rM(x,y,a),$$

where $0 \le r < 1$ and

$$M(x,y,a) = \max\left\{d(x,y,a), \frac{d(x,fx,a)d(y,fy,a)}{1+d(fx,fy,a)}\right\},$$

for all elements $x,y,a \in X$ with x, y comparable. If f is continuous, or, whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Example 4 Let $X = \{A,B,C,D\}$ be ordered by $A \succeq B \succeq C$, with all other pairs of distinct points incomparable. Define $d: X^3 \to \mathbb{R}$ by

$$d(A,B,C) = 0, \qquad d(A,B,D) = 1, \qquad d(A,C,D) = 4, \qquad d(B,C,D) = 2,$$

with symmetry in all variables and with $d(x,y,z) = 0$ when at least two of the arguments are equal. Then it is easy to check that (X,d) is a complete b_2-metric space with $s = \frac{4}{3}$.

Consider the mapping $f: X \to X$ given as

$$f = \begin{pmatrix} A & B & C & D \\ A & A & B & D \end{pmatrix}$$

and a comparison function $\psi(t) = \frac{2}{3}t$. Then f is a nondecreasing mapping w.r.t. $\preceq$ and there exists $x_0 \in X$ such that $x_0 \preceq fx_0$. The only nontrivial cases for checking the contractive condition (3.15) are when $a = D$ and $x = A$, $y = C$ or $x = B$, $y = C$ (or *vice versa*). Then we have

$$sd(fA,fC,D) = \frac{4}{3}d(A,B,D) = \frac{4}{3} < \frac{2}{3}\cdot 4 = \psi(4) = \psi\bigl(d(A,C,D)\bigr) \le \psi\bigl(M(A,C,D)\bigr),$$

resp.

$$sd(fB,fC,D) = \frac{4}{3}d(A,B,D) = \frac{4}{3} = \frac{2}{3}\cdot 2 = \psi(2) = \psi\bigl(d(B,C,D)\bigr) \le \psi\bigl(M(B,C,D)\bigr).$$

Hence, all the conditions of Theorem 3 are fulfilled. The mapping f has two fixed points (A and D).

3.3 Results for almost generalized weakly contractive mappings

Berinde in [18–21] initiated the concept of almost contractions and obtained many interesting fixed point theorems. Results with similar conditions were obtained, *e.g.*, in [22] and [23]. In this section, we define the notion of almost generalized $(\psi,\varphi)_{s,a}$-contractive mapping and we prove some new results. In particular, we extend Theorems 2.1, 2.2 and 2.3 of Ćirić *et al.* in [24] to the setting of b_2-metric spaces.

Recall that Khan *et al.* introduced in [25] the concept of an altering distance function as follows.

Definition 7 [25] A function $\varphi:[0,+\infty)\to[0,+\infty)$ is called an altering distance function, if the following properties hold:

1. φ is continuous and nondecreasing.
2. $\varphi(t)=0$ if and only if $t=0$.

Let (X,d) be a b_2-metric space and let $f:X\to X$ be a mapping. For $x,y,a\in X$, set

$$M_a(x,y)=\max\left\{d(x,y,a),d(x,fx,a),d(y,fy,a),\frac{d(x,fy,a)+d(y,fx,a)}{2s}\right\}$$

and

$$N_a(x,y)=\min\{d(x,fx,a),d(x,fy,a),d(y,fx,a),d(y,fy,a)\}.$$

Definition 8 Let (X,d) be a b_2-metric space. We say that a mapping $f:X\to X$ is an almost generalized $(\psi,\varphi)_{s,a}$-contractive mapping if there exist $L\geq 0$ and two altering distance functions ψ and φ such that

$$\psi\bigl(sd(fx,fy,a)\bigr)\leq\psi\bigl(M_a(x,y)\bigr)-\varphi\bigl(M_a(x,y)\bigr)+L\psi\bigl(N_a(x,y)\bigr) \tag{3.24}$$

for all $x,y,a\in X$.

Now, let us prove our new result.

