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Refined Inference on the Scale Parameter of the Generalized Logistic Distribution Based on Adjusted Profile Likelihood Functions

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Abstract: We consider inference based on the profile likelihood function for the scale parameter of the generalized logistic distribution. This distribution is a generalization of the logistic distribution, a symmetric distribution like the normal distribution, and it has several applications in various fields. The generalization allows for possible left or right skewness, which makes it more flexible for modeling purposes. Inference procedures based on the profile likelihood of the scale parameter do not perform very well when the sample size is small, therefore, we derived adjustments to the profile likelihood for the generalized logistic distribution using results from higher-order likelihood theory. We obtained an adjustment based on the empirical covariances of certain scores of the profile likelihood function. Another adjustment is derived using ancillary statistics. The performance of the adjustments is investigated for point estimation of the scale parameter of the generalized logistic distribution using the bias and mean squared error criteria. Using an extensive simulation study, we found the adjustments are very successful in reducing the bias and the mean squared error of the maximum profile likelihood estimator in most situations. Moreover, we studied the performance of the profile likelihood ratio test and its adjustments using the criterion of the attainment of nominal sizes. We found that, when the sample size is small, the profile likelihood ratio test has empirical sizes that are highly inflated. Therefore, the test will be invalid in such situations. Simulation results show that the adjusted versions of the profile likelihood produce tests that attain the nominal sizes even for very small samples. This also applies to confidence intervals derived from these tests. In conclusion, both adjustments of the profile likelihood have significantly better performance than the unadjusted profile likelihood and are recommended, especially for small samples. In particular, the adjustment based on ancillary statistics appears to have the best overall performance in all situations considered. We applied the methods in this paper to real data on Carbon fibers.

Keywords: profile likelihood; generalized logistic distribution; Barndorff-Nielsen's adjustment

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1. Introduction

The standard logistic distribution is a symmetric distribution that is close in shape to the standard normal distribution, but it has heavier tails. Balakrishnan and Leung [1] introduced the generalized logistic distribution as a generalization to the standard logistic distribution. They obtained this distribution by compounding the extreme value distribution with the Gamma distribution.

The Two parameter generalized logistic distribution with shape parameter α and scale parameter β has cumulative distribution function and probability density function given respectively by

$$F(x,\alpha,\beta) = \left(1 - e^{-\beta x}\right)^{-\alpha},\ x > 0,\ \alpha > 0,\ \beta > 0, \tag{1}$$

$$f(x,\alpha,\beta) = \alpha\beta\left(1 - e^{-\beta x}\right)^{-\alpha-1} e^{-\beta x},\ x > 0,\ \alpha > 0,\ \beta > 0. \tag{2}$$

This distribution is skewed. However, for $\alpha = 1$ it reduces to the logistic distribution, and it is symmetric, while it is positively skewed for $\alpha > 1$ and it is negatively skewed for $0 < \alpha < 1$. This distribution is unimodal and log-concave for all values of α, see Alkasasbeh and Raqab [2]. The earliest known application of this distribution was by Verhulst [3] to represent population growth. Ahuja and Nash [4] studied the relation between the Gompertz and Verhulst generalized logistic distributions and the family of Pearson curves and studied their moments and cumulants. Recently, this distribution has received attention from several authors in the literature, particularly because of its direct relation with the logistic distribution and it many applications in reliability, survival analysis, actuarial modeling, and economics. Asgharzadeh [5] studied point and interval estimation of the parameters of this distribution under progressively type II censored data. Sreekumar and Thomas [6] consider estimation of the location and scale parameters using U-statistics constructed by using best linear functions of order statistics as kernels. They compare the performance of their estimators with the maximum likelihood estimators. Characterization of type 1 generalized logistic distribution was studied by [7]. Various characteristics of this distribution were studied by [8], including the moment generating function, the characteristic function, the moments, and the Renyi entropy. Lagos-Alvariz [9] proposed a Bayesian approach for the estimation of the generalized logistic distribution.

In this paper, we will consider estimation of the scale parameter of type 1 generalized logistic distribution, treating the shape parameter as a nuisance parameter. The scale parameter is important in statistical distributions as it can be considered the "unit of measurement" of the available data. Moreover, it is viewed as a measure of the spread of the distribution. The larger the scale parameter, the larger the spread of the distribution and vice versa, see [10,11]. Yang and Xie [12] considered modified profile likelihood for inference about the shape parameter of the Weibull distribution. They used the idea of parameter orthogonality of [13]. Ferrari et al. [14] further investigated profile likelihood inference for the shape parameter of the Weibull distribution and considered adjustments developed by [15] and its approximations proposed by [16,17]. Sewailem and Baklizi [18] extended the work of [14] to the Lomax distribution. Other relevant work on modified profile likelihood functions includes [19] on Gumbel mixture model, [20] on fixed effects panel data models, [21,22] on the β-model and fixed effects models, respectively.

