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SOME MAJORIZATION INTEGRAL INEQUALITIES FOR FUNCTIONS DEFINED ON RECTANGLES VIA STRONG CONVEXITY

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Abstract. In this paper, we have extended some integral majorization types and generalized Favard's inequalities from functions defined on intervals to functions defined on rectangles via strong convexity and apply the results to establish some new integral inequalities for functions defined on rectangles.

1. Introduction

Hardy *et al.* [11] introduced the notion of majorization and showed that a necessary and sufficient condition that $u \prec v$ is that there exists a doubly stochastic matrix $\tilde{\text{P}}$ such that u=v$\tilde{\text{P}}$. Schur [24] proved that the eigen values majorize the diagonal elements of a positive semidefinite Hermitian matrix. Majorization theory has interesting applications in different fields of mathematics, such as linear algebra, geometry, probability, statistics, group theory, optimization, etc. In 2012, Niezgada [20] extended the Hardy–Littlewood–Polya theorem on majorization from convex functions to invex ones and considered some variants for pseudo invex and quasi invex functions. For more details, we can refer to [1, 4, 5, 6, 9, 13, 15, 22, 24, 29, 28].

In 1969, Karamardian [12] introduced the concept of strongly convex function. However, there are references citing Polyak [23] has introduced strongly convex functions as a generalization of convex functions, see [16, 21]. Karamardian [12] showed that every bidifferentiable function is strongly convex if and only if its Hessian matrix is strongly positive definite. For more details, one can refer to [16, 18, 19, 21, 25].

Adil Khan *et al.* [7] introduced a new class of functions known as coordinate strongly convex function. It is well known that a twice differentiable function $\psi : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ is coordinate strongly convex with respect to $\mu_1, \mu_2 > 0$ on Δ if and only if the partial mappings $\psi_y : [a, b] \to \mathbb{R}$ defined by $\psi_y(u) = \psi(u, y)$ and $\psi_x : [c, d] \to \mathbb{R}$ defined by $\psi_x(v) = \psi(x, v)$, satisfied $\psi_y''(u) \geq 2\mu_1$ and $\psi_x''(v) \geq 2\mu_2$ for all $u \in [a, b]$ and $v \in [c, d]$ (see, [7]). Further, Wu *et al.* [27] established some defined versions of majorization inequality involving twice differentiable convex

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functions by using Taylor's theorem with mean value form of the remainder and also given some interesting applications. In particular, many important inequalities can be found in the literature [2, 3, 8, 17]

Recently, Zaheer Ullah *et al.* [28] established a monotonicity property for the function involving the strongly convex function and proved the classical majorization theorem by using strongly convex functions for majorized n-tuples. Zaheer Ullah *et al.* [29] obtained integral majorization type and generalized Favard's inequalities for the class of strongly convex functions. Further, Wu *et al.* [26] established some majorization integral inequalities and Favard type inequalities for functions defined on rectangles.

The main purpose of this paper is to extend several integral majorization type and generalized Favard's inequalities from functions defined on intervals to functions defined on rectangles via strong convexity. The results obtained in this paper are the generalizations of the results given in [29, 26].

1.1. **Preliminaries.** Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. We denote the usual inner product by $\langle .,. \rangle$ and for $x \in \mathbb{R}^n$, $\|.\|$ denote the norm defined by $\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$.

Definition 1. [12] *A function $\psi : X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be strongly convex on a convex set $X \subseteq \mathbb{R}^n$ if there exist a constant $\mu > 0$ such that*

$$\psi(tx + (1-t)y) \leq t\psi(x) + (1-t)\psi(y) - \mu t(1-t)\|y - x\|^2 \tag{1.1}$$

for any $x, y \in X$ and $t \in [0,1]$.

From (1.1), we clearly see that

$$\psi(x) - \psi(y) \geq \nabla_+\psi(y)(x-y) + \mu\|y-x\|^2 \quad \text{for any } x, y \in X, \tag{1.2}$$

where

$$\nabla\psi(y)(x-y) = \left\langle \frac{\partial\psi_+(y)}{\partial y}, (x-y) \right\rangle, \qquad \frac{\partial\psi_+(y)}{\partial y} = \left(\frac{\partial\psi_+(y)}{\partial y_1}, \frac{\partial\psi_+(y)}{\partial y_2}, ..., \frac{\partial\psi_+(y)}{\partial y_n} \right),$$

$x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$.

Definition 2. [7] *A function $\psi : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is said to be coordinate strongly convex if the partial mappings $\psi_y : [a,b] \to \mathbb{R}$ defined as $\psi_y(u) = \psi(u,y)$ for all $y \in [c,d]$ and $\psi_x : [c,d] \to \mathbb{R}$ defined as $\psi_x(v) = \psi(x,v)$ for all $x \in [a,b]$ are strongly convex.*

A vector $\mathbf{x} = (x_1, x_2, ..., x_n)$ is said to be majorized by a vector $\mathbf{y} = (y_1, y_2, ..., y_n)$, in symbol $y \succ x$, if $x_1 \geq ... \geq x_n$, $y_1 \geq ... \geq y_n$, $\sum_{i=1}^{m} y_i \geq \sum_{i=1}^{m} x_i$, $m = 1, 2, ..., n-1$ and $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i$. It means that the sum of m largest entries of x does not exceed the sum of m largest entries of y; for all $m = 1, 2, ..., n$.

