Results in Physics 28 (2021) 104557

Contents lists available at ScienceDirect

Results in Physics

journal homepage: www.elsevier.com/locate/rinp

Exact traveling wave solutions for two prolific conformable M-Fractional differential equations via three diverse approaches

Imran Siddique [a], Mohammed M.M. Jaradat [b,*], Asim Zafar [c], Khush Bukht Mehdi [a], M.S. Osman [d]

[a] Department of Mathematics, University of Management and Technology, Lahore 54770, Pakistan
[b] Department of Mathematics, Statistics and Physics, Qatar University, Doha, Qatar
[c] Department of Mathematics, COMSATS University Islamabad, Vehari Campus, Pakistan
[d] Department of Mathematics, Faculty of Science, Cairo University, Giza 12613, Egypt

ARTICLE INFO

Keywords:
Generalized reaction duffing model
Diffusion reaction equation
M−fractional derivative
Exact traveling wave solutions

ABSTRACT

In this paper, we obtain the exact traveling solutions of the M-fractional generalized reaction Duffing model and density dependent M-fractional diffusion reaction equation by using three fertile, $(G'/G, 1/G)$, modified (G'/G^2) and $(1/G')$-expansion methods. These methods contribute a variety of exact traveling wave solutions to the scientific literature. The obtained solutions are also verified for the aforesaid equations through symbolic soft computations. Furthermore, some results are explained through numerical simulations that show the novelty of our work. Moreover, we observe that all the solutions are new and an excellent contribution in the existing literature of solitary wave theory.

Introduction

Nonlinear fractional differential equations (NFDEs) occur more frequently in engineering applications and different research areas [1–8]. Then, many real-life problems can be modeled by ordinary or partial differential equations involving the derivatives of fractional order. In order to better understand and apply these physical phenomena in practical scientific research, it is important to find their exact solutions. Finding exact solutions of most of the NFDEs is not easy, so searching and constructing exact solutions of NFDEs is a continuing investigation.

Recently, many powerful methods for obtaining exact solutions of nonlinear partial differential equations (NLPDEs) have been presented such as, Hirota's bilinear method [9], modified expansion function method [10], sine–cosine method [11], tanh-method [12], Adomin decomposition method [13,14], variational iteration method [15,16], homotopy perturbation method [17], homotopy analysis method [18], Laplace iterative method [19], nonlinear Schrodinger equation [20–22], Boussinesq fractional type model [23]. Periodic-type solutions have been investigated by implementing the Variational principal method of the KMN equation in the (2 + 1) – dimensional form [24], Riccati equation method [25], tanh-expansion method [26], extended direct algebraic method [27–31], Kudryashov method [32], Exp-function method [33], the modified extended exp-function method [34], F-expansion method [35], the Backlund transformation method [36,37], reductive perturbation method [38], the extended tanh-method [39,40], Jacobi elliptic function expansion methods [41 –43], the residual power series method [44], extended sinh- Gordon equation expansion method [45] and different other methods [46–50].

The primary prospect of this paper is to determine the exact solutions of the fractional generalized reaction Duffing model [51]

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}+p\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}+qu(x,t)+ru^2(x,t)+su^3(x,t)=0,\ t>0,\ 0<\alpha\leqslant1, \tag{1}$$

and nonlinear fractional diffusion–reaction equation

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}+ku(x,t)\frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}}=D\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}+au(x,t)-bu^2(x,t),\ t>0,\ 0<\alpha\leqslant1, \tag{2}$$

with M-fractional derivative [52] based on three different methods, the $(G'/G, 1/G)$ −expansion method [53,54], the modified (G'/G^2) −expansion method [55] and the $(1/G')$ −expansion method [56]. These methods are frequently used to find the different types of the

* Corresponding author.
E-mail address: mmjst4@qu.edu.qa (M.M.M. Jaradat).