In this work, we will consider profile likelihood inference on the scale parameter of the generalized logistic distribution. It is well known that the maximum likelihood estimator is generally a biased estimator, especially in small samples. Moreover, the likelihood ratio test that is used for hypotheses testing and confidence interval estimation is a large sample test, whose validity depends on asymptotic theory. Therefore, it is desirable to have some adjustments to the profile likelihood function in the hope of reducing the bias of the maximum likelihood estimator and improving the performance and validity of the likelihood ratio test in small samples. The rest of the paper will be as follows. We will obtain the profile likelihood function for the scale parameter of the generalized logistic distribution and derive some adjustments for it in the hope of obtaining sharper inferences. We studied the performance of the derived estimators using simulation. The sizes of the likelihood ratio tests based on the profile likelihood and its adjustments are studied by simulation. The profile likelihood function is obtained in Section 2. The adjustments are derived for the generalized logistic distribution in Section 3. The profile likelihood ratio test and its modifications are presented in Section 4. An extensive simulation study to investigate and compare the inference procedures is described in Section 6. The findings and conclusions are given in the final section.

2. Profile Likelihood Function for the Generalized Logistic Distribution

Let $x_1, \ldots, x_n$ be a random sample from the generalized logistic distribution with pdf given in (1), the likelihood function of the parameters α and β is given by:

$$L(\alpha,\beta) = (\alpha\beta)^n e^{-\beta\sum_{i=1}^{n} x_i} \prod_{i=1}^{n}\left(1-e^{-\beta x_i}\right)^{-\alpha-1}, \ \alpha > 0,\ \beta > 0. \tag{3}$$

The loglikelihood function is given by:

$$l(\alpha,\beta) = nln(\alpha) + nln(\beta) - \beta\sum_{i=1}^{n} x_i - (\alpha+1)\sum_{i=1}^{n} ln\left(\left(1-e^{-\beta x_i}\right)\right). \tag{4}$$

Differentiating with respect to α we obtain:

$$l_\alpha(\alpha,\beta) = \frac{\partial l(\alpha,\beta)}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} ln\left(\left(1-e^{-\beta x_i}\right)\right). \tag{5}$$

Equating this derivative to zero we obtain the root $\hat{\alpha}$ as a function of β as follows:

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} ln\left(\left(1-e^{-\beta x_i}\right)\right)} \tag{6}$$

Substituting this root in the loglikelihood function we obtain a function of β alone. This function is the profile likelihood function $l_P(\beta)$ as follows:

$$l_P(\beta) = nln\left(\frac{n}{\sum_{i=1}^{n} ln\left(\left(1+e^{-\beta x_i}\right)\right)}\right) + nln(\beta) - \beta\sum_{i=1}^{n} x_i - n - \sum_{i=1}^{n} ln\left(\left(1+e^{-\beta x_i}\right)\right). \tag{7}$$

Maximizing the profile likelihood, we obtain the maximum profile likelihood estimator $\hat{\beta}$ of β, which coincides with the maximum likelihood estimator. Note that we have eliminated the nuisance parameter α. However, it is well known that the profile likelihood is not a genuine likelihood, therefore, it does not enjoy some of the desirable properties of the usual likelihood function. For example, the profile likelihood score function does not have zero expectation, see [23]. Therefore, several attempts were made in the literature to improve the statistical properties of the profile likelihood function. Some of these attempts are based on using certain adjustments to the profile likelihood function. One of the widely used is Barndorff-Nielsen's [15] modified profile likelihood. In the next section, we shall introduce this adjustment. Since it is usually very complicated to calculate, we will present some approximations of this adjustment.

3. Adjusted Profile Likelihood Functions

Inference on the interest parameter is generally affected by the presence of nuisance parameters. The usual approaches to dealing with nuisance parameters are through conditioning or marginalization [23]. However, in many cases, it is not possible to obtain the required conditional or marginal likelihood. Therefore, several adjustments to the profile likelihood function appeared in the literature to improve its performance. Barndorff-Nielsen [15] obtained an adjustment to approximate the conditional or marginal likelihood of the interest parameter, if it exists, so that inferences on the interest parameter can be improved. Consider the profile loglikelihood function $l_P(\beta)$. The Barndorff-Nielsen's adjustment applied to the generalized logistic distribution is as follows:

$$l_{BN}(\beta) = l_p(\beta) - log\left|\frac{\partial\hat{\alpha}_\beta}{\partial\hat{\alpha}}\right| - \frac{1}{2}log\left|j_{\alpha\alpha}\left(\hat{\alpha}_\beta,\beta\right)\right|, \tag{8}$$

where $j_{\alpha\alpha}\left(\hat{\alpha}_\beta,\beta\right) = -\frac{\partial^2 l\left(\hat{\alpha}_\beta,\beta\right)}{\partial\alpha^2} = \frac{n}{\alpha^2}$ and $\frac{\partial\,\hat{\alpha}_\beta}{\partial\hat{\alpha}}$ is the partial derivative of $\hat{\alpha}_\beta$ with respect to $\hat{\alpha}$. The difficulty lies in computing $\left|\frac{\partial\hat{\alpha}_\beta}{\partial\hat{\alpha}}\right|$. To avoid this problem, several approximations to Barndorff-Nielsen's adjustment were proposed in the literature. We will consider two approximations suggested by [16,17] and Fraser and Reid [24], respectively.