Definition 3. [22] *Let f and g be two decreasing real-valued integrable functions on the interval $[a,b]$. Then f is said to majorize g, in symbol, $f \succ g$, if the inequality*

$$\int_a^x g(u)du \leq \int_a^x f(u)du \quad \textit{holds for all } \ x \in [a,b)$$

and

$$\int_a^b g(u)du = \int_a^b f(u)du.$$

Theorem 1.1. [10] *Let f be a nonnegative continuous concave function on $[a,b]$, not identically zero, and let ψ be a convex function on $[0,2\bar{f}]$, where*

$$\bar{f} = \frac{2}{b-a}\int_a^b f(u)du.$$

Then,

$$\frac{1}{b-a}\int_a^b \psi\{f(u)\}du \le \int_0^1 \psi(s\bar{f}).$$

Lemma 1.2. [7] *Every strongly convex function $\psi : \Delta = [a,b]\times[c,d] \to \mathbb{R}$ is coordinate strongly convex, but the converse is not true in general.*

Lemma 1.3. [14] *Let ϑ be a weight function on $[a,b]$.*

(a) *If l is decreasing function on $[a,b]$, then*

$$\int_a^b l(u)\vartheta(u)du \int_a^x \vartheta(u)du \le \int_a^x l(u)\vartheta(u)du \int_a^b \vartheta(u)du \quad \text{for all } x \in [a,b].$$

(b) *If l is increasing function on $[a,b]$, then*

$$\int_a^x l(u)\vartheta(u)du \int_a^b \vartheta(u)du \le \int_a^b l(u)\vartheta(u)du \int_a^x \vartheta(u)du \quad \text{for all } x \in [a,b].$$

Lemma 1.4. [29] *Let g be a real-valued function defined on $[a,b]$. Then the following statements are true.*

(a) *If g be a strongly concave function with modulus μ, then*
(i) *the function $P_1(u) = g(u)/(u-a) - \mu u$ is decreasing on $(a,b]$;*
(ii) *the function $Q_1(u) = g(u)/(b-u) + \mu u$ is increasing on $[a,b)$.*
(b) *If g be a strongly convex function with modulus μ, then*
(i) *the function $P_1(u) = g(u)/(u-a) - \mu u$ is increasing on $(a,b]$, if $g(a)=0$;*
(ii) *the function $Q_1(u) = g(u)/(b-u) + \mu u$ is decreasing on $[a,b)$, if $g(b)=0$.*

Theorem 1.5. [29] *Let $\mu > 0, \psi : [0,\infty) \to \mathbb{R}$ be a continuous strongly convex function with modulus μ, and let f, g and w be three positive and integrable functions defined on $[a,b]$ such that*

$$\int_a^x g(u)w(u)du \le \int_a^x f(u)w(u)du \quad \text{for all } x \in [a,b) \tag{1.3}$$

and

$$\int_a^b g(u)w(u)du = \int_a^b f(u)w(u)du. \tag{1.4}$$

Then the following statements are true.

(a) *If g is decreasing on $[a,b]$, then we have*

$$\int_a^b \psi\{f(u)\}w(u)du \ge \int_a^b \psi\{g(u)\}w(u)du + \mu\int_a^b \{g(u)-f(u)\}^2 w(u)du. \tag{1.5}$$

(b) *If f is increasing on $[a,b]$, then we have*

$$\int_a^b \psi\{g(u)\}w(u)du \ge \int_a^b \psi\{f(u)\}w(u)du + \mu\int_a^b \{g(u)-f(u)\}^2 w(u)du. \tag{1.6}$$

2. Main Results

In this section, first, we prove some majorization integral inequalities for functions defined on rectangles via strong convexity.

Theorem 2.1. *Let w and ρ be positive continuous functions on $[a,b]$ and $[c,d]$, respectively, and let f, g and h, k be positive differentiable functions on $[a,b]$ and $[c,d]$ respectively. Suppose that $\psi : [0,\infty)\times[0,\infty) \to \mathbb{R}$ is a strongly convex function with modulus $\mu > 0$ and that*

$$\int_a^x g(u)w(u)du \le \int_a^x f(u)w(u)du \quad \text{for all } x \in [a,b),$$

$$\int_c^y k(v)\rho(v)dv \le \int_c^y h(v)\rho(v)dv \quad \text{for all } y \in [c,d),$$

$$\int_a^b g(u)w(u)du = \int_a^b f(u)w(u)du$$

and

$$\int_c^d k(v)\rho(v)dv = \int_c^d h(v)\rho(v)dv.$$

(a) *If g and k are decreasing functions on $[a,b]$ and $[c,d]$ respectively, then*

$$\int_a^b \int_c^d \psi(g(u),k(v))w(u)\rho(v)dudv \le \int_a^b \int_c^d \psi(f(u),h(v))w(u)\rho(v)dudv - \mu \int_a^b \int_c^d \|(g(u),k(v)) - (f(u),h(v))\|^2 w(u)\rho(v)dudv.$$

(b) *If f and h are increasing functions on $[a,b]$ and $[c,d]$ respectively, then*

$$\int_a^b \int_c^d \psi(f(u),h(v))w(u)\rho(v)dudv \le \int_a^b \int_c^d \psi(g(u),k(v))w(u)\rho(v)dudv - \mu \int_a^b \int_c^d \|(g(u),k(v)) - (f(u),h(v))\|^2 w(u)\rho(v)dudv.$$