https://doi.org/10.1016/j.rinp.2021.104557
Received 23 May 2021; Received in revised form 11 July 2021; Accepted 13 July 2021
Available online 4 August 2021

exact solutions of the NLPDEs. The main concept of the two variables $(G'/G,\ 1/G)$ −expansion method is that the exact traveling wave solutions of nonlinear evolution equations (NLEEs) can be written as a polynomial in two variables (G'/G) and $(1/G)$, in which $G = G(\eta)$ satisfies a second order linear ODE $G''(\eta) + \lambda G(\eta) = \mu$, where λ and μ are constants. For these methods, the degree of the polynomials can be evaluated by a taking homogenous balance between the highest-order derivatives and nonlinear terms in the given nonlinear PDEs. Besides, the coefficients of the polynomial can be determined by solving a set of algebraic equations resulting from the process of using the methods. For examples, Hafiz [57] determined the close form solutions of the fractional generalized reaction Duffing model and the density dependent fractional diffusion reaction equation by $(G'/G,\ 1/G)$-expansion method. Traveling wave solutions of the Zakharov and nonlinear Kdv-mKdv equations have been found by Li et al [53] and Zayed et al [54]. The general solutions of the fifth order NLEEs and the Burger KP-equation have been obtained in [58,59]. Exact traveling wave solution of nonlinear fractional evolution equations obtained by Sirisubtawee [60]. By using the modified (G'/G^2) −expansion method, traveling wave solutions have been found for the nonlinear Schrodinger equation along the third-order dispersion [55]. Different types of traveling wave solutions of the Fokas-Lenells equations have been determined by this method [61]. Aljahdaly found the general exact traveling wave solutions to the NLEEs in [62]. Dragon and Donmez [63] obtained the traveling wave solutions of the Gardner equation and their applications to the different physical plasma. Exact solutions of nonlinear and supernonlinear traveling wave's solutions for Sharma-Tasso-Olver (STO) equations are obtained by Ali et al. [64].

This article organized it as follows: In Sect. 2, we present the M-fractional derivative and its properties. The descriptions of strategies are given in Sec.3. In Sec.4, we present a mathematical analysis of the models and its solutions via proposed methods. Some conclusions are drawn in the last section.

M-fractional derivatives and its properties

Definition. Assume that$f : (0,\infty)\to R$, then, the newItruncated M-fractional derivative of function $f$of order α is defined as:

$$D_M^{\alpha,\beta} f(t) = \lim_{\varepsilon\to 0}\frac{f\left(t\in_\beta(\varepsilon t^{1-\alpha})\right) - f(t)}{\varepsilon},\ \text{for all } t>0,\ 0<\alpha<1,\ \beta>0,$$

where $\in_\beta(\cdot)$ is a truncated Mittag-Leffler function of one parameter [52].

Properties. Let $\alpha\in(0,\ 1],\ \beta>0$ and $f = f(t),\ g = g(t)$ be α- differentiable, at a point t > 0, then:

1. $D_M^{\alpha,\beta}(af + bg) = aD_M^{\alpha,\beta}f + bD_M^{\alpha,\beta}g$ for all $a,\ b\in R$.
2. $D_M^{\alpha,\beta}(c) = 0$, where $f(t) = c$, is a constant.
3. $D_M^{\alpha,\beta}(f.g) = gD_M^{\alpha,\beta}(f) + fD_M^{\alpha,\beta}(g)$
4. $D_M^{\alpha,\beta}\left(\frac{f}{g}\right) = \frac{gD_M^{\alpha,\beta}(f)+fD_M^{\alpha,\beta}(g)}{g^2}$
5. Furthermore; if the function f is differentiable; then

$$D_M^{\alpha,\beta} f(t) = \frac{t^{1-\alpha}}{\Gamma(\beta+1)}\frac{df}{dt}. \tag{3}$$

6. $D_M^{\alpha,\beta}(f\circ g)(t) = f'(g(t))D_M^{\alpha,\beta}g(t)$, for f differentiable at $g(t)$. This characterization also fulfills the Chain rule.

Description of strategies

$(G'/G,\ 1/G)$ −Expansion method

In this section, we describe the main steps of the $(G'/G,\ 1/G)$ −expansion method [53,54] for finding travelling wave solutions of nonlinear evolution equations. Let us consider the second order linear ordinary differential equation (ODE):

$$G''(\eta) + \lambda G(\eta) = \mu, \tag{4}$$

where $\phi = G'/G$ and $\psi = 1/G$, then we attain

$$\phi' = -\phi^2 + \mu\psi - \lambda,\ \psi' = -\phi\psi. \tag{5}$$

Case 1. When $\lambda<0$, the general solutions of Eq. (4) is given as

$$G(\eta) = A_1 sinh\left(\sqrt{-\lambda}\,\eta\right) + A_2 cosh\left(\sqrt{-\lambda}\,\eta\right) + \frac{\mu}{\lambda}, \tag{6}$$

and we have

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma+\mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right), \tag{7}$$

where A_1 and A_2 are arbitrary integration constants and$\sigma = A_1^2 - A_2^2$.