The approximation of [16,17] is given by

$$\bar{l}_{BN} = l_p(\beta) + \frac{1}{2} log \left| j_{\alpha\alpha}(\hat{\alpha}_\beta, \beta) \right| - log \left| I_\alpha(\hat{\alpha}_\beta, \beta \,; \hat{\alpha} \,, \hat{\beta}) \right|, \tag{9}$$

where,

$$I_\theta(\alpha, \beta \,; \alpha_0, \beta_0) = E_{(\alpha_0, \beta_0)}\{ l_\alpha(\alpha, \beta) l_\alpha(\alpha_0, \beta_0) \}, \tag{10}$$

with $l_\alpha(\alpha,\beta) = \frac{\partial l(\alpha,\beta)}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^n ln\left(\left(1 + e^{-\beta x_i}\right)\right)$. Note that $I_\alpha(\alpha, \beta \,; \alpha_0, \beta_0)$ represents the covariance between $l_\alpha(\alpha, \beta)$ and $l_\alpha(\alpha_0, \beta_0)$. The computation of this covariance, however, involved some complicated integrals. An alternative approach followed in this paper is suggested by [17]. It is based on empirical covariances approximation as follows

$$\check{I}(\hat{\alpha}_\beta, \beta, \hat{\alpha}, \hat{\beta}) = \sum_{j=1}^n l_\alpha^{(j)}(\hat{\alpha}_\beta, \beta)\, l_\alpha^{(j)}(\hat{\alpha}, \hat{\beta}\,), \tag{11}$$

were, $l_\alpha^{(j)} = \frac{1}{\alpha} - ln\left(\left(1 + e^{-\beta x_j}\right)\right)$ is the score function of the jth observation. The corresponding modified maximum profile likelihood estimator is denoted by $\hat{\beta}_1$.

The other adjustment, proposed by Fraser and Reid [24] and Fraser et al. [25], is given by:

$$\widetilde{l}_{BN}(\beta) = l_P(\beta) + \frac{1}{2} log|j_{\alpha\alpha}(\hat{\alpha}_\beta, \beta)| - log|l_{\alpha;x}(\hat{\alpha}_\beta, \beta)\hat{V}_\alpha|, \tag{12}$$

where, $l_{\alpha;x}(\alpha, \beta) = \frac{\partial l_\alpha(\alpha,\beta)}{\partial \mathrm{x}^T}$ is the score function for $\mathrm{x}^T = (x_1, \ldots, x_n)$. It is given by

$$l_{\alpha;x}(\alpha, \beta) = \frac{\partial l_\alpha(\alpha, \beta)}{\partial \mathrm{x}^T} = \left(\frac{\beta e^{-\beta x_1}}{1 + e^{-\beta x_1}}, \ldots, \frac{\beta e^{-\beta x_n}}{1 + e^{-\beta x_n}} \right). \tag{13}$$

The vector of ancillary directions $\hat{V}_\alpha$ is given by:

$$\hat{V}_\alpha = \left(- \frac{\partial F_1(x_1; \hat{\alpha}, \hat{\beta}) / \partial \hat{\alpha}}{f_1(x_1; \hat{\alpha}, \hat{\beta})}, \ldots\ldots\ldots\ldots, - \frac{\partial F_n(x_n; \hat{\alpha}, \hat{\beta}) / \partial \hat{\alpha}}{f_n(x_n; \hat{\alpha}, \hat{\beta})} \right)^T, \tag{14}$$

where $f_j(x; \alpha, \beta)$ and $F_j(x; \alpha, \beta)$ are the probability density function and the cumulative distribution function of x_j, respectively. For the generalized logistic distribution, we have

$$- \frac{\partial F(x_j, \hat{\alpha}, \hat{\beta}) / \partial \hat{\alpha}}{f(x_j, \hat{\alpha}, \hat{\beta})} = \left(1 + e^{-\hat{\beta} x_j}\right)^{-\hat{\alpha}} ln\left(1 + e^{-\hat{\beta} x_j}\right)$$

The corresponding modified maximum profile likelihood estimator is denoted by $\hat{\beta}_2$.

4. Sizes of Adjusted Profile Likelihood Ratio Tests

Consider testing the hypothesis $H_0 : \beta = \beta_0$ vs. $H_1 : \beta \neq \beta_0$. The profile loglikelihood function can be used to perform an asymptotic test that rejects the null hypothesis at significance level γ if

$$LR = 2(l_P(\hat{\beta}) - l_P(\beta_0)) > \chi^2_{\gamma,1}. \tag{15}$$

This is an approximate test based on the asymptotic chi-squared distribution. However, if the sample size is not large enough, the actual size of the test may not be close to the nominal size γ. Tests based on the adjusted profile likelihood function are expected to have faster convergence to the asymptotic distribution and hence smaller sample sizes for their validity. We will consider tests that are based on the two adjustments introduced earlier in this paper. Specifically, we have the test based on the empirical covariances adjustment to the profile loglikelihood function that rejects the null hypothesis at significance level γ if

$$LR_1 = 2(l_P^*(\hat{\beta}) - l_P^*(\beta_0)) > \chi^2_{\gamma,1}. \tag{16}$$

The other modified test is based on the ancillary statistic adjustment, and it rejects the null hypothesis at significance level γ if

$$LR_2 = 2\left(l_P^{**}(\hat{\beta}) - l_P^{**}(\beta_0)\right) > \chi^2_{\gamma,1} \tag{17}$$

The performance of the profile likelihood ratio test and its two adjustments in terms of attaining the nominal sizes will be investigated and compared using simulation, as will be explained in Section 6.