Proof. (a) By the definition of strong convexity, we have

$$\psi(x,y) - \psi(w,z) \ge \langle \nabla_+\psi(w,z), (x-w, y-z)\rangle + \mu\|(w,z) - (x,y)\|^2 \quad \text{for all } (x,y),(w,z) \in [0,\infty)\times[0,\infty),$$

that is,

$$\psi(x,y) - \psi(w,z) \ge \frac{\partial\psi_+(w,z)}{\partial w}(x-w) + \frac{\partial\psi_+(w,z)}{\partial z}(y-z) + \mu\|(w,z)-(x,y)\|^2 \quad \text{for all } (x,y),(w,z) \in [0,\infty)\times[0,\infty). \tag{2.1}$$

Putting $x = f(u), y = h(v), w = g(u)$ and $z = k(v)$ in (2.1), we get

$$\psi(f(u),h(v)) - \psi(g(u),k(v)) \ge \frac{\partial\psi_+(g(u),k(v))}{\partial g(u)}(f(u)-g(u)) + \frac{\partial\psi_+(g(u),k(v))}{\partial k(v)}(h(v)-k(v)) + \mu\|((g(u),k(v)) - (f(u),h(v))\|^2.$$

Suppose $\xi_v^1(u) = \frac{\partial \psi_+(\alpha,\beta)}{\partial \alpha}\Big|_{\alpha=g(u),\beta=k(v)}$, $\xi_v^2(u) = \frac{\partial \psi_+(\alpha,\beta)}{\partial \beta}\Big|_{\alpha=g(u),\beta=k(v)}$, $\xi_v^3(u) = \frac{\partial^2 \psi_+(\alpha,\beta)}{\partial^2 \alpha}\Big|_{\alpha=g(u),\beta=k(v)}$, $\xi_v^4(u) = \frac{\partial^2 \psi_+(\alpha,\beta)}{\partial^2 \beta}\Big|_{\alpha=g(u),\beta=k(v)}$. Then, we get

$$\begin{aligned}\psi(f(u),h(v)) - \psi(g(u),k(v)) \geq \xi_v^1(u)(f(u)-g(u)) + \xi_v^2(u)(h(v)-k(v)) \\ + \mu\|((g(u),k(v)) - (f(u),h(v))\|^2. \qquad (2.2)\end{aligned}$$

Assume that,

$$F(x) = \int_a^x (f(u)-g(u))w(u)du \quad \text{for all } x \in [a,b]$$

and

$$G(y) = \int_c^y (h(v)-k(v))\rho(v)dv \quad \text{for all } y \in [c,d].$$

From the assumptions of Theorem 2.1, we have
$F(x) \geq 0,\ G(y) \geq 0$ for all $x \in [a,b],\ y \in [c,d]$ and $F(a) = F(b) = G(c) = G(d) = 0$. Multiplying both sides by $w(u)\rho(v)$ in (2.2), we have

$$\begin{aligned}[\psi(f(u),h(v)) - \psi(g(u),k(v))]w(u)\rho(v) \geq \xi_v^1(u)[(f(u)-g(u))]w(u)\rho(v) \\ + \xi_v^2(u)[(h(v)-k(v))]w(u)\rho(v) \\ + \mu\|(g(u),k(v)) - (f(u),h(v))\|^2 w(u)\rho(v). \\ (2.3)\end{aligned}$$

Integrating both sides (2.3), we have

$$\begin{aligned}&\int_a^b \int_c^d [\psi(f(u),h(v)) - \psi(g(u),k(v))]w(u)\rho(v)dudv \\ &\geq \int_a^b \int_c^d \xi_v^1(u)[(f(u)-g(u))]w(u)\rho(v)dudv \\ &\quad + \int_a^b \int_c^d \xi_v^2(u)[(h(v)-k(v))]w(u)\rho(v)dudv \\ &\quad + \mu \int_a^b \int_c^d \|(g(u),k(v)) - (f(u),h(v))\|^2 w(u)\rho(v)dudv. \qquad (2.4)\end{aligned}$$

Using Fubini's theorem in above inequality, we get

$$\begin{aligned}&\int_a^b \int_c^d [\psi(f(u),h(v)) - \psi(g(u),k(v))]w(u)\rho(v)dudv \\ &\geq \int_c^d \rho(v)\left[\int_a^b \xi_v^1(u)dF(u)\right]dv \\ &\quad + \int_a^b w(u)\left[\int_c^d \xi_v^2(u)dG(v)\right]du \\ &\quad + \mu \int_a^b \int_c^d \|(g(u),k(v)) - (f(u),h(v))\|^2 w(u)\rho(v)dudv. \qquad (2.5)\end{aligned}$$

