Mathematics and Statistics 10(6): 1304-1312, 2022 http://www.hrpub.org
DOI: 10.13189/ms.2022.100616

Maximum Likelihood Estimation in the Inverse Weibull Distribution with Type II Censored Data

Fatima A. Alshaikh, Ayman Baklizi*

Department of Mathematics, Statistics and Physics, College of Arts and Science, Qatar University, 2713, Doha, Qatar

Received April 29, 2022; Revised August 23, 2022; Accepted September 10, 2022

Cite This Paper in the Following Citation Styles

***(a):** [1] Fatima A. Alshaikh, Ayman Baklizi , "Maximum Likelihood Estimation in the Inverse Weibull Distribution with Type II Censored Data," Mathematics and Statistics, Vol. 10, No. 6, pp. 1304 - 1312, 2022. DOI: 10.13189/ms.2022.100616.*

***(b):** Fatima A. Alshaikh, Ayman Baklizi (2022). Maximum Likelihood Estimation in the Inverse Weibull Distribution with Type II Censored Data. Mathematics and Statistics, 10(6), 1304 - 1312. DOI: 10.13189/ms.2022.100616.*

Abstract We consider maximum likelihood estimation for the parameters and certain functions of the parameters in the Inverse Weibull (IW) distribution based on type II censored data. The functions under consideration are the Mean Residual Life (MRL), which is very important in reliability studies, and Tail Value at Risk (TVaR), which is an important measure of risk in actuarial studies. We investigated the performance of the MLE of the parameters and derived functions under various experimental conditions using simulation techniques. The performance criteria are the bias and the mean squared error of the estimators. Recommendations on the use of the MLE in this model are given. We found that the parameter estimators are almost unbiased, while the MRL and TVaR estimators are asymptotically unbiased. Moreover, the mean squared error of all estimators decreased for larger sample sizes and it increased when the censoring proportion is increased for a fixed sample size. The conclusion is that the maximum likelihood method of estimation works well for the parameters and the derived functions of the parameter like the MRL and TVaR. Two examples on a real data set are presented to illustrate the application of the methods used in this paper. The first one is on survival time of pigs while the other is on fire losses.

Keywords Maximum Likelihood Estimation, Mean Residual Life, Tail Value at Risk, Inverse Weibull Distribution, Type II Censoring

1. Introduction

The Weibull distribution is one of the most important distributions in reliability theory and survival analysis. It received and still receives considerable attention in the literature, see for example the recent contributions of Ikbal et al. (2022) and Mohamed et l. (2022). A closely related distribution is the Inverse Weibull (IW) distribution, it is useful for modeling in several important implementations in reliability engineering, infant mortality, product useful life, wear out periods, life testing, and service records (Alkarni and others 2020). The IW distribution and some related distributions have been recently studied by several authors from a Bayesian and likelihood perspectives. Kundu and Howlader (2010) had considered the Bayesian inference and forecast difficulties of the inverse Weibull distribution depending on type II censored data, while Kumar (2019) estimated parameters and reliability characteristics IW distribution depending on random censoring model by both Maximum likelihood and Bayesian estimation methods. The case of type 1 hybrid censored data was considered by Kazemi and Azizpoor (2021) where they studied the Bayesian and classical inference of this distribution.

Cooray and colleagues (2010) compare the Weibull and Inverse Weibull Composite Distributions for modelling reliability data, while Bhattacharyya (1985) analyzed and studied maximum likelihood and associated estimators using type II censored data. In the inverse Weibull distribution, Calabria and Pulcini (1990) calculated and compared the maximum likelihood and least square estimations. Helu (2015), on the other hand, uses the maximum likelihood, approximate maximum likelihood, and least squares methods to investigate the performance of parameters of the inverse Weibull distribution with progressively first-failure censoring. Sultan, Alsadat, and Kundu (2014) use Bayesian and maximum likelihood estimations to investigate the performance of the inverse Weibull parameters under progressive type-II censoring.

The probability density function (pdf) of IW distribution for random variable X and two parameters α and λ is given by

$$f(x;\alpha,\lambda) = \alpha\lambda\exp\{-\lambda x^{-\alpha}\}x^{-(\alpha+1)}$$

where $x > 0, \alpha > 0, \lambda > 0$ (1)