5. Real Data Example

Bader and Priest [26] gave a data set on strength measurements in GPA for single carbon fibers. The data given represent the strength measured in GPA for single carbon fibers of 10 mm in gauge lengths. The data set consists of 63 observations and is further considered by Alkasasbeh and Raqab (2009), who checked the fit of the data to the generalized logistic distribution using the Kolmogorov–Smirnov statistic. They found that the generalized logistic distribution provides a good fit for the data. Since the data set is relatively large and the differences between the inferences based on the profile likelihood and its adjustments will be too small. We randomly selected a subsample of size 15 from this data set, and they are as follows:

3.264, 3.220, 3.145, 2.474, 2.350, 3.125, 2.132, 3.223, 3.871, 2.624, 2.659, 2.454, 1.901, 2.525, 4.225

We obtained the point estimators and the 95% confidence intervals of the scale parameter based on the profile likelihood and its two adjustments. In addition, we obtained the values of the test statistics and the corresponding p-values for testing $H_0 : \beta = 3$ vs. $H_1 : \beta \neq 3$. The results are given in the table below (Table 1).

Table 1. The results of the likelihood inference procedures for the carbon fibers data.

Method	Point Estimation	95% Confidence Interval	Test Statistic (p-Value)
Profile Likelihood	1.9588	(1.2804, 2.8019)	5.6315 (0.0176)
Empirical Covariances Adjustment	1.8859	(1.2512, 2.7245)	6.4126(0.0113)
Ancillary Directions Adjustment	1.8789	(1.2114, 2.7135)	6.5397(0.0105)

The point estimators based on the adjusted profile likelihood are very close to each other. The confidence interval based on the profile likelihood is the widest, while the intervals based on the adjusted likelihood are narrower, especially the interval based on the ancillary statistic adjustment. For the testing problem, the values of the test statistics and p-values based on adjusted likelihood are close to each other and smaller than the ones based on the profile likelihood. These observations are further examined in the simulation study in the next section.

6. Simulation Study

The performance of the maximum profile likelihood estimator and the estimators based on the adjusted profile likelihood are investigated and compared through a simulation study. The criteria of comparison are the bias and the mean squared errors of the estimators. Similarly, the empirical size performance of the profile likelihood ratio test and the adjusted profile likelihood ratio tests are investigated and compared based on the attainment of the nominal sizes. To fulfill this goal, an extensive simulation study is carried out. The scale parameter is kept fixed at $\beta = 1$, because the inferential procedures based on the profile likelihood are scale invariant. The shape parameter was varied among the values $\alpha = 0.2,\ 0.5,\ 0.8$ (left skewness), $\alpha = 1$ (symmetry), and $\alpha = 1.5,\ 2.0,\ 2.5,\ 3.0,\ 4.0$ (right skewness). The sample size is varied from very small to fairly large, specifically, we take $n = 5,\ 10,\ 15,\ 20,\ 25,\ 30,\ 50,\ 70,\ 100$. For the testing hypotheses problem, we choose the nominal size to be $\gamma = 0.01,\ 0.05,\ 0.10$. The biases, mean squared error, and empirical sizes are obtained using $N =$ 10,000 replications. The results are given in Tables 2 and 3.

Table 2. Biases and Mean Squared Errors of the Estimators.