This implies,

$$\begin{aligned}
&\int_a^b\int_c^d[\psi(f(u),h(v))-\psi(g(u),k(v))]w(u)\rho(v)dudv\\
&\geq\int_c^d\rho(v)\left[\xi_v^1(u)F(u)\big|_a^b-\int_a^b\xi_v^3(u)g'(u)F(u)du\right]dv\\
&\quad+\int_a^b w(u)\left[\xi_v^2(u)G(v)\big|_c^d-\int_c^d\xi_v^4(u)k'(v)G(v)dv\right]du\\
&\quad+\mu\int_a^b\int_c^d\|(g(u),k(v))-(f(u),h(v))\|^2w(u)\rho(v)dudv, \qquad (2.6)
\end{aligned}$$

which yields,

$$\begin{aligned}
&\int_a^b\int_c^d[\psi(f(u),h(v))-\psi(g(u),k(v))]w(u)\rho(v)dudv\\
&\geq-\int_c^d\int_a^b\xi_v^3(u)g'(u)F(u)\rho(v)dudv-\int_a^b\int_c^d\xi_v^4(u)k'(v)G(v)w(u)dvdu\\
&\quad+\mu\int_a^b\int_c^d\|(g(u),k(v))-(f(u),h(v))\|^2w(u)\rho(v)dudv. \qquad (2.7)
\end{aligned}$$

From Lemma 1.2, ψ is coordinate strongly convex function, therefore $\xi_v^3(u)\geq 0,\ \xi_v^4(u)\geq 0$.
Since, g and k are decreasing functions, therefore $g'(u)\leq 0$ and $k'(v)\leq 0$.
Thus, it follows that

$$-\int_c^d\int_a^b\xi_v^3(u)g'(u)F(u)\rho(v)dudv\geq 0 \qquad (2.8)$$

and

$$-\int_a^b\int_c^d\xi_v^4(u)k'(v)G(v)w(u)dvdu\geq 0. \qquad (2.9)$$

From (2.7), (2.8) and (2.9), we get

$$\begin{aligned}
&\int_a^b\int_c^d[\psi(f(u),h(v))-\psi(g(u),k(v))]w(u)\rho(v)dudv\\
&-\mu\int_a^b\int_c^d\|(g(u),k(v))-(f(u),h(v))\|^2w(u)\rho(v)dudv\geq 0. \qquad (2.10)
\end{aligned}$$

This implies,

$$\begin{aligned}
\int_a^b\int_c^d\psi(g(u),k(v))w(u)\rho(v)dudv\leq&\int_a^b\int_c^d\psi(f(u),h(v))w(u)\rho(v)dudv\\
&-\mu\int_a^b\int_c^d\|(g(u),k(v))-(f(u),h(v))\|^2w(u)\rho(v)dudv.
\end{aligned}$$

(b) The proof of Theorem 2.1(b) is similar to Theorem 2.1(a).

□

Theorem 2.2. *Let w and ρ be positive continuous functions on $[a,b]$ and $[c,d]$, respectively, and let f,g and h,k be positive differentiable functions on $[a,b]$ and*

$[c,d]$, respectively. Suppose that $\psi:[0,\infty)\times[0,\infty)\to\mathbb{R}$ is a strongly convex function with modulus μ.

(a) *Let $\frac{f}{g}$ and $\frac{h}{k}$ be decreasing functions on $[a,b]$ and $[c,d]$, respectively. If f and h are increasing functions on $[a,b]$ and $[c,d]$, respectively, then*

$$\begin{aligned}&\int_a^b\int_c^d \psi\left(\frac{f(u)}{\int_a^b f(u)w(u)du},\frac{h(v)}{\int_c^d h(v)\rho(v)dv}\right)w(u)\rho(v)dudv\\ &\leq \int_a^b\int_c^d \psi\left(\frac{g(u)}{\int_a^b g(u)w(u)du},\frac{k(v)}{\int_c^d k(v)\rho(v)dv}\right)w(u)\rho(v)dudv\\ &-\mu\int_a^b\int_c^d\left\|\left(\frac{g(u)}{\int_a^b g(u)w(u)du},\frac{k(v)}{\int_c^d k(v)\rho(v)dv}\right)-\left(\frac{f(u)}{\int_a^b f(u)w(u)du},\frac{h(v)}{\int_c^d h(v)\rho(v)dv}\right)\right\|^2\\ &\times w(u)\rho(v)dudv. \qquad (2.11)\end{aligned}$$

(b) *Let $\frac{f}{g}$ and $\frac{h}{k}$ be increasing functions on $[a,b]$ and $[c,d]$, respectively. If g and k are increasing functions on $[a,b]$ and $[c,d]$, respectively, then*

$$\begin{aligned}&\int_a^b\int_c^d \psi\left(\frac{g(u)}{\int_a^b g(u)w(u)du},\frac{k(v)}{\int_c^d k(v)\rho(v)dv}\right)w(u)\rho(v)dudv\\ &\leq \int_a^b\int_c^d \psi\left(\frac{f(u)}{\int_a^b f(u)w(u)du},\frac{h(v)}{\int_c^d h(v)\rho(v)dv}\right)w(u)\rho(v)dudv\\ &-\mu\int_a^b\int_c^d\left\|\left(\frac{g(u)}{\int_a^b g(u)w(u)du},\frac{k(v)}{\int_c^d k(v)\rho(v)dv}\right)-\left(\frac{f(u)}{\int_a^b f(u)w(u)du},\frac{h(v)}{\int_c^d h(v)\rho(v)dv}\right)\right\|^2\\ &\times w(u)\rho(v)dudv. \qquad (2.12)\end{aligned}$$