Case 2. When $\lambda>0$, the general solutions of Eq. (4) is clearly

$$G(\eta) = A_1 sin\left(\sqrt{\lambda}\,\eta\right) + A_2 cos\left(\sqrt{\lambda}\,\eta\right) + \frac{\mu}{\lambda}, \tag{8}$$

and we have

$$\psi^2 = \frac{\lambda}{\lambda^2\sigma-\mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right), \tag{9}$$

where A_1 and A_2 are arbitrary integration constants and$\sigma = A_1^2 + A_2^2$.

Case 3. When $\lambda = 0$, the general solutions of Eq. (4) is

$$G(\eta) = \frac{\mu}{2}\eta^2 + A_1\eta + A_2, \tag{10}$$

and we have

$$\psi^2 = \frac{1}{A_1^2 - 2\mu A_2}\left(\phi^2 - 2\mu\psi\right), \tag{11}$$

where A_1 and A_2 are arbitrary integration constants.

Let us consider the nonlinear partial differential equation (NLPDE) is in the form

$$Q(u,\ u_t,\ u_x,\ u_{tt},\ u_{xt},\ u_{xx},\ \ldots) = 0, \tag{12}$$

where $u = u(x,t)$ is an unknown function, Q is a polynomial of $u(x,t)$ and its various partial derivatives. The main steps of $(G'/G,\ 1/G)$ −expansion method are:

Step 1: By coordinates transformation

$$\eta = x - vt,\ u(x,t) = U(\eta). \tag{13}$$

Here,v is the speed of traveling wave.

The wave variable allow us to reduce Eq. (12) into a nonlinear ODE for$U = U(\eta)$:

$$R(U,\ U',\ U''\ U''',\ \ldots) = 0, \tag{14}$$

where R is a polynomial of $U(\eta)$ and its total derivatives with respect toη.

Step 2: Assume that the solutions of Eq. (14) can be expressed by a polynomial in two variables ϕ and ψ as:

$$U(\eta) = \sum_{i=0}^{m} a_i\phi^i + \sum_{i=1}^{m} b_i\phi^{i-1}\psi, \tag{15}$$

where $a_i(i = 0,\ 1,\ \ldots,\ m)$ and $b_i(i = 1,\ \ldots,\ m)$ are constants to be determined later, and the positive integer m can be determined by using the homogenous balance between the highest order derivatives and the nonlinear terms appearing in ODE (14).

Step 3: Substituting Eq. (15) into Eq. (14) along with Eqs. (5) and (7), the left hand side of Eq. (14) can be converted into a polynomial in terms of ϕ and ψ, in which the degree of ψ is not larger than 1. Equating each coefficient of the polynomial to zero yields a system of algebraic equations which can be solved by using the software MATHEMATICA to get the values of $a_i(i = 0,\ 1, ..., m),\ b_i(i = 1, ..., m),\ \nu,\ \ \mu,\ \ \lambda(\lambda < 0)$, A_1 and A_2.

Step 4: Substituting the values of $a_i(i = 0, 1, ..., m),\ b_i(i = 1, ..., m), \nu$, $\mu,\ \lambda(\lambda < 0),\ A_1$ and A_2obtained into (15); one can attain the traveling wave solutions expressed by the hyperbolic functions of Eq. (14).

Step 5: Similar to step 3 and step 4, substituting (15) into Eq. (14) along with (5) and (9) (or (5) and (11)), we attain the exact travelling wave solutions of Eq. (14) expressed by trigonometric functions (or expressed by rational functions).

The modified (G'/G^2)- expansion method

Here, we will describe the basic steps of modified (G'/G^2)- expansion method [55]

Step 1: Consider Eqs. (12), (13) and (14).

Step 2: Extend the solutions of Eq. (14) in the following form

$$U(\eta) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G^2}\right)^i, \tag{16}$$

where $a_i(i = 0, 1, 2, 3, ..., m)$ are constants and find to be later. It is important that $a_i \neq 0$.