α	n	$Bias(\hat{\beta})$	$Bias(\hat{\beta}_1)$	$Bias(\hat{\beta}_2)$	$MSE(\hat{\beta})$	$MSE(\hat{\beta}_1)$	$MSE(\hat{\beta}_2)$
0.2	5	4.537	4.481	4.501	39.572	39.454	39.517
0.2	10	2.540	2.511	2.518	21.509	21.454	21.479
0.2	15	1.469	1.449	1.452	11.540	11.511	11.521
0.2	20	0.881	0.866	0.868	6.138	6.121	6.125
0.2	25	0.570	0.558	0.559	3.314	3.303	3.305
0.2	30	0.363	0.353	0.354	1.656	1.649	1.650
0.2	50	0.152	0.146	0.147	0.296	0.294	0.294
0.2	70	0.092	0.088	0.088	0.110	0.109	0.109
0.2	100	0.056	0.053	0.053	0.051	0.050	0.050
0.5	5	1.984	1.875	1.894	15.702	15.437	15.542
0.5	10	0.623	0.573	0.578	3.620	3.556	3.572
0.5	15	0.279	0.247	0.249	0.950	0.926	0.930
0.5	20	0.158	0.134	0.135	0.299	0.288	0.289
0.5	25	0.113	0.095	0.096	0.186	0.180	0.181
0.5	30	0.094	0.079	0.080	0.096	0.092	0.092
0.5	50	0.050	0.041	0.041	0.041	0.040	0.040
0.5	70	0.034	0.028	0.028	0.027	0.026	0.026
0.5	100	0.023	0.018	0.019	0.016	0.016	0.016
0.8	5	1.057	0.932	0.935	6.670	6.393	6.457
0.8	10	0.293	0.238	0.238	0.956	0.905	0.912
0.8	15	0.142	0.108	0.108	0.211	0.196	0.197
0.8	20	0.096	0.070	0.070	0.097	0.090	0.090
0.8	25	0.072	0.052	0.052	0.065	0.060	0.060
0.8	30	0.061	0.045	0.045	0.048	0.045	0.045
0.8	50	0.033	0.024	0.024	0.024	0.023	0.023
0.8	70	0.023	0.016	0.016	0.016	0.016	0.016
0.8	100	0.017	0.012	0.012	0.011	0.010	0.010
1	5	0.724	0.598	0.591	3.730	3.484	3.516
1	10	0.197	0.141	0.140	0.349	0.311	0.314
1	15	0.118	0.083	0.082	0.121	0.108	0.108
1	20	0.080	0.055	0.054	0.067	0.061	0.061
1	25	0.063	0.043	0.042	0.050	0.047	0.047
1	30	0.053	0.036	0.036	0.040	0.037	0.037
1	50	0.031	0.021	0.021	0.021	0.020	0.020
1	70	0.022	0.015	0.015	0.014	0.014	0.014
1	100	0.014	0.009	0.009	0.009	0.009	0.009
1.5	5	0.472	0.348	0.321	1.426	1.226	1.204
1.5	10	0.157	0.102	0.096	0.157	0.131	0.130
1.5	15	0.098	0.062	0.060	0.080	0.070	0.070
1.5	20	0.070	0.044	0.043	0.051	0.046	0.045
1.5	25	0.054	0.033	0.032	0.038	0.035	0.034
1.5	30	0.044	0.027	0.026	0.030	0.027	0.027
1.5	50	0.026	0.016	0.015	0.016	0.015	0.015
1.5	70	0.017	0.010	0.010	0.011	0.010	0.010
1.5	100	0.014	0.009	0.009	0.007	0.007	0.007
2	5	0.414	0.292	0.254	0.878	0.706	0.661
2	10	0.150	0.095	0.087	0.138	0.114	0.111
2	15	0.088	0.052	0.049	0.063	0.055	0.054
2	20	0.065	0.039	0.037	0.045	0.040	0.039

Table 2. *Cont.*

α	n	$Bias(\hat{\beta})$	$Bias(\hat{\beta}_1)$	$Bias(\hat{\beta}_2)$	$MSE(\hat{\beta})$	$MSE(\hat{\beta}_1)$	$MSE(\hat{\beta}_2)$
2	25	0.050	0.029	0.028	0.031	0.029	0.029
2	30	0.043	0.026	0.025	0.025	0.023	0.023
2	50	0.025	0.015	0.014	0.014	0.013	0.013
2	70	0.018	0.010	0.010	0.010	0.009	0.009
2	100	0.012	0.007	0.007	0.006	0.006	0.006
3	5	0.378	0.258	0.211	0.616	0.464	0.407
3	10	0.143	0.089	0.078	0.118	0.096	0.092
3	15	0.088	0.052	0.047	0.060	0.052	0.050
3	20	0.064	0.038	0.035	0.040	0.036	0.035
3	25	0.052	0.031	0.029	0.029	0.026	0.026
3	30	0.040	0.023	0.021	0.024	0.022	0.022
3	50	0.024	0.013	0.013	0.013	0.012	0.012
3	70	0.017	0.009	0.009	0.008	0.008	0.008
3	100	0.012	0.007	0.007	0.006	0.006	0.006
4	5	0.391	0.269	0.217	0.675	0.511	0.443
4	10	0.146	0.092	0.079	0.123	0.100	0.095
4	15	0.089	0.053	0.048	0.058	0.050	0.048
4	20	0.066	0.039	0.036	0.039	0.034	0.034
4	25	0.052	0.031	0.028	0.029	0.026	0.026
4	30	0.042	0.024	0.022	0.023	0.021	0.021
4	50	0.024	0.013	0.013	0.012	0.012	0.012
4	70	0.016	0.008	0.008	0.008	0.008	0.008
4	100	0.011	0.006	0.005	0.005	0.005	0.005

Table 3. Empirical sizes of the profile likelihood ratio tests.