Proof. (a) Applying Lemma 1.3(a) with substitution $\vartheta(u)=g(u)w(u)$ and $l(u)=\frac{f(u)}{g(u)}$, we have

$$\int_a^b f(u)w(u)du\int_a^x g(u)w(u)du \leq \int_a^x f(u)w(u)du\int_a^b g(u)w(u)du \quad \text{for all } x\in[a,b],$$

which yields,

$$\int_a^x\left(\frac{g(u)}{\int_a^b g(u)w(u)du}\right)w(u)du \leq \int_a^x\left(\frac{f(u)}{\int_a^b f(u)w(u)du}\right)w(u)du \quad \text{for all } x\in[a,b]. \qquad (2.13)$$

Also, substituting $\vartheta(v)=k(v)\rho(v)$ and $l(v)=\frac{h(v)}{k(v)}$ in Lemma 1.3(a), we have

$$\int_c^d h(v)\rho(v)dv\int_c^y k(v)\rho(v)dv \leq \int_c^y h(v)\rho(v)dv\int_c^d k(v)\rho(v)dv \quad \text{for all } y\in[c,d],$$

which yields,

$$\int_c^y\left(\frac{k(v)}{\int_c^d k(v)\rho(v)dv}\right)\rho(v)dv \leq \int_c^y\left(\frac{h(v)}{\int_c^d h(v)\rho(v)dv}\right)\rho(v)dv \quad \text{for all } y\in[c,d]. \qquad (2.14)$$

Additionally, it is easy to observe that from (2.13) and (2.14), we have

$$\int_a^b \left(\frac{g(u)}{\int_a^b g(u)w(u)du} \right) w(u)du = \int_a^b \left(\frac{f(u)}{\int_a^b f(u)w(u)du} \right) w(u)du, \tag{2.15}$$

$$\int_c^d \left(\frac{k(v)}{\int_c^d k(v)\rho(v)dv} \right) \rho(v)dv = \int_c^d \left(\frac{h(v)}{\int_c^d h(v)\rho(v)dv} \right) \rho(v)dv. \tag{2.16}$$

Applying (2.13), (2.14), (2.15) and (2.16) in Theorem 2.1(b), we have

$$\begin{aligned}
&\int_a^b \int_c^d \psi \left(\frac{f(u)}{\int_a^b f(u)w(u)du}, \frac{h(v)}{\int_c^d h(v)\rho(v)dv} \right) w(u)\rho(v)dudv \\
&\leq \int_a^b \int_c^d \psi \left(\frac{g(u)}{\int_a^b g(u)w(u)du}, \frac{k(v)}{\int_c^d k(v)\rho(v)dv} \right) w(u)\rho(v)dudv \\
&- \mu \int_a^b \int_c^d \left\| \left(\frac{g(u)}{\int_a^b g(u)w(u)du}, \frac{k(v)}{\int_c^d k(v)\rho(v)dv} \right) - \left(\frac{f(u)}{\int_a^b f(u)w(u)du}, \frac{h(v)}{\int_c^d h(v)\rho(v)dv} \right) \right\|^2 \\
&\times w(u)\rho(v)dudv. \qquad (2.17)
\end{aligned}$$

(b) The proof of Theorem 2.2(b) is similar to Theorem 2.2(a).

□

Next, we establish some Favard's type inequalities for functions defined on rectangles via strong convexity.

Theorem 2.3. (a) *Let g and k be strongly concave functions with modulus μ_1 and μ_2 on $[a, b]$ and $[c, d]$, respectively, such that $f(u) = g(u) - \mu_1 u(u - a)$ and $h(v) = k(v) - \mu_2 v(v - c)$ are positive increasing functions. Also suppose ψ be a strongly convex function with modulus μ on $[0, 2\bar{g}_1] \times [0, 2\bar{k}_1]$, $\bar{z}_1 = a(1 - \zeta) + b\zeta$, $\bar{z}_2 = c(1 - \tau) + d\tau$, $\bar{g}_1 = \frac{(b-a)\int_a^b f(u)w(u)du}{2\int_a^b (u-a)w(u)du}$ and $\bar{k}_1 = \frac{(d-c)\int_c^d h(v)\rho(v)dv}{2\int_c^d (v-c)\rho(v)dv}$.*
Then

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \psi(f(u), h(v))w(u)\rho(v)dudv \\
&\leq \int_0^1 \int_0^1 \psi(2\bar{g}_1\zeta, 2\bar{k}_1\tau)w(\bar{z}_1)\rho(\bar{z}_2)d\zeta d\tau \\
&- \mu \int_0^1 \int_0^1 \|(2\bar{g}_1\zeta, 2\bar{k}_1\tau) - (g(\bar{z}_1) - \mu_1\bar{z}_1(b - a)\zeta, k(\bar{z}_2) \\
&- \mu_2\bar{z}_2(d - c)\tau)\|^2 w(\bar{z}_1)\rho(\bar{z}_2)d\zeta d\tau. \qquad (2.18)
\end{aligned}$$

If g and k be strongly convex functions with modulus μ_1 and μ_2 on $[a, b]$ and $[c, d]$, respectively, such that $f(u) = g(u) - \mu_1 u(u - a)$ and $h(v) = k(v) - \mu_2 v(v - c)$ are positive increasing functions and $g(a) = k(c) = 0$, then the reverse inequality in (2.18) holds.