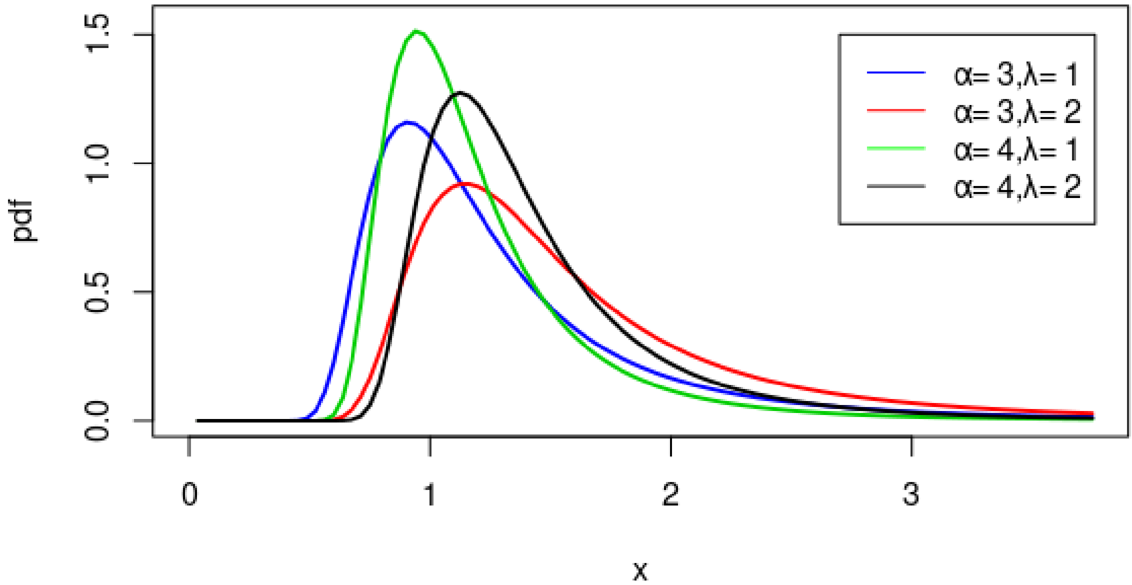

Figure 1. The pdf of IW distribution with different values of parameters

The IW distribution's cumulative distribution function (CDF) is given by

$$F(x;\alpha,\lambda) = \exp\{-\lambda x^{-\alpha}\} \quad (2)$$

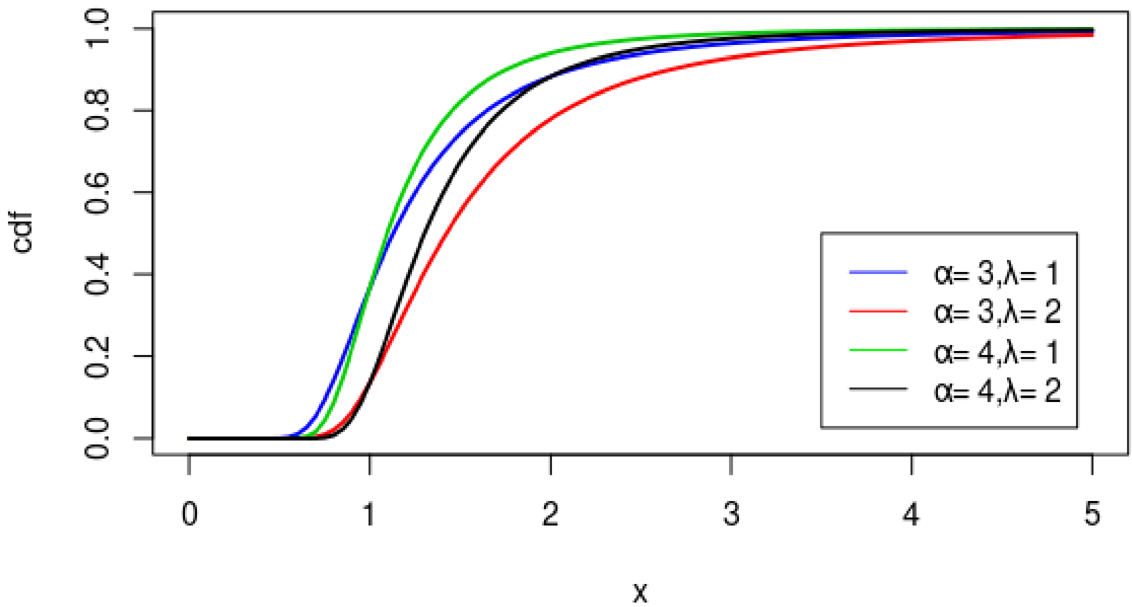

Figure 2. The cdf of IW distribution with different values of parameters

The reliability (survivor) function (RF) of the IW distribution is given by

$$S(x) = 1 - \exp\{-\lambda x^{-\alpha}\} \quad (3)$$

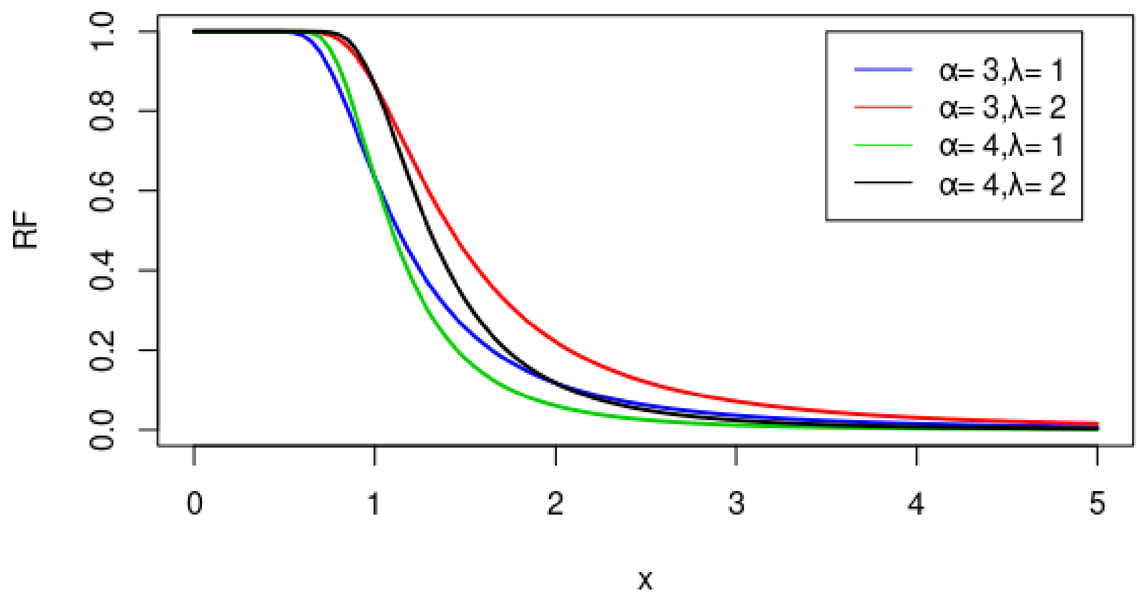

Figure 3. The RF of Inverse Weibull distribution with different values of parameters

Mean Residual Life (MRL) is a function of time t calculating the expected additional lifetime given that a component has survived until time t. It plays an important role in reliability and life testing. This function is an attractive alternative to the survival function or the hazard function of a survival time in practice. It provides the remaining life expectancy of a subject surviving up to time t.

Gupta (1981) offered a new theory to explain the MRL function, as well as some characterizations of the exponential distribution. Tang, Lu, and Chew (1999) investigated the overall behavior of the MRL in terms of failure rates for both continuous and discontinuous lifetime distributions, whereas Hall and Wellner (2017) expanded Yang's estimation of the mean residual life function (1978).

TVaR is a statistical measure of risk associated with the more general value at risk (VaR) approach, which measures the maximum amount of loss that is expected with an investment portfolio over a specified period, with a degree of confidence. It is a measure of risk important on actuarial studies.