Theorem 5 *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a b_2-metric d on X such that (X,d) is a b_2-complete b_2-metric space. Let $f:X\to X$ be a continuous mapping, nondecreasing with respect to $\preceq$. Suppose that f satisfies condition* (3.24), *for all elements $x,y,a\in X$, with x, y comparable. If there exists $x_0\in X$ such that $x_0\preceq fx_0$, then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.*

Proof Starting with the given x_0, define a sequence $\{x_n\}$ in X such that $x_{n+1}=fx_n$, for all $n\geq 0$. Since $x_0\preceq fx_0=x_1$ and f is nondecreasing, we have $x_1=fx_0\preceq x_2=fx_1$, and by induction

$$x_0\preceq x_1\preceq\cdots\preceq x_n\preceq x_{n+1}\preceq\cdots.$$

If $x_n = x_{n+1}$, for some $n \in \mathbb{N}$, then $x_n = fx_n$ and hence x_n is a fixed point of f. So, we may assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. By (3.24), we have

$$\begin{aligned} \psi\big(d(x_n, x_{n+1}, a)\big) &\le \psi\big(sd(x_n, x_{n+1}, a)\big) \\ &= \psi\big(sd(fx_{n-1}, fx_n, a)\big) \\ &\le \psi\big(M_a(x_{n-1}, x_n)\big) - \varphi\big(M_a(x_{n-1}, x_n)\big) + L\psi\big(N_a(x_{n-1}, x_n)\big), \end{aligned} \tag{3.25}$$

where

$$\begin{aligned} &M_a(x_{n-1}, x_n) \\ &\quad= \max\left\{d(x_{n-1}, x_n, a), d(x_{n-1}, fx_{n-1}, a), d(x_n, fx_n, a), \right. \\ &\quad\quad \left. \frac{d(x_{n-1}, fx_n, a) + d(x_n, fx_{n-1}, a)}{2s}\right\} \\ &\quad= \max\left\{d(x_{n-1}, x_n, a), d(x_n, x_{n+1}, a), \frac{d(x_{n-1}, x_{n+1}, a)}{2s}\right\} \\ &\quad\le \max\left\{d(x_{n-1}, x_n, a), d(x_n, x_{n+1}, a), \right. \\ &\quad\quad \left. \frac{d(x_{n-1}, x_{n+1}, x_n) + d(x_{n+1}, a, x_n) + d(a, x_{n-1}, x_n)}{2}\right\} \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} &N_a(x_{n-1}, x_n) \\ &\quad= \min\big\{d(x_{n-1}, fx_{n-1}, a), d(x_{n-1}, fx_n, a), d(x_n, fx_{n-1}, a), d(x_n, fx_n, a)\big\} \\ &\quad= \min\big\{d(x_{n-1}, x_n, a), d(x_{n-1}, x_{n+1}, a), 0, d(x_n, x_{n+1}, a)\big\} = 0. \end{aligned} \tag{3.27}$$

From (3.25)–(3.27) and the properties of ψ and φ, we get

$$\begin{aligned} &\psi\big(d(x_n, x_{n+1}, a)\big) \\ &\quad\le \psi\left(\max\left\{d(x_{n-1}, x_n, a), d(x_n, x_{n+1}, a), \right.\right. \\ &\quad\quad \left.\left. \frac{d(x_{n-1}, x_{n+1}, x_n) + d(x_{n+1}, a, x_n) + d(a, x_{n-1}, x_n)}{2}\right\}\right) \\ &\quad\quad - \varphi\left(\max\left\{d(x_{n-1}, x_n, a), d(x_n, x_{n+1}, a), \frac{d(x_{n-1}, x_{n+1}, a)}{2s}\right\}\right). \end{aligned} \tag{3.28}$$

If

$$\begin{aligned} &\max\left\{d(x_{n-1}, x_n, a), d(x_n, x_{n+1}, a), \frac{d(x_{n-1}, x_{n+1}, x_n) + d(x_{n+1}, a, x_n) + d(a, x_{n-1}, x_n)}{2}\right\} \\ &\quad= d(x_n, x_{n+1}, a), \end{aligned}$$

then by (3.28) we have

$$\begin{aligned}\psi\bigl(d(x_n,x_{n+1},a)\bigr) &\le \psi\bigl(d(x_n,x_{n+1},a)\bigr)\\ &\quad - \varphi\left(\max\left\{d(x_{n-1},x_n,a), d(x_n,x_{n+1},a), \frac{d(x_{n-1},x_{n+1},a)}{2s}\right\}\right),\end{aligned}$$

which gives a contradiction.