(b) *Let g and k be strongly concave functions with modulus μ_1 and μ_2 on $[a, b]$ and $[c, d]$, respectively, such that $f(u) = g(u) + \mu_1 u(b - u)$ and $h(v) = k(v) + \mu_2 v(d - v)$ are positive decreasing functions. Also suppose ψ be a* *strongly convex function with modulus* μ *on* $[0, 2\bar{g_2}] \times [0, 2\bar{k_2}]$, $\bar{\omega}_1 = a\zeta + b(1-\zeta)$, $\bar{\omega}_2 = c\tau + d(1-\tau)$, $\bar{g}_2 = \frac{(b-a)\int_a^b f(u)w(u)du}{2\int_a^b (b-u)w(u)du}$ *and* $\bar{k}_2 = \frac{(d-c)\int_c^d h(v)\rho(v)dv}{2\int_c^d (d-v)\rho(v)dv}$. *Then*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \psi(f(u),h(v))w(u)\rho(v)dudv\\
&\leq \int_0^1\int_0^1 \psi(2\bar{g_2}\zeta, 2\bar{k_2}\tau)w(\bar{\omega}_1)\rho(\bar{\omega}_2)d\zeta d\tau\\
&\quad -\mu\int_0^1\int_0^1 \|(2\bar{g_2}\zeta, 2\bar{k_2}\tau) - (g(\bar{\omega}_1) - \mu_1\bar{\omega}_1(b-a)\zeta, k(\bar{\omega}_2)\\
&\quad -\mu_2\bar{\omega}_2(d-c)\tau)\|^2 w(\bar{\omega}_1)\rho(\bar{\omega}_2)d\zeta d\tau. \qquad (2.19)
\end{aligned}$$

If g *and* k *be strongly convex functions with modulus* μ_1 *and* μ_2 *on* $[a, b]$ *and* $[c, d]$, *respectively, such that* $f(u) = g(u) + \mu_1 u(b-u)$ *and* $h(v) = k(v) + \mu_2 v(d-v)$ *are positive increasing functions and* $g(a) = k(c) = 0$, *then the reverse inequality in (2.19) holds.*

Proof. (a) From Lemma 1.4, we know that the function $P_1(u) = \frac{g(u)}{u-a} - \mu_1 u$ and $P_2(v) = \frac{k(v)}{v-c} - \mu_2 v$ is decreasing then substituting $\vartheta(u) = (u-a)w(u)$ and $l(u) = \frac{g(u)}{u-a} - \mu_1 u$ in Lemma 1.3(a), we get

$$\int_a^b f(u)w(u)du\int_a^x (u-a)w(u)du \leq \int_a^x f(u)w(u)du\int_a^b (u-a)w(u)du \quad \text{for all } x \in [a,b],$$

that is,

$$\int_a^x \frac{(u-a)}{(b-a)}2\bar{g}_1 w(u)du \leq \int_a^x f(u)w(u)du \quad \text{for all } x \in [a,b]. \qquad (2.20)$$

Also, substitute $\vartheta(v) = (v-c)\rho(v)$ and $l(v) = \frac{k(v)}{v-c} - \mu_2 v$ in Lemma 1.3(a), we obtain

$$\int_c^d h(v)\rho(v)dv\int_c^y (v-c)\rho(v)dv \leq \int_c^y h(v)\rho(v)dv\int_c^d (v-c)\rho(v)dv \quad \text{for all } y \in [c,d],$$

that is,

$$\int_c^y \frac{(v-c)}{(d-c)}2\bar{k_1}\rho(v)dv \leq \int_c^y h(v)\rho(v)dv \quad \text{for all } y \in [c,d]. \qquad (2.21)$$

Since f and h are increasing functions, therefore from (2.20), (2.21) and Theorem 2.1(b), we obtain

$$\begin{aligned}
\int_a^b\int_c^d \psi(f(u),h(v))w(u)\rho(v)dudv &\leq \int_a^b\int_c^d \psi\left(\frac{u-a}{b-a}2\bar{g}_1, \frac{v-c}{d-c}2\bar{k_1}\right)w(u)\rho(v)dudv\\
&\quad -\mu\int_a^b\int_c^d \left\|\left(\frac{u-a}{b-a}2\bar{g}_1, \frac{v-c}{d-c}2\bar{k_1}\right) - (f(u),h(v))\right\|^2 w(u)\rho(v)dudv.
\end{aligned} \qquad (2.22)$$

Multiplying on both sides by $\frac{1}{(b-a)(d-c)}$ in (2.22), we get

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \psi(f(u),h(v))w(u)\rho(v)dudv$$
$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \psi\left(\frac{u-a}{b-a}2\bar{g_1},\frac{v-c}{d-c}2\bar{k_1}\right)w(u)\rho(v)dudv$$
$$-\frac{\mu}{(b-a)(d-c)}\int_a^b\int_c^d \left\|\left(\frac{u-a}{b-a}2\bar{g_1},\frac{v-c}{d-c}2\bar{k_1}\right)-(f(u),h(v))\right\|^2 w(u)\rho(v)dudv.$$