Jorion (1996) presented statistical methods for measuring estimation error in VaR and demonstrated how to increase the accuracy of VaR estimates. Peng (2009) provided the risk measurement techniques value at risk (VaR) and tail value at risk (TVaR) under uncertainty, while Christoffersen, Hahn, and Inoue (2001) tested the VaR measure and chose the best among two models.

The likelihood equations under type II Censored data are derived in section 2. In section 3, The MRL and TVaR functions and their MLE are derived. Methods of evaluating estimators are explained in Section 4. In Sect. 5, simulation studies are used to evaluate estimators. Two real data examples are studied at section 6. In section 7, the conclusions and suggestions are given.

2. The Likelihood Function under Type II Censoring

For type II censored data, suppose that n items are put on a life testing, $X_1, \dots, X_n$, and we observe only the first r failure times, $X_1 < X_2 < \cdots < X_r$. If X_i have pdf $f(x)$ and survival function $S(x)$, below the assumptions that the lifetime distribution of the items is random variable, the likelihood function of the observed data without the multiplicative constant are written as (Lawless, 2011):

$$L(\theta) = \{\prod_{i=1}^{r} f(x_i)\} S(x_r)^{n-r} \tag{4}$$

By substituting eq (1) and (3) in eq (4) the likelihood function for IW distribution will be:

$$L(x|\alpha,\lambda) = \alpha^r \lambda^r \exp\left\{-\lambda \sum_{i=1}^{r} x_i^{-\alpha}\right\} \prod_{i=1}^{r} x_i^{-(\alpha+1)} (1 - \exp\{-\lambda x_r^{-\alpha}\})^{(n-r)} \tag{5}$$

and the log likelihood function will be:

$$l(data|\alpha,\lambda) = r\log\alpha + r\log\lambda - \lambda \sum_{i=1}^{r} x_i^{-\alpha} - (\alpha+1)\sum_{i=1}^{r} \log x_i + (n-r)\log(1 - \exp\{-\lambda x_r^{-\alpha}\}) \tag{6}$$