		$\gamma = 0.01$			$\gamma = 0.05$			$\gamma = 0.10$		
α	n	LR	LR_1	LR_2	LR	LR_1	LR_2	LR	LR_1	LR_2
0.2	5	0.003	0.003	0.003	0.017	0.016	0.016	0.054	0.046	0.046
0.2	10	0.003	0.003	0.003	0.038	0.035	0.035	0.115	0.108	0.109
0.2	15	0.007	0.007	0.007	0.067	0.064	0.065	0.146	0.144	0.143
0.2	20	0.009	0.009	0.009	0.073	0.071	0.071	0.136	0.136	0.136
0.2	25	0.014	0.014	0.014	0.071	0.071	0.072	0.130	0.128	0.129
0.2	30	0.012	0.011	0.011	0.062	0.061	0.061	0.120	0.120	0.120
0.2	50	0.011	0.011	0.011	0.058	0.058	0.058	0.111	0.110	0.110
0.2	70	0.013	0.013	0.013	0.055	0.055	0.055	0.103	0.102	0.102
0.2	100	0.010	0.011	0.011	0.050	0.050	0.050	0.103	0.102	0.102
0.5	5	0.010	0.005	0.006	0.065	0.044	0.046	0.147	0.111	0.117
0.5	10	0.014	0.011	0.011	0.073	0.065	0.066	0.133	0.124	0.125
0.5	15	0.014	0.013	0.013	0.063	0.057	0.057	0.115	0.111	0.111
0.5	20	0.012	0.010	0.010	0.056	0.053	0.053	0.111	0.106	0.107
0.5	25	0.013	0.012	0.012	0.057	0.055	0.055	0.112	0.108	0.108
0.5	30	0.011	0.010	0.010	0.057	0.055	0.055	0.111	0.108	0.108
0.5	50	0.011	0.011	0.011	0.054	0.052	0.052	0.103	0.100	0.100
0.5	70	0.012	0.012	0.012	0.057	0.056	0.056	0.109	0.109	0.109
0.5	100	0.010	0.010	0.010	0.053	0.053	0.053	0.103	0.102	0.103
0.8	5	0.018	0.007	0.007	0.089	0.057	0.060	0.165	0.124	0.125
0.8	10	0.018	0.015	0.015	0.069	0.059	0.060	0.127	0.111	0.112
0.8	15	0.015	0.012	0.013	0.061	0.053	0.054	0.117	0.109	0.109
0.8	20	0.012	0.011	0.011	0.060	0.054	0.054	0.110	0.100	0.100

Table 3. *Cont.*

		γ = 0.01			γ = 0.05			γ = 0.10		
α	n	LR	LR_1	LR_2	LR	LR_1	LR_2	LR	LR_1	LR_2
0.8	25	0.012	0.011	0.011	0.057	0.054	0.054	0.110	0.105	0.105
0.8	30	0.012	0.010	0.010	0.055	0.052	0.052	0.105	0.098	0.098
0.8	50	0.011	0.011	0.011	0.053	0.051	0.051	0.104	0.101	0.101
0.8	70	0.009	0.010	0.010	0.051	0.050	0.050	0.101	0.098	0.098
0.8	100	0.010	0.010	0.010	0.047	0.046	0.046	0.099	0.097	0.097
1	5	0.023	0.011	0.011	0.092	0.059	0.063	0.162	0.117	0.121
1	10	0.016	0.011	0.011	0.065	0.052	0.053	0.122	0.105	0.104
1	15	0.013	0.011	0.011	0.060	0.052	0.052	0.112	0.101	0.102
1	20	0.011	0.010	0.010	0.056	0.049	0.050	0.111	0.099	0.100
1	25	0.011	0.010	0.010	0.054	0.050	0.050	0.107	0.100	0.100
1	30	0.011	0.010	0.010	0.056	0.051	0.051	0.113	0.106	0.106
1	50	0.011	0.011	0.011	0.052	0.051	0.051	0.101	0.099	0.098
1	70	0.012	0.012	0.012	0.053	0.053	0.052	0.105	0.103	0.103
1	100	0.010	0.010	0.010	0.051	0.049	0.050	0.101	0.100	0.100
1.5	5	0.028	0.014	0.014	0.096	0.058	0.059	0.165	0.115	0.114
1.5	10	0.016	0.010	0.010	0.065	0.051	0.050	0.122	0.102	0.100
1.5	15	0.016	0.010	0.011	0.064	0.054	0.053	0.116	0.102	0.104
1.5	20	0.014	0.012	0.011	0.061	0.053	0.053	0.118	0.107	0.107
1.5	25	0.014	0.011	0.011	0.056	0.050	0.049	0.108	0.102	0.101
1.5	30	0.011	0.010	0.010	0.055	0.052	0.052	0.110	0.101	0.101
1.5	50	0.012	0.010	0.010	0.054	0.051	0.051	0.104	0.097	0.098
1.5	70	0.010	0.010	0.009	0.051	0.049	0.049	0.101	0.099	0.099
1.5	100	0.011	0.011	0.010	0.053	0.052	0.052	0.102	0.101	0.101
2	5	0.030	0.014	0.014	0.104	0.062	0.061	0.174	0.116	0.114
2	10	0.019	0.013	0.012	0.071	0.055	0.055	0.134	0.105	0.104
2	15	0.013	0.010	0.010	0.057	0.049	0.047	0.112	0.095	0.095
2	20	0.015	0.011	0.011	0.064	0.055	0.055	0.115	0.106	0.105
2	25	0.011	0.010	0.010	0.054	0.048	0.048	0.108	0.099	0.099
2	30	0.011	0.009	0.009	0.055	0.050	0.050	0.108	0.102	0.101
2	50	0.011	0.010	0.010	0.053	0.049	0.049	0.105	0.099	0.099
2	70	0.010	0.009	0.009	0.052	0.049	0.049	0.106	0.101	0.101
2	100	0.010	0.009	0.009	0.050	0.048	0.048	0.103	0.102	0.102
3	5	0.031	0.014	0.013	0.101	0.059	0.058	0.170	0.116	0.111
3	10	0.017	0.012	0.012	0.070	0.052	0.050	0.127	0.103	0.101
3	15	0.015	0.011	0.010	0.064	0.050	0.051	0.119	0.106	0.105
3	20	0.014	0.012	0.011	0.061	0.051	0.050	0.116	0.107	0.106
3	25	0.012	0.010	0.009	0.057	0.052	0.052	0.109	0.099	0.098
3	30	0.013	0.011	0.011	0.058	0.054	0.053	0.113	0.101	0.101
3	50	0.011	0.010	0.010	0.054	0.050	0.050	0.107	0.101	0.100
3	70	0.011	0.010	0.010	0.050	0.048	0.048	0.099	0.095	0.094
3	100	0.010	0.009	0.009	0.053	0.051	0.050	0.104	0.101	0.101
4	5	0.031	0.014	0.013	0.102	0.059	0.058	0.171	0.115	0.111
4	10	0.019	0.013	0.013	0.077	0.058	0.058	0.140	0.111	0.110
4	15	0.016	0.012	0.011	0.061	0.050	0.051	0.118	0.100	0.099
4	20	0.014	0.011	0.011	0.058	0.052	0.052	0.113	0.099	0.099
4	25	0.014	0.011	0.010	0.059	0.049	0.049	0.113	0.103	0.103
4	30	0.012	0.009	0.009	0.060	0.053	0.052	0.114	0.106	0.105
4	50	0.013	0.011	0.012	0.057	0.054	0.054	0.108	0.104	0.103
4	70	0.011	0.010	0.010	0.055	0.052	0.051	0.109	0.105	0.104
4	100	0.009	0.008	0.008	0.048	0.046	0.045	0.093	0.092	0.091