If $d(x_{n-1},x_{n+1},x_n) = 0$, then

$$\begin{aligned}&\max\left\{d(x_{n-1},x_n,a), d(x_n,x_{n+1},a), \frac{d(x_{n-1},x_{n+1},x_n) + d(x_{n+1},a,x_n) + d(a,x_{n-1},x_n)}{2}\right\}\\ &\quad = d(x_{n-1},x_n,a),\end{aligned}$$

therefore (3.28) becomes

$$\begin{aligned}\psi\bigl(d(x_n,x_{n+1},a)\bigr) &\le \psi\bigl(d(x_n,x_{n-1},a)\bigr)\\ &\quad - \varphi\left(\max\left\{d(x_{n-1},x_n,a), d(x_n,x_{n+1},a), \frac{d(x_{n-1},x_{n+1},a)}{2s}\right\}\right)\\ &\le \psi\bigl(d(x_n,x_{n-1},a)\bigr). \end{aligned} \tag{3.29}$$

Thus, $\{d(x_n,x_{n+1},a) : n \in \mathbb{N} \cup \{0\}\}$ is a nonincreasing sequence of positive numbers. Hence, there exists $r \ge 0$ such that

$$\lim_n d(x_n,x_{n+1},a) = r.$$

Letting $n \to \infty$ in (3.29), we get

$$\psi(r) \le \psi(r) - \varphi\left(\max\left\{r, r, \lim_n \frac{d(x_{n-1},x_{n+1},a)}{2s}\right\}\right) \le \psi(r).$$

Therefore,

$$\varphi\left(\max\left\{r, r, \lim_n \frac{d(x_{n-1},x_{n+1},a)}{2s}\right\}\right) = 0,$$

and hence $r = 0$. Thus, we have

$$\lim_n d(x_n,x_{n+1},a) = 0, \tag{3.30}$$

for each $a \in X$.

Note that if $d(x_{n-1},x_{n+1},x_n) \ne 0$ and

$$\begin{aligned}&\max\left\{d(x_{n-1},x_n,a), d(x_n,x_{n+1},a), \frac{d(x_{n-1},x_{n+1},x_n) + d(x_{n+1},a,x_n) + d(a,x_{n-1},x_n)}{2}\right\}\\ &\quad = \frac{d(x_{n-1},x_{n+1},x_n) + d(x_{n+1},a,x_n) + d(a,x_{n-1},x_n)}{2}.\end{aligned}$$

Then, by (3.28) and taking $a = x_{n-1}$, we have

$$\begin{aligned}\psi\left(d(x_n,x_{n+1},x_{n-1})\right) &\le \psi\left(\frac{d(x_{n-1},x_{n+1},x_n)+d(x_{n+1},x_{n-1},x_n)+d(x_{n-1},x_{n-1},x_n)}{2}\right) \\ &\quad - \varphi\left(\max\left\{d(x_{n-1},x_n,x_{n-1}), d(x_n,x_{n+1},x_{n-1}), \frac{d(x_{n-1},x_{n+1},x_{n-1})}{2s}\right\}\right),\end{aligned}$$

which gives $d(x_{n-1},x_{n+1},x_n) = 0$, a contradiction.

Next, we show that $\{x_n\}$ is a b_2-Cauchy sequence in X. For this purpose, we use the following relation (see (3.9) and (3.18)):

$$d(x_i,x_j,x_k) = 0, \tag{3.31}$$

for all $i,j,k \in N$ (note that this can obtained as $\{d(x_n,x_{n+1},a) : n \in \mathbb{N} \cup \{0\}\}$ is a nonincreasing sequence of positive numbers).