with derivatives

$$l_\alpha(x|\alpha,\lambda) = \frac{r}{\alpha} + \lambda \sum_{i=1}^{r} (x_i^{-\alpha} \log x_i) - \sum_{i=1}^{r} \log x_i + (n-r)\frac{\lambda x_r^{-\alpha} \log x_r}{\exp\{\lambda x_r^{-\alpha}\} - 1} \tag{7}$$

$$l_\lambda(x|\alpha,\lambda) = \frac{r}{\lambda} - \sum_{i=1}^{r} x_i^{-\alpha} + (n-r)\frac{x_r^{-\alpha}}{\exp\{\lambda x_r^{-\alpha}\} - 1} \tag{8}$$

3. Deriving MRL and TVaR Functions

Deriving MRL Function

MRL function of IW distribution can be derived using eq (1) and (3) and doing some calculation to become as the following:

$$m(t) = \int_t^\infty \frac{xf(x)}{S(t)}dx = \frac{\lambda^{\alpha^{-1}}[\Gamma(1-\alpha^{-1}) - \Gamma(1-\alpha^{-1}, \lambda t^{-\alpha})]}{1-\exp\{-\lambda t^{-\alpha}\}} \quad (9)$$

The MLE of the MRL is given by:

$$\widehat{m}(t) = \frac{\hat{\lambda}^{\hat{\alpha}^{-1}}[\Gamma(1-\hat{\alpha}^{-1}) - \Gamma(1-\hat{\alpha}^{-1}, \hat{\lambda} t^{-\hat{\alpha}})]}{1-\exp\{-\hat{\lambda} t^{-\hat{\alpha}}\}}, \ \alpha > 1 \quad (10)$$

Deriving TVaR Function

The Tail Value-at-Risk with confidence level p is defined as:

$$TVaR_p(X) = \frac{\int_{VaR_p(X)}^\infty xf(x)dx}{1-p} \quad (11)$$

After we found the integration, the function of TVaR will be:

$$TVaR_p(X) = VaR_p(X) + m\left(VaR_p(X)\right) \quad (12)$$

For the continuous distribution, the VaR with confidence level p is usually defined as follows:

$$Prob\left(X > VaR_p(x)\right) = 1 - p \quad (13)$$

By doing some calculation $VaR_p(x)$ for IW distribution become:

$$VaR_p(x) = \left(\frac{-\lambda}{\ln p}\right)^{\alpha^{-1}} \quad (14)$$

And

$$m\left(VaR_p(x)\right) = \frac{\lambda^{\alpha^{-1}}[\Gamma(1-\alpha^{-1}) - \Gamma(1-\alpha^{-1}, \lambda VaR_p(x)^{-\alpha})]}{1-\exp\{-\lambda VaR_p(x)^{-\alpha}\}} \quad (15)$$

then by substituting eq (14) and (15) in eq (12) and doing some calculus TVaR function of IW distribution will be:

$$TVaR_p(x) = \left(\frac{-\lambda}{\ln p}\right)^{\alpha^{-1}} + \frac{\lambda^{\alpha^{-1}}[\Gamma(1-\alpha^{-1}) - \Gamma(1-\alpha^{-1}, -\ln p)]}{1-p} \quad (16)$$

and the MLE of the $TVaR$ is given by:

$$\widehat{TVaR} = \left(\frac{-\hat{\lambda}}{\ln P}\right)^{\hat{\alpha}^{-1}} + \frac{\hat{\lambda}^{\hat{\alpha}^{-1}}[\Gamma(1-\hat{\alpha}^{-1}) - \Gamma(1-\hat{\alpha}^{-1}, -\ln P)]}{1-P} \quad (17)$$

4. Methods of Evaluating Estimators

Bias

The bias of an estimator $\hat{\theta}$ of a parameter θ can be defined by the difference between the expected value of $\hat{\theta}$ and the rale value θ as the following

$$Bias(\hat{\theta}) = E(\hat{\theta}) - \theta \quad (18)$$

If the bias for estimator is equal to 0, then it will be unbiased and satisfies $E(\hat{\theta}) = \theta$ for all θ.