7. Findings and Conclusions

Results regarding the performance of the maximum profile likelihood estimator $\hat{\beta}$, the estimator based on the empirical covariances adjustment to the profile likelihood $\hat{\beta}_1$, and the estimator based on the ancillary statistics adjustment $\hat{\beta}_2$ are given in Table 2. The conclusions are clear-cut. The estimators based on adjusted likelihood have clearly better performance than $\hat{\beta}$ in all situations considered. The bias and MSE are both reduced by adjustment, the reduction is especially clear for values of the shape parameter (α) greater than 1. That is, when the parent distribution is positively skewed. The performance of the two estimators based on adjusted profile likelihoods appears to be similar. For values of the shape parameter less than 1 (left-skewed distribution), the estimator $\hat{\beta}_1$ has a smaller bias and slightly larger MSE than $\hat{\beta}_2$. For $\alpha > 1$ (right-skewed distribution), the estimator $\hat{\beta}_2$ clearly has the best performance where its bias and MSE are the smallest for all sample sizes. For $\alpha = 1$ (symmetry), the performance of $\hat{\beta}_1$ and $\hat{\beta}_2$ are very similar, but both are substantially better than $\hat{\beta}$. As an overall result for the comparison between adjustments, it appears that $\hat{\beta}_2$, which is based on the adjustment based on ancillary directions proposed by Fraser and Reid [24] and Fraser et al. [25], has the best overall performance, especially for the MSE criterion.

For the testing hypotheses problem, the results are given in Table 3. The adjusted profile likelihood ratio tests appear to attain the nominal sizes better than the unadjusted test. This is especially clear for sample sizes of less than 30. The improvement becomes clearer as α moves away from zero. The empirical size performance of the unadjusted test is poor for sample sizes like 5, 10, or 15 as the empirical size becomes much larger than the nominal size. This means that unadjusted test is invalid in such cases and should be avoided. This also has a reflection on the performance of confidence intervals for the scale parameter. This follows from the dual relation between confidence interval estimation and hypotheses testing, see, for example, [27]. This means that for small sample sizes, the intervals based on the profile likelihood are too short and do not achieve the nominal coverage probability of the interval. On the other hand, the intervals based on the adjusted profile likelihood will remain valid under these conditions. The performance of the two adjustments is very similar for the testing and confidence interval estimation problems.

To facilitate the comparison of the biases, mean squared error, and error probabilities of 95% confidence intervals for different values of the shape parameter (α), we constructed Figure 1 for sample size $n = 10$. The pattern is similar for other values of the sample size. From Figure 1, we observe that the biases and mean squared error decrease for the increasing value of the shape parameter. It is clear that the bias of the maximum profile likelihood estimator is considerably reduced by using the adjustments, especially the one based on ancillary directions. The MSE performance of the estimators is generally the same for the estimators, with the adjustments giving slightly smaller MSE in general. The error probability of the confidence interval based on the profile likelihood is generally greater than that of the adjustments, and it exceeds the nominal error probability.