The function $G = G(\eta)$ satisfies the following Riccati equation

$$\left(\frac{G'}{G^2}\right)' = \lambda_1 \left(\frac{G'}{G^2}\right)^2 + \lambda_0, \tag{17}$$

where λ_0 and λ_1 are constants. We gain the below solutions to Eq. (17) due to different conditions of λ_0:

When $\lambda_0\lambda_1 < 0$,

$$\left(\frac{G'}{G^2}\right) = -\frac{\sqrt{|\lambda_0\lambda_1|}}{\lambda_1} + \frac{\sqrt{|\lambda_0\lambda_1|}}{2}\left[\frac{C_1 sinh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right) + C_2 cosh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}{C_1 cosh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right) + C_2 sinh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}\right] \tag{18}$$

When $\lambda_0\lambda_1 > 0$,

$$\left(\frac{G'}{G^2}\right) = \sqrt{\frac{\lambda_0}{\lambda_1}}\left[\frac{C_1 cos\left(\sqrt{\lambda_0\lambda_1}\,\eta\right) + C_2 sin\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}{C_1 sin\left(\sqrt{\lambda_0\lambda_1}\,\eta\right) - C_2 sin\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}\right] \tag{19}$$

When $\lambda_0 = 0$ and $\lambda_1 \neq 0$,

$$\left(\frac{G'}{G^2}\right) = -\frac{C_1}{\lambda_1(C_1\eta + C_2)}, \tag{20}$$

where C_1 and C_2 are arbitrary constants.

Step 3: Replacing Eq. (16) into Eq. (14) along with Eq. (17) and tracing all coefficients of each $\left(\frac{G'}{G^2}\right)^i$ to zero, then solving that algebraic equations generated in the terms $a_i,\ \lambda_0,\ \lambda_1,\ \nu$ and other parameters.

Step 4: Replacing Eq. (16) of which $\alpha_i,\ \nu$ and other parameters that are found in step 3 into Eq. (13), we get the solutions of Eq. (12).

$(1/G')$- expansion method

Here, we will describe the basic steps of $(1/G')$ –expansion method [56].

Step 1: Consider Eqs. (12)–(14).

Step 2: Extend the solution of Eq. (14) in the following form

$$U(\eta) = \sum_{i=0}^{m} a_i \left(\frac{1}{G'}\right)^i, \tag{21}$$

where $G = G(\eta)$ and satisfies the following second order linear ODE

$$G''(\eta) + \lambda G'(\eta) + \mu = 0, \tag{22}$$

where $a_i(i = 1, ..., m), \lambda$ and μ are constants to be determined later and the positive integer m is a homogenous balance number. The solution of the differential Eq. (22) is

$$G(\eta) = C_1 e^{-\lambda\eta} - \frac{\mu}{\lambda} + C_2. \tag{23}$$

Then

$$\left(\frac{1}{G'}\right) = \frac{\lambda}{-\mu + \lambda C_1[cosh(\lambda\eta) - sinh(\lambda\eta)]}, \tag{24}$$

can be written, where C_1 and C_2 are arbitrary constants.

Step 3: By substituting Eq. (21) into Eq. (14) and using Eq. (22), the left hand side of Eq. (14) can be converted into a polynomial in term of $(1/G')$ Equating each coefficient of the polynomial to zero yields a system of algebraic equations. By solving the algebraic equations with symbolic computation, we obtain $a_i(i = 1, ..., m),\ \lambda$ and μ.

Mathematical analyses of the models and its solutions

For fractional generalized reaction Duffing model

Let's assume the transformations:

$$u(x, t) = U(\eta),\ \eta = \frac{\Gamma(\beta + 1)}{\alpha}(kx^\alpha - ct^\alpha), \tag{25}$$

where k and c are constants. By using Eq. (25) into Eq. (1), we get the following ODE

$$c^2U'' + pk^2U'' + qU + rU^2 + sU^3 = 0. \tag{26}$$

In the following subsections, the proposed methods are applied to extract the required solutions:

Solutions with the $(G'/G,\ 1/G)$ –expansion method

By applying the homogenous balance technique between the terms U'' and U^3 into Eq. (26), we get $m = 1$.For $m = 1$, Eq. (15) reduces into:

$$U(\eta) = a_0 + a_1\phi(\eta) + b_1\psi(\eta), \tag{27}$$

where $a_0,\ a_1$ and b_1 are unknown parameters.