Mean Square Error

The mean square error (MSE) of $\hat{\theta}$ of θ, which also called the risk function of an estimator or the quadratic loss function, can be defined as the following

$$MSE_{\hat{\theta}} = E(\hat{\theta} - \theta)^2 \quad (19)$$

The MSE measures the average squared difference between the estimator and the parameter, a rather acceptable measure of performing for the estimator. Although, the mean absolute error function $E(|\hat{\theta} - \theta|)$ can be used to measure the estimator as an acceptable option, MSE has at least two improvements more than other distance measures, it is analytically tractable and has the interpretation.

$$MSE_{\hat{\theta}} = E(\hat{\theta} - \theta)^2 = Var(\hat{\theta}) + (E(\hat{\theta}) - \theta)^2$$
$$= Var(\hat{\theta}) + \left(Bias(\hat{\theta})\right)^2 \quad (20)$$

Therefore, MSE has two parts, one measure the estimator variability and the other measures its bias. If the estimator has small variance and bias together, it will have good MSE properties. Moreover, the unbiased estimator has MSE equal to its variance.

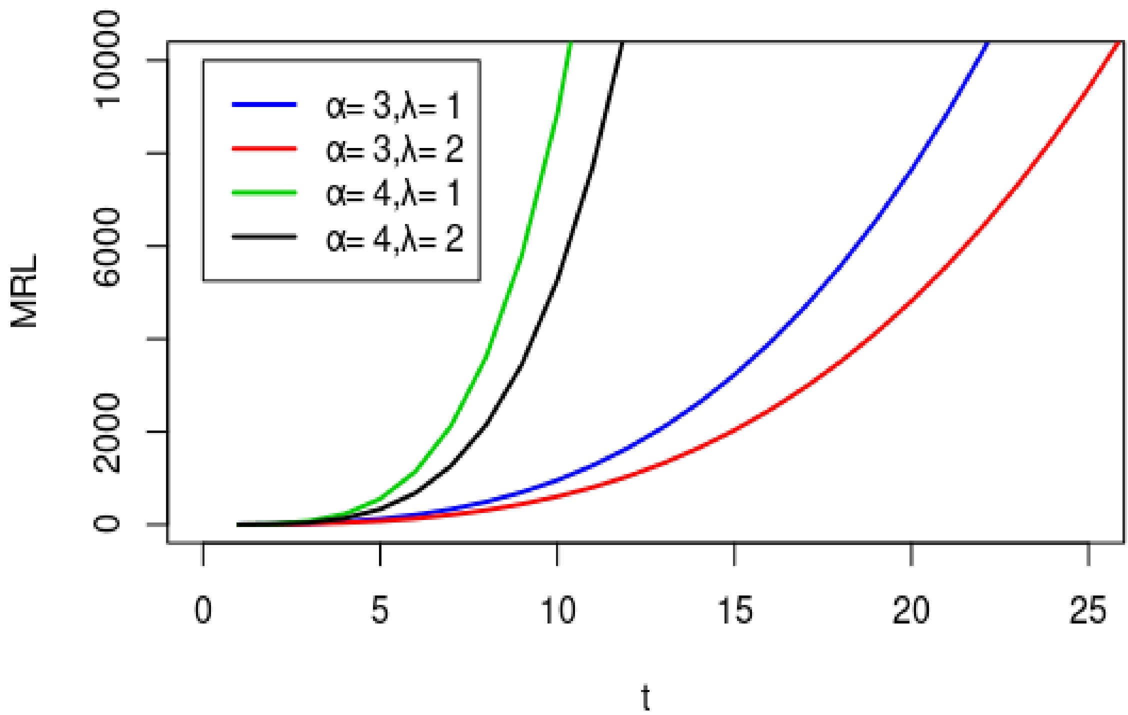

Figure 4. The MRL of Inverse Weibull distribution with different values of parameters

Table 1. Bias and MSE when $\alpha = 3$ and $\lambda = 1$

n	r		$\alpha = 3$	$\lambda = 1$	$t = 3$	$p = 0.95$
			α	λ	MRL	$TVaR$
50	30	Bias	0.1554	0.0039	10.031	-0.1015
		MSE	0.2372	0.0273	763.89	4.3665
	40	Bias	0.0927	0.0123	6.01567	0.0409
		MSE	0.1601	0.0262	372.396	3.5447
	50	Bias	0.0812	0.0140	4.90653	0.0521
		MSE	0.1342	0.0241	247.491	3.1406
80	48	Bias	0.0927	0.0018	5.48199	-0.0446
		MSE	0.1352	0.0166	265.205	2.8826
	64	Bias	0.0587	0.0109	3.23551	0.0462
		MSE	0.0909	0.0147	122.998	2.1294
	80	Bias	0.0593	0.0095	2.99512	-0.0007
		MSE	0.0760	0.0147	102.461	1.8340
100	60	Bias	0.0768	0.0017	4.25582	-0.0479
		MSE	0.1032	0.0132	170.291	2.2294
	80	Bias	0.0603	0.0081	2.94169	-0.0074
		MSE	0.0744	0.0120	91.5138	1.7257
	100	Bias	0.0392	0.0138	1.98704	0.0750
		MSE	0.0583	0.0122	65.3629	1.6088