Suppose the contrary, that is, $\{x_n\}$ is not a b_2-Cauchy sequence. Then there exist $a \in X$ and $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i, \qquad d(x_{m_i},x_{n_i},a) \ge \varepsilon. \tag{3.32}$$

This means that

$$d(x_{m_i},x_{n_i-1},a) < \varepsilon. \tag{3.33}$$

Using (3.33) and taking the upper limit as $i \to \infty$, we get

$$\limsup_{n\to\infty} d(x_{m_i},x_{n_i-1},a) \le \varepsilon. \tag{3.34}$$

On the other hand, we have

$$d(x_{m_i},x_{n_i},a) \le sd(x_{m_i},x_{n_i},x_{m_i+1}) + sd(x_{n_i},a,x_{m_i+1}) + sd(a,x_{m_i},x_{m_i+1}).$$

Using (3.30), (3.31), (3.32), and taking the upper limit as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \le \limsup_{n\to\infty} d(x_{m_i+1},x_{n_i},a). \tag{3.35}$$

Again, using the rectangular inequality, we have

$$d(x_{m_i+1},x_{n_i-1},a) \le sd(x_{m_i+1},x_{n_i-1},x_{m_i}) + sd(x_{n_i-1},a,x_{m_i}) + sd(a,x_{m_i+1},x_{m_i}),$$

and

$$d(x_{m_i},x_{n_i},a) \le sd(x_{m_i},x_{n_i},x_{n_i-1}) + sd(x_{n_i},a,x_{n_i-1}) + sd(a,x_{m_i},x_{n_i-1}).$$

Taking the upper limit as $i \to \infty$ in the first inequality above, and using (3.30), (3.31), and (3.34) we get

$$\limsup_{n\to\infty} d(x_{m_i+1}, x_{n_i-1}, a) \leq \varepsilon s. \tag{3.36}$$

Similarly, taking the upper limit as $i \to \infty$ in the second inequality above, and using (3.30), (3.31), and (3.33), we get

$$\limsup_{n\to\infty} d(x_{m_i}, x_{n_i}, a) \leq \varepsilon s. \tag{3.37}$$

From (3.24), we have

$$\begin{aligned} &\psi\big(sd(x_{m_i+1}, x_{n_i}, a)\big) \\ &\quad = \psi\big(sd(fx_{m_i}, fx_{n_i-1}, a)\big) \\ &\quad \leq \psi\big(M_a(x_{m_i}, x_{n_i-1})\big) - \varphi\big(M_a(x_{m_i}, x_{n_i-1})\big) + L\psi\big(N_a(x_{m_i}, x_{n_i-1})\big), \end{aligned} \tag{3.38}$$

where

$$\begin{aligned} &M_a(x_{m_i}, x_{n_i-1}) \\ &\quad = \max\biggl\{d(x_{m_i}, x_{n_i-1}, a), d(x_{m_i}, fx_{m_i}, a), d(x_{n_i-1}, fx_{n_i-1}, a), \\ &\qquad \frac{d(x_{m_i}, fx_{n_i-1}, a) + d(fx_{m_i}, x_{n_i-1}, a)}{2s}\biggr\} \\ &\quad = \max\biggl\{d(x_{m_i}, x_{n_i-1}, a), d(x_{m_i}, x_{m_i+1}, a), d(x_{n_i-1}, x_{n_i}, a), \\ &\qquad \frac{d(x_{m_i}, x_{n_i}, a) + d(x_{m_i+1}, x_{n_i-1}, a)}{2s}\biggr\}, \end{aligned} \tag{3.39}$$

and

$$\begin{aligned} &N_a(x_{m_i}, x_{n_i-1}) \\ &\quad = \min\bigl\{d(x_{m_i}, fx_{m_i}, a), d(x_{m_i}, fx_{n_i-1}, a), d(x_{n_i-1}, fx_{m_i}, a), d(x_{n_i-1}, fx_{n_i-1}, a)\bigr\} \\ &\quad = \min\bigl\{d(x_{m_i}, x_{m_i+1}, a), d(x_{m_i}, x_{n_i}, a), d(x_{n_i-1}, x_{m_i+1}, a), d(x_{n_i-1}, x_{n_i}, a)\bigr\}. \end{aligned} \tag{3.40}$$