Applying change of variable technique in above inequality, we have

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \psi(f(u),h(v))w(u)\rho(v)dudv$$
$$\leq \frac{1}{4\bar{g_1}\bar{k_1}}\int_0^{2\bar{g_1}}\int_0^{2\bar{k_1}} \psi(x,y)w\left(a+\frac{(b-a)x}{2\bar{g_1}}\right)\rho\left(c+\frac{(d-c)y}{2\bar{k_1}}\right)dxdy$$
$$-\frac{\mu}{4\bar{g_1}\bar{k_1}}\int_0^{2\bar{g_1}}\int_0^{2\bar{k_1}}\left\|(x,y)-\left(f\left(a+\frac{(b-a)x}{2\bar{g_1}}\right),h\left(c+\frac{(d-c)y}{2\bar{k_1}}\right)\right)\right\|^2$$
$$\times w\left(a+\frac{(b-a)x}{2\bar{g_1}}\right)\rho\left(c+\frac{(d-c)y}{2\bar{k_1}}\right)dxdy. \tag{2.23}$$

Since,

$$f\left(a+\frac{(b-a)x}{2\bar{g_1}}\right)=g\left(a+\frac{(b-a)x}{2\bar{g_1}}\right)-\mu_1\left(a+\frac{(b-a)x}{2\bar{g_1}}\right)\frac{(b-a)x}{2\bar{g_1}} \tag{2.24}$$

and

$$h\left(c+\frac{(d-c)y}{2\bar{k_1}}\right)=k\left(c+\frac{(d-c)y}{2\bar{k_1}}\right)-\mu_2\left(c+\frac{(d-c)y}{2\bar{k_1}}\right)\frac{(d-c)y}{2\bar{k_1}}. \tag{2.25}$$

From (2.23), (2.24) and (2.25), we get

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \psi(f(u),h(v))w(u)\rho(v)dudv$$
$$\leq \frac{1}{4\bar{g_1}\bar{k_1}}\int_0^{2\bar{g_1}}\int_0^{2\bar{k_1}} \psi(x,y)w\left(a+\frac{(b-a)x}{2\bar{g_1}}\right)\rho\left(c+\frac{(d-c)y}{2\bar{k_1}}\right)dxdy$$
$$-\frac{\mu}{4\bar{g_1}\bar{k_1}}\int_0^{2\bar{g_1}}\int_0^{2\bar{k_1}}\left\|(x,y)-\left(g\left(a+\frac{(b-a)x}{2\bar{g_1}}\right)-\mu_1\left(a+\frac{(b-a)x}{2\bar{g_1}}\right)\frac{(b-a)x}{2\bar{g_1}},k\left(c+\frac{(d-c)y}{2\bar{k_1}}\right)\right.\right.$$
$$\left.\left.-\mu_2\left(c+\frac{(d-c)y}{2\bar{k_1}}\right)\frac{(d-c)y}{2\bar{k_1}}\right)\right\|^2 w\left(a+\frac{(b-a)x}{2\bar{g_1}}\right)\rho\left(c+\frac{(d-c)y}{2\bar{k_1}}\right)dxdy. \tag{2.26}$$

Applying change of variable technique, we obtain

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \psi(f(u),h(v))w(u)\rho(v)dudv$$
$$\leq \int_0^1\int_0^1 \psi(2\bar{g_1}\zeta,2\bar{k_1}\tau)w(a+(b-a)\zeta)\rho(c+(d-c)\tau)d\zeta d\tau$$
$$-\mu\int_0^1\int_0^1 \|(2\bar{g_1}\zeta,2\bar{k_1}\tau)-(g(a+(b-a)\zeta)-\mu_1(a+(b-a)\zeta)(b-a)\zeta,k(c+(d-c)\tau)$$
$$-\mu_2(c+(d-c)\tau)(d-c)\tau)\|^2 w(a+(b-a)\zeta)\rho(c+(d-c)\tau)d\zeta d\tau. \tag{2.27}$$

This implies,

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \psi(f(u),h(v))w(u)\rho(v)dudv$$
$$\leq \int_0^1\int_0^1 \psi(2\bar{g_1}\zeta, 2\bar{k_1}\tau)w(\bar{z}_1)\rho(\bar{z}_2)d\zeta d\tau$$
$$-\mu\int_0^1\int_0^1 \|(2\bar{g_1}\zeta, 2\bar{k_1}\tau)-(g(\bar{z}_1)-\mu_1\bar{z}_1(b-a)\zeta, k(\bar{z}_2)$$
$$-\mu_2\bar{z}_2(d-c)\tau)\|^2 w(\bar{z}_1)\rho(\bar{z}_2)d\zeta d\tau. \tag{2.28}$$

Similarly, we can proved the reverse inequality of (2.18).

(b) The proof of Theorem 2.3(b) is similar to Theorem 2.3(a).

□

Theorem 2.4. *Let $f(u) = g(u) - \mu_1 u(u-a)$, $h(v) = k(v) - \mu_2 v(v-c)$ are increasing functions on $(0,1)$ and $\frac{f}{P}$, $\frac{h}{Q}$ be decreasing functions on $(0,1)$. Also suppose f, h, P, Q, w and ρ are positive functions on $(0,1)$, and $fw, h\rho, Pw$ and $Q\rho$ are integrable on $(0,1)$ such that*

$$\phi = \frac{\int_0^1 f(u)w(u)du}{\int_0^1 P(u)w(u)du} \geq 0 \text{ and } \varphi = \frac{\int_0^1 h(v)\rho(v)dv}{\int_0^1 Q(v)\rho(v)dv} \geq 0. \tag{2.29}$$

Suppose ψ be a strongly convex function with modulus μ. Then,

$$\int_0^1\int_0^1 \psi(mf(u),nh(v))w(u)\rho(v)dudv \leq \int_0^1\int_0^1 \psi(m\phi P(u), n\varphi Q(v))w(u)\rho(v)dudv$$
$$-\mu\int_0^1\int_0^1 \|(m\phi P(u), n\varphi Q(v)) - (mf(u), nh(v))\|^2 w(u)\rho(v)dudv$$

for all $m, n > 0$.