Table 2. Bias and MSE when $\alpha = 3$ and $\lambda = 2$

n	r		$\alpha = 3$	$\lambda = 2$	$t = 3$	$p = 0.95$
			α	λ	MRL	$TVaR$
50	30	Bias	0.1371	0.0747	3.9560	-0.0504
		MSE	0.2268	0.1205	133.71	6.7936
	40	Bias	0.0985	0.0629	2.6629	-0.0054
		MSE	0.1571	0.1153	67.635	5.6399
	50	Bias	0.0798	0.0686	2.0157	0.0685
		MSE	0.1266	0.1066	44.203	4.8844
80	48	Bias	0.0930	0.0473	2.3099	-0.0729
		MSE	0.1296	0.0642	47.333	4.1338
	64	Bias	0.0585	0.0361	1.5465	0.0116
		MSE	0.0936	0.0625	30.347	3.6004
	80	Bias	0.0528	0.0414	1.2788	0.0270
		MSE	0.0796	0.0624	22.969	3.0685
100	60	Bias	0.0637	0.0328	1.6621	-0.0144
		MSE	0.0970	0.0489	32.936	3.4572
	80	Bias	0.0500	0.0442	1.1272	0.0421
		MSE	0.0689	0.0513	19.522	2.7110
	100	Bias	0.0420	0.0419	0.9458	0.0611
		MSE	0.0598	0.0487	16.608	2.4701

Table 3. Bias and MSE when $\alpha = 4$ and $\lambda = 1$

n	r		$\alpha = 4$	$\lambda = 1$	$t = 3$	$p = 0.95$
			α	λ	MRL	$TVaR$
50	30	Bias	0.2072	0.0041	50.383	-0.0767
		MSE	0.4084	0.0270	24848	1.5921
	40	Bias	0.1499	0.0105	33.053	-0.0272
		MSE	0.2974	0.0260	9417.95	1.2653
	50	Bias	0.1041	0.0182	22.882	0.0403
		MSE	0.2392	0.0246	5581.064	1.1035
80	48	Bias	0.1289	0.0001	27.239	-0.0509
		MSE	0.2449	0.0168	6245.5	1.0660
	64	Bias	0.0889	0.0091	17.596	0.0064
		MSE	0.1737	0.0161	3158.0	0.8618
	80	Bias	0.0708	0.0093	13.168	0.0079
		MSE	0.1366	0.0146	1843.0	0.6880
100	60	Bias	0.0881	0.0057	17.527	-0.0073
		MSE	0.1711	0.0130	3289.7	0.7624
	80	Bias	0.0733	0.0066	13.435	-0.0041
		MSE	0.1340	0.0121	1918.5	0.6368
	100	Bias	0.0517	0.0073	9.6997	0.0096
		MSE	0.1043	0.0118	1266.0	0.5467

Table 4. Bias and MSE when $\alpha = 4$ and $\lambda = 2$

n	r		$\alpha = 4$	$\lambda = 2$	$t = 3$	$p = 0.95$
			α	λ	MRL	$TVaR$
50	30	Bias	0.1496	0.0568	16.025	-0.0716
		MSE	0.3253	0.1007	3287.1	1.5976
	40	Bias	0.1406	0.0608	13.872	-0.0471
		MSE	0.2952	0.1771	1638.0	1.8115
	50	Bias	0.1157	0.0679	10.584	0.0003
		MSE	0.2335	0.1093	1076.0	1.5376
80	48	Bias	0.1446	0.0368	13.015	-0.1227
		MSE	0.2503	0.0654	1277.7	1.4162
	64	Bias	0.0969	0.0459	7.7929	-0.0349
		MSE	0.1604	0.0663	549.52	1.0979
	80	Bias	0.0574	0.0370	5.1540	0.0212
		MSE	0.1258	0.0583	367.54	0.9567
100	60	Bias	0.0883	0.0303	7.9176	-0.0375
		MSE	0.1688	0.0471	622.40	1.0865
	80	Bias	0.0843	0.0455	6.4937	-0.0114
		MSE	0.1356	0.0499	434.22	0.8968
	100	Bias	0.0606	0.0329	4.6457	-0.0162
		MSE	0.1014	0.0498	272.12	0.7237

5. Simulation Study

Simulation studies are very important because they used to obtain experimental results to estimate the efficiency of statistical methods. For that, researchers can study several properties of statistical methods.

In this paper, simulation of 2000 iterations are used to estimate bias and mean square error (MSE) for parameters, MRL, and TVaR with different sample size n ($50, 80, 100$), first r faller times r ($0.6n$, $0.8n, n$), confidence significant level $p = 0.95$, and time t = 3.