Taking the upper limit as $i \to \infty$ in (3.39) and (3.40) and using (3.30), (3.34), (3.36), and (3.37), we get

$$\begin{aligned} &\limsup_{n\to\infty} M_a(x_{m_i-1}, x_{n_i-1}) \\ &\quad = \max\biggl\{\limsup_{n\to\infty} d(x_{m_i}, x_{n_i-1}, a), 0, 0, \\ &\qquad \frac{\limsup_{n\to\infty} d(x_{m_i}, x_{n_i}, a) + \limsup_{n\to\infty} d(x_{m_i+1}, x_{n_i-1}, a)}{2s}\biggr\} \\ &\quad \leq \max\biggl\{\varepsilon, \frac{\varepsilon s + \varepsilon s}{2s}\biggr\} = \varepsilon. \end{aligned} \tag{3.41}$$

So, we have

$$\limsup_{n\to\infty} M_a(x_{m_i-1}, x_{n_i-1}) \le \varepsilon, \tag{3.42}$$

and

$$\limsup_{n\to\infty} N_a(x_{m_i}, x_{n_i-1}) = 0. \tag{3.43}$$

Now, taking the upper limit as $i \to \infty$ in (3.38) and using (3.35), (3.42), and (3.43) we have

$$\begin{aligned}\psi\left(s\cdot\frac{\varepsilon}{s}\right) &\le \psi\left(s\limsup_{n\to\infty} d(x_{m_i+1}, x_{n_i}, a)\right) \\ &\le \psi\left(\limsup_{n\to\infty} M_a(x_{m_i}, x_{n_i-1})\right) - \liminf_{n\to\infty}\varphi\left(M_a(x_{m_i}, x_{n_i-1})\right) \\ &\le \psi(\varepsilon) - \varphi\left(\liminf_{n\to\infty} M_a(x_{m_i}, x_{n_i-1})\right),\end{aligned}$$

which further implies that

$$\varphi\left(\liminf_{n\to\infty} M_a(x_{m_i}, x_{n_i-1})\right) = 0,$$

so $\liminf_{n\to\infty} M_a(x_{m_i}, x_{n_i-1}) = 0$, a contradiction to (3.32). Thus, $\{x_{n+1} = fx_n\}$ is a b_2-Cauchy sequence in X.

As X is a b_2-complete space, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$, that is,

$$\lim_n x_{n+1} = \lim_n fx_n = u.$$

Now, using continuity of f and the rectangle inequality, we get

$$d(u, fu, a) \le sd(u, fu, fx_n) + sd(fu, a, fx_n) + sd(a, u, fx_n).$$

Letting $n \to \infty$, we get

$$d(u, fu, a) \le s\lim_n d(u, fu, fx_n) + s\lim_n d(fu, a, fx_n) + s\lim_{n\to\infty} d(a, u, fx_n) = 0.$$

Therefore, we have $fu = u$. Thus, u is a fixed point of f.

The uniqueness of fixed point can be proved as in Theorem 1. □

Note that the continuity of f in Theorem 5 can be replaced by a property of the space.