In conclusion, the adjusted profile likelihood gives inference procedures with better performance, especially for small sample sizes and under moderate left skewness, symmetry, or right skewness of the generalized logistic distribution. The adjustments to the profile likelihood function derived and applied in this paper were successful in reducing the effect of the nuisance shape parameter on inference with the scale parameter of the generalized logistic distribution. This is reflected by a smaller bias of the resulting maximum likelihood estimator in addition to producing confidence intervals that are valid and attain the nominal sizes.

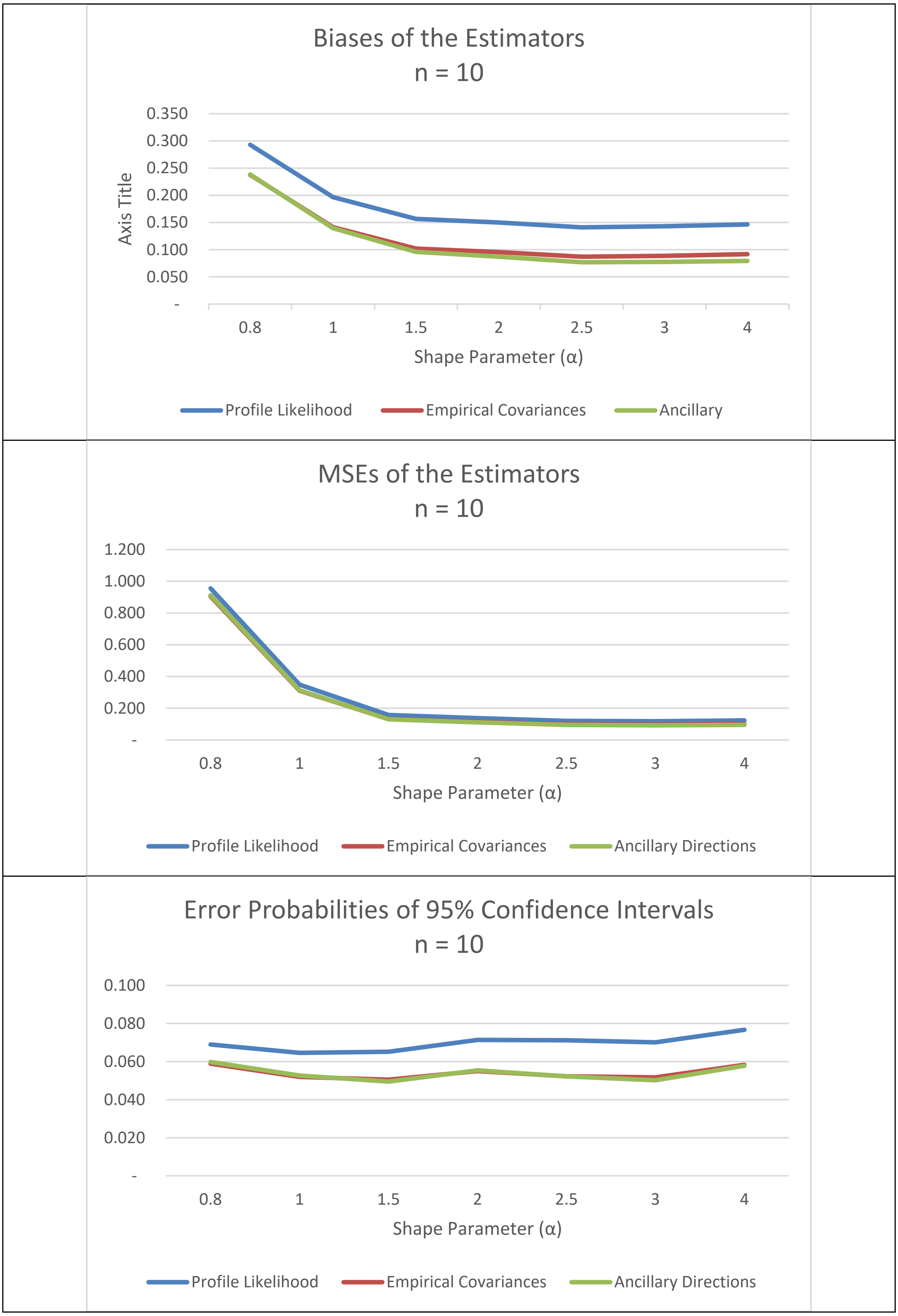

Figure 1. Biases, MSEs, and Error Probabilities for Profile Likelihood and its Adjustments.

The work in this paper can be extended to cover various types of censored data like type 1, type 2, or progressive censoring that frequently appear in industrial life testing experiments and survival analysis. Another direction is to investigate bias reduction techniques for the maximum likelihood estimator based on the Jackknife or asymptotic expansions, in addition to computer-intensive methods for the construction of confidence intervals for the scale parameter. Fuzzy inference may be considered for this situation. Recent references include Srikanth Reddy [28] and Tang et al. [29]. It is also of interest to investigate the performance of the profile likelihood function and its adjustments in more complicated models arising in reliability studies like multicomponent systems [30,31] and the multicomponent stress-strength model [32] when the component log-lifetimes follow the generalized logistic distribution.

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