Tables 1 to 4 present bias and MSE with different values of shape and scales. These tables conclude that the estimated parameters and TVaR are approximately unbiased. However, estimated MRL is bias for small sample size. But when sample size increase, bias decrease until it is approximately unbiased. These tables show same inference about MSE. For parameters, MSE are very small, less than 0.5, and became smaller when the sample size increased. Which mean that, the variances estimators for parameters are small. Also, TVaR's MSE for small sample size are small, and they decrease when the sample size increase. However, MSE for MRL are very huge for the small sample size. But then, they are decreased remarkably when the sample size increased.

6. Real Data Analysis

The methods in this paper are applied in Two real data analysis. The first real data, showed in Table 5, is from Bjerkedal (1960) represents the survival times of 72 guinea pigs (in days) injected with different amounts of tubercle bacilli.

Table 5. The survival times (in days) of guinea pigs injected with different doses of tubercle bacilli.

12	15	22	24	24	32	32	33	34	38
38	43	44	48	52	53	54	54	55	56
57	58	58	59	60	60	60	60	61	62
63	65	65	67	68	70	70	72	73	75
76	76	81	83	84	85	87	91	95	96
98	99	109	110	121	127	129	131	143	146
146	175	175	211	233	258	258	263	297	341
341	376								

By applying the above data in R program, after divided each of them by 1000 to make calculation easier, the estimators for first faller time $r = 43$ are $\hat{\alpha} = 1.29$, $\hat{\lambda} = 0.025$, $\widehat{MRL} = 27.88$, and $\widehat{TVaR} = 3.93$. When $r = 58$ estimators would be $\hat{\alpha} = 1.36$, $\hat{\lambda} = 0.019$, $\widehat{MRL} = 30.77$, and $\widehat{TVaR} = 3.15$. For $r = 72$, the estimators become $\hat{\alpha} = 1.41$, $\hat{\lambda} = 0.016$, $\widehat{MRL} = 33.74$, and $\widehat{TVaR} = 2.81$. Table 6 present values of MRL and TVaR with different initial shape and scale values.

Table 6. Values of MRL and TVaR

t	p	α	λ	MRL	TVaR
3	0.95	3	1	26.22	21.74
		3	2	16.78	27.39
		4	1	72.05	19.72
		4	2	43.06	23.45

The second real data, showed in Table 7, is from Cummins, J., Dionne, G. and McDonald, J. (1990) represents the data on aggregate fire loss at a major university reported in Cummins and FreiFelder (1978).

Table 7. Fire loss experience of a major university

Year	Total losses	Year	Total losses
1950	71280	1962	14790
1951	3671	1963	9480
1952	18664	1964	8676
1953	8784	1965	114198
1954	3966	1966	5150
1955	30892	1967	105864
1956	631626	1968	32814
1957	11464	1969	41340
1958	127194	1970	46284
1959	4950	1971	12230
1960	30452	1972	19418
1961	8028		

If the above data is entered into the R software without modification, the shape estimator will be $\hat{\alpha} = 0.93$. To calculate MRL and TVaR functions, $\hat{\alpha}$ should be bigger than 1, see section 2.2. To avoid this problem, the lower term for shape was modified to 1.001 instead of 0.001 in the optima function used to maximize likelihood function.

The MRL and TVaR values are 26.22 and 21.74, respectively, after applying the data in R software and dividing each of them by 10000 to make calculation easier, with shape = 3, scale =1, first failure time $r = 14$, and significant level $\gamma = 0.05$, 0.1. The estimated parameters are also $\hat{\alpha} = 1.001$ *and* $\hat{\lambda} = 1.215$, with the MRL estimator $\widehat{MRL} = 3645.06$ and TVaR estimator $\widehat{TVaR} = 24201.58$.

Our estimators' values would be $\hat{\alpha} = 1.001$, $\hat{\lambda} = 1.207$, $\widehat{MRL} = 3640.75$, and $\widehat{TVaR} = 24050.69$ for $r = 19$. The estimators' values for $r = 23$ would be $\hat{\alpha} = 1.001$, $\hat{\lambda} = 1.207$, $\widehat{MRL} = 3640.62$, and $\widehat{TVaR} = 24046.11$.

7. Conclusion

The performance of the maximum likelihood estimators for parameters, Mean Residual Life (MRL), and Tail Value at Risk (TVaR) of the Inverse Weibull (IW) distribution are derived and studied in this paper using type II censored data.

We may conclude from simulation and real data that the parameters are approximately unbiased for small and big sample sizes. However, MRL and TVaR appear to be unbiased estimators with a large sample size, that is, they are asymptotically unbiased.

The mean squared error (MSE) of the estimators tend to decrease when the sample size increases. On the other hand, the MSE of all estimators increases as the proportion of censoring increases with a fixed sample size.