Theorem 6 *Under the hypotheses of Theorem* 5, *without the continuity assumption on f, assume that whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x \in X$, one has $x_n \preceq x$, for all $n \in \mathbb{N}$. Then f has a fixed point in X. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.*

Proof Following similar arguments to those given in the proof of Theorem 5, we construct an increasing sequence $\{x_n\}$ in X such that $x_n \to u$, for some $u \in X$. Using the assumption on X, we have $x_n \preceq u$, for all $n \in \mathbb{N}$. Now, we show that $fu = u$. By (3.24), we have

$$\begin{aligned}\psi\big(sd(x_{n+1},fu,a)\big) &= \psi\big(sd(fx_n,fu,a)\big)\\ &\leq \psi\big(M_a(x_n,u)\big) - \varphi\big(M_a(x_n,u)\big) + L\psi\big(N_a(x_n,u)\big),\end{aligned} \tag{3.44}$$

where

$$\begin{aligned}&M_a(x_n,u)\\ &\quad= \max\left\{d(x_n,u,a), d(x_n,fx_n,a), d(u,fu,a), \frac{d(x_n,fu,a)+d(fx_n,u,a)}{2s}\right\}\\ &\quad= \max\left\{d(x_n,u,a), d(x_n,x_{n+1},a), d(u,fu,a), \frac{d(x_n,fu,a)+d(x_{n+1},u,a)}{2s}\right\}\end{aligned} \tag{3.45}$$

and

$$\begin{aligned}N_a(x_n,u) &= \min\big\{d(x_n,fx_n,a), d(x_n,fu,a), d(u,fx_n,a), d(u,fu,a)\big\}\\ &= \min\big\{d(x_n,x_{n+1},a), d(x_n,fu,a), d(u,x_{n+1},a), d(u,fu,a)\big\}.\end{aligned} \tag{3.46}$$

Letting $n \to \infty$ in (3.45) and (3.46) and using Lemma 1, we get

$$\begin{aligned}\frac{\frac{1}{s}d(u,fu,a)}{2s} &\leq \liminf_{n\to\infty} M_a(x_n,u) \leq \limsup_{n\to\infty} M_a(x_n,u)\\ &\leq \max\left\{d(u,fu,a), \frac{sd(u,fu,a)}{2s}\right\} = d(u,fu,a),\end{aligned} \tag{3.47}$$

and

$$N_a(x_n,u) \to 0.$$

Again, taking the upper limit as $i \to \infty$ in (3.44) and using Lemma 1 and (3.47) we get

$$\begin{aligned}\psi\big(d(u,fu,a)\big) &= \psi\left(s\cdot\frac{1}{s}d(u,fu,a)\right) \leq \psi\left(s\limsup_{n\to\infty} d(x_{n+1},fu,a)\right)\\ &\leq \psi\left(\limsup_{n\to\infty} M_a(x_n,u)\right) - \liminf_{n\to\infty}\varphi\big(M_a(x_n,u)\big)\\ &\leq \psi\big(d(u,fu,a)\big) - \varphi\left(\liminf_{n\to\infty} M_a(x_n,u)\right).\end{aligned}$$

Therefore, $\varphi(\liminf_{n\to\infty} M_a(x_n,u)) \leq 0$, equivalently, $\liminf_{n\to\infty} M_a(x_n,u) = 0$. Thus, from (3.47) we get $u = fu$ and hence u is a fixed point of f. □

Corollary 4 *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a b_2-metric d on X such that (X,d) is a b_2-complete b_2-metric space. Let $f : X \to X$ be a nondecreasing* *continuous mapping with respect to* $\preceq$. *Suppose that there exist* $k \in [0,1)$ *and* $L \geq 0$ *such that*

$$d(fx,fy,a) \leq \frac{k}{s}\max\left\{d(x,y,a),d(x,fx,a),d(y,fy,a),\frac{d(x,fy,a)+d(y,fx,a)}{2s}\right\} + \frac{L}{s}\min\bigl\{d(x,fx,a),d(y,fx,a)\bigr\},$$

for all elements $x,y,a \in X$ *with* x, y *comparable. If there exists* $x_0 \in X$ *such that* $x_0 \preceq fx_0$, *then* f *has a fixed point. Moreover, the set of fixed points of* f *is well ordered if and only if* f *has one and only one fixed point.*

Proof Follows from Theorem 5 by taking $\psi(t) = t$ and $\varphi(t) = (1-k)t$, for all $t \in [0,+\infty)$. □

Corollary 5 *Under the hypotheses of Corollary* 4, *without the continuity assumption of* f, *let for any nondecreasing sequence* $\{x_n\}$ *in* X *such that* $x_n \to x \in X$ *we have* $x_n \preceq x$, *for all* $n \in \mathbb{N}$. *Then* f *has a fixed point in* X.