Proof. From $P > 0$ and (2.29), substituting $\vartheta(u) = P(u)w(u)$ and $l(u) = f(u)/P(u)$, in Lemma 1.3(a), we get

$$\int_0^x m\phi P(u)w(u)du \leq \int_0^x mf(u)w(u)du. \tag{2.30}$$

Similarly, $Q > 0$ and (2.29), substituting $\vartheta(v) = Q(v)\rho(v)$ and $l(v) = h(v)/Q(v)$, in Lemma 1.3(a), we get

$$\int_0^y n\varphi Q(v)\rho(v)dv \leq \int_0^y nh(v)\rho(v)dv. \tag{2.31}$$

Since f and h are increasing functions, therefore by using Theorem 2.1(b), we get

$$\int_0^1\int_0^1 \psi(mf(u),nh(v))w(u)\rho(v)dudv \leq \int_0^1\int_0^1 \psi(m\phi P(u), n\varphi Q(v))w(u)\rho(v)dudv$$
$$-\mu\int_0^1\int_0^1 \|(m\phi P(u), n\varphi Q(v)) - (mf(u), nh(v))\|^2 w(u)\rho(v)dudv.$$

□

Theorem 2.5. *Let $\psi : [0,\infty) \to \mathbb{R}$ be a continuous strongly convex function with modulus μ, $f(u) = g(u) - \mu_1 u(u-a)$ and $h(v) = k(v) - \mu_2 v(v-c)$, P, w and Q, ρ are positive integrable functions on [a,b] and [c,d] respectively, $z_1(u) = \frac{f(u)}{\int_a^b f(u)w(u)du}$, $z_2(u) =$*

$\frac{P(u)}{\int_a^b P(u)w(u)du}$, $\omega_1(v) = \frac{h(v)}{\int_c^d h(v)\rho(v)dv}$, $\omega_2(v) = \frac{Q(v)}{\int_c^d Q(v)\rho(v)dv}$. *Then the following statements are true.*

(a) *If f and h are increasing on $[a,b]$ and $[c,d]$, respectively, and f/P and h/Q are decreasing on $[a,b]$ and $[c,d]$, respectively, then*

$$\int_a^b \int_c^d \psi(z_1(u),\omega_1(v))w(u)\rho(v)dudv \leq \int_a^b \int_c^d \psi(z_2(u),\omega_2(v))w(u)\rho(v)dudv - \mu \int_a^b \int_c^d \|(z_2(u),\omega_2(v)) - (z_1(u),\omega_1(v))\|^2 w(u)\rho(v)dudv. \tag{2.32}$$

(b) *If P and Q are increasing on $[a,b]$ and $[c,d]$, respectively, and f/P and h/Q are increasing on $[a,b]$ and $[c,d]$, respectively, then*

$$\int_a^b \int_c^d \psi(z_2(u),\omega_2(v))w(u)\rho(v)dudv \leq \int_a^b \int_c^d \psi(z_1(u),\omega_1(v))w(u)\rho(v)dudv - \mu \int_a^b \int_c^d \|(z_2(u),\omega_2(v)) - (z_1(u),\omega_1(v))\|^2 w(u)\rho(v)dudv. \tag{2.33}$$

Proof. (a) Since $P > 0$, then substituting $\vartheta(u) = P(u)w(u)$ and $l(u) = f(u)/P(u)$, in Lemma 1.3(a), we get

$$\int_a^x z_2(u)w(u)du \leq \int_a^x z_1(u)w(u)du. \tag{2.34}$$

Similarly, $Q > 0$ and (2.29), substituting $\vartheta(v) = Q(v)\rho(v) and l(v) = h(v)/Q(v)$, in Lemma 1.3(a), we get

$$\int_c^y \omega_2(v)\rho(v)dv \leq \int_c^y \omega_1(u)\rho(v)dv. \tag{2.35}$$

f and h are increasing functions, then applying Theorem 2.1(b), we have

$$\int_a^b \int_c^d \psi(z_1(u),\omega_1(v))w(u)\rho(v)dudv \leq \int_a^b \int_c^d \psi(z_2(u),\omega_2(v))w(u)\rho(v)dudv - \mu \int_a^b \int_c^d \|(z_2(u),\omega_2(v)) - (z_1(u),\omega_1(v))\|^2 w(u)\rho(v)dudv. \tag{2.36}$$

(b) The proof of Theorem 2.5(b) is similar to Theorem 2.5(a).

□

3. Conclusion

We established some integral majorization type inequalities on rectangles via strong convexity and obtained some significant results by using Theorem 2.1. The results obtained in this paper are the generalization of the previously known results. Our results may have further applications in future research work.

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