The suggestion for future research is to apply these methods to different forms of censoring data, such as hybrid censoring or progressive type II censoring. Researchers could also investigate the impacts of different values of Times t on MRL, as well as the effects of confidence levels $p = 0.99, 0.995$ on TVaR, and their estimators.

Acknowledgments

The authors would like to thank the referees for their suggestions and thoughtful comments that resulted in a much-improved version of the paper. This research was supported by a grant from the Office of Research Support at Qatar University, Grant no. QUST-1-CAS-2022-318.

Conflict of Interest

The authors have no relevant financial or nonfinancial interests to disclose.

REFERENCES

[1] Alkarni, S., Afify, A., Elbatal, I. & Elgarhy, M. (2020). The Extended Inverse Weibull Distribution: Properties and Applications. *Complexity*, Vol. 2020.

[2] Bhattacharyya, G. (1985). The Asymptotic of Maximum Likelihood and Related Estimators Based on Type II Censored Data. *Journal of the American Statistical Association*, Vol. 80, No. 390: 398-404.

[3] Bjerkedal, T. (1960). Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli. *American Journal of Hygiene*, vol. 72, 130-148.

[4] Calabria, R. and Pulcini, G. (1990). On the Maximum Likelihood and Least-Squares Estimation in the inverse Weibull Distribution. *Statistical Applicata*, Vol. 2, n. 1: 53-66.

[5] Christoffersen, P., Hahn, J., and Inoue, A. (2001). Testing and Comparing Value-at-Risk Measures. *Journal of Empirical Finance*, Vol. 8, No. 3: 325-342.

[6] Cooray, K., Gunasekera, S., and Ananda, M. (2010). Weibull and Inverse Weibull Composite Distribution for Modeling Reliability Data. *Model Assisted Statistics and Applications*, Vol. 5, No. 2: 109 – 115.

[7] Cummins, J., Dionne, G. and McDonald, J. (1990). Applications of the GB2 family of distributions in modeling insurance loss processes. *Mathematics and Economics*, 9: 57-272.

[8] Gupta, R. (1981). On the Mean Residual Life Function in Survival Studies. *Statistical Distributions in Scientific Work*, Vol. 5: 327-334.

[9] Hall, W. and Wellner, J. (2017). Estimation of Mean Residual Life. *math.ST*, Vol.0: 1- 6.

[10] Helu, A. (2015). On the Maximum Likelihood and Least Squares Estimation for the Inverse Weibull Parameters with Progressively First-Failure Censoring. *Open Journal of Statistics*, Vol. 5, No. 1: 75-89.

[11] Jorion, P. (1996). Risk2: Measuring the Risk in Value at Risk. *Financial Analysts Journal*, Vol. 52, No. 6: 47-56.

[12] Kazemi, M. & Azizpoor, M. (2021). Estimation of the Inverse Weibull Distribution Parameters under Type-I Hybrid Censoring. *Austrian Journal of Statistics*, 50: 38 – 51.

[13] Kumar, K. & Kumar, I. (2019). Estimation in Inverse Weibull Distribution Based on Randomly Censored Data. *STATISTICA*, 79: 1, 47 – 74.

[14] Kundu, D. & Howlader, H. (2010). Bayesian inference and prediction of the inverse Weibull distribution for Type-II censored data. *Computational Statistics and Data Analysis*, 54: 1547–1558.

[15] Mohamed, Noura S., Moshira A. Ismail, Sanaa A. Ismail, "Study of the New Finite Mixture of Weibull Extension Model: Identifiability, Properties and Estimation," Mathematics and Statistics, Vol. 10, No. 1, pp. 35 - 60, 2022. DOI: 10.13189/ms.2022.100104.

[16] Mohd Ikbal, Nawal Adlina, Syafrina Abdul Halim, Norhaslinda Ali, "Estimating Weibull Parameters Using Maximum Likelihood Estimation and Ordinary Least Squares: Simulation Study and Application on Meteorological Data," Mathematics and Statistics, Vol. 10, No. 2, pp. 269 - 292, 2022. DOI: 10.13189/ms.2022.10020 1.

[17] Peng, J. (2009). Value at Risk and Tail Value at Risk in Uncertain Environment. *Proceedings of the Eighth International Conference on Information and Management Sciences*: 787-793.

[18] Sultan, K., Alsadat, N., and Kundu, D. (2014). Bayesian and Maximum Likelihood Estimations of the Inverse Weibull Parameters Under Progressive Type-II Censoring. *Journal of Statistical Computation and Simulation*, Vol. 84, No. 10: 2248-2265.

[19] Tang, L., Lu, Y., and Chew, E. (1999). Mean Residual Life of Lifetime Distributions. *IEEE Transactions on Reliability*, Vol. 48, No. 1: 73-78.