4 An application to integral equations

As an application of our results, inspired by [26], we will consider the following integral equation:

$$x(t) = h(t) + \int_0^T g(t,s)F\bigl(s,x(s)\bigr)\,ds, \quad t \in I = [0,T]. \tag{4.1}$$

Consider the set $X = C_{\mathbb{R}}(I)$ of all real continuous functions on I, ordered by the natural relation

$$x \preceq y \quad \Longleftrightarrow \quad x(t) \leq y(t) \quad \text{for all } t \in I,$$

and take arbitrary real $p > 1$. We will use the following assumptions.

(I) $h : I \to \mathbb{R}$, $g : I \times \mathbb{R} \to [0,+\infty)$ and $F : I \times \mathbb{R} \to \mathbb{R}$ are continuous functions;

(II) for $x,y \in X$,

$$x \preceq y \quad \Longrightarrow \quad \int_0^T g(\cdot,s)F\bigl(s,x(s)\bigr)\,ds \leq \int_0^T g(\cdot,s)F\bigl(s,y(s)\bigr)\,ds;$$

(III) for some $0 \leq r < 1$ and all $x,y,a \in X$, with x and y comparable (w.r.t. $\preceq$),

$$\begin{aligned} 3^{p-1}\Biggl[\max_{0\leq t\leq T}\min\Biggl\{&\left|\int_0^T g(t,s)\bigl[F\bigl(s,x(s)\bigr)-F\bigl(s,y(s)\bigr)\bigr]\,ds\right|, \\ &\left|h(t)+\int_0^T g(t,s)F\bigl(s,y(s)\bigr)\,ds - a(t)\right|, \\ &\left|h(t)+\int_0^T g(t,s)F\bigl(s,x(s)\bigr)\,ds - a(t)\right|\Biggr\}\Biggr]^p \\ \leq r\Bigl[\max_{0\leq t\leq T}\min\bigl\{&\bigl|x(t)-y(t)\bigr|,\bigl|y(t)-a(t)\bigr|,\bigl|x(t)-a(t)\bigr|\bigr\}\Bigr]^p; \end{aligned}$$

(IV) there exists $x_0 \in X$ such that $x_0(t) \leq h(t) + \int_0^T g(t,s)F(s,x_0(s))\,ds$ for all $t \in I$.
Let $d : X \times X \times X \to [0,\infty)$ be defined by

$$d(x,y,z) = \Big[\max_{0\leq t\leq T} \min\big\{\big|x(t)-y(t)\big|, \big|y(t)-z(t)\big|, \big|x(t)-z(t)\big|\big\}\Big]^p.$$

Then (X,d) is a b_2-complete b_2-metric space, with $s \leq 3^{p-1}$ (similarly as in Example 2). We have the following result.

Theorem 7 *Let the functions h, g, F satisfy conditions* (I)-(IV) *and let the space* $(X,\preceq,d)$ *satisfy the requirement that if $\{x_n\}$ is a sequence in X, nondecreasing w.r.t. $\preceq$, and converging* (*in d*) *to some $u \in X$, then $x_n \preceq u$ for all $n \in \mathbb{N}$. Then the integral equation* (4.1) *has a solution in X.*

Proof Define the mapping $f : X \to X$ by

$$fx(t) = h(t) + \int_0^T g(t,s)F\big(s,x(s)\big)\,ds, \quad t \in I.$$

Then all the conditions of Corollary 3 are fulfilled. In particular, condition (III) implies that, for all $x,y,a \in X$, with x, y comparable, we have

$$sd(fx,fy,a) \leq 3^{p-1}d(fx,fy,a) \leq rd(x,y,a) \leq rM(x,y,a).$$

Hence, using Corollary 3, we conclude that there exists a fixed point $x \in X$ of f, which is obviously a solution of (4.1). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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