Case 1. For $\lambda < 0$, substituting Eq. (27) into Eq. (26) along with Eqs. (5) and (7) yields a polynomial equation and setting each coefficient polynomial to zero gives a system of algebraic equations for $a_0,\ a_1,\ b_1$, $\mu,\ \sigma,\ \lambda,\ p,\ q,\ s,\ c$ and k. Solving the obtained system of algebraic eqations by using symbolic computation software MATHEMATICA, we obtain the following results:

$$a_0 = \frac{-3(c^2\lambda + k^2p\lambda)}{2r},\ a_1 = \pm\frac{3i(c^2 + k^2p)\sqrt{\lambda}}{2r}, b_1 = \pm\frac{3(c^2 + k^2p)\sqrt{\mu^2 - \lambda^2\sigma}}{2r},$$
$$q = (c^2 + k^2p)\lambda,\ s = \frac{2r}{9(c^2 + k^2p)\lambda}. \tag{28}$$

Substituting Eq. (28) into Eq. (27), we get the hyperbolic traveling wave solutions of Eq. (1) as follows:

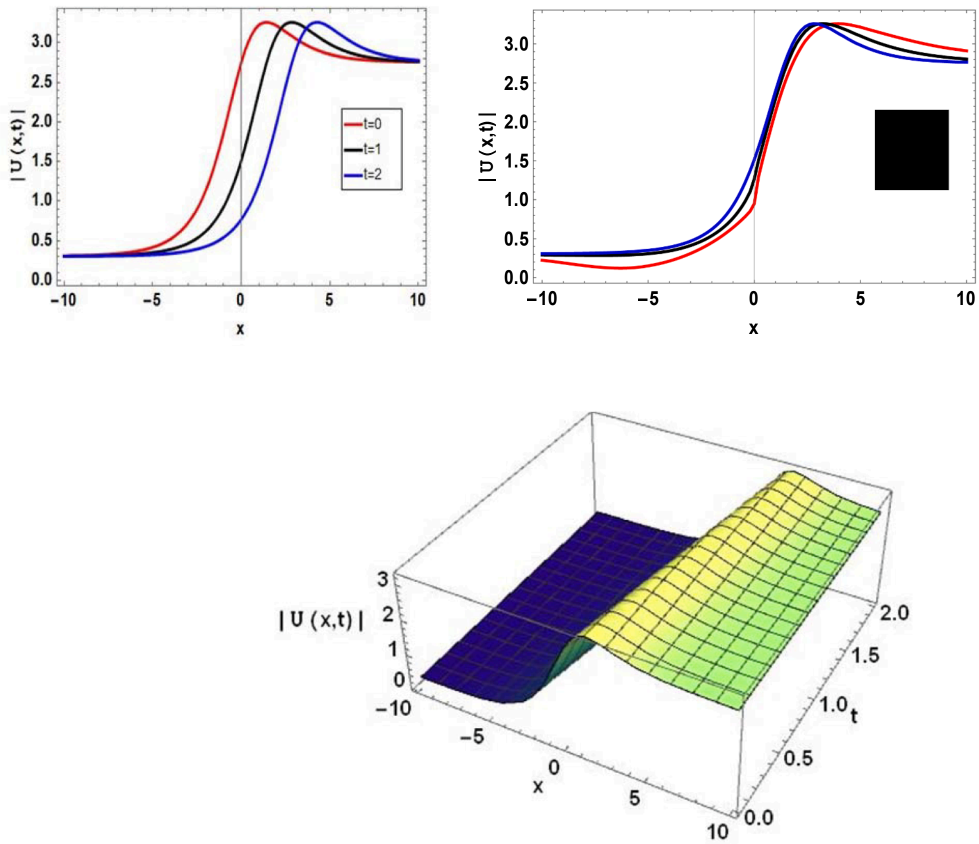

Fig. 1. 2D and 3D graphics of case 1 for hyperbolic traveling wave solution (30) at.$\{k = 0.7, p = -0.05, r = 1, \lambda = -0.8, A_2 = 1, \beta = 2, c = 1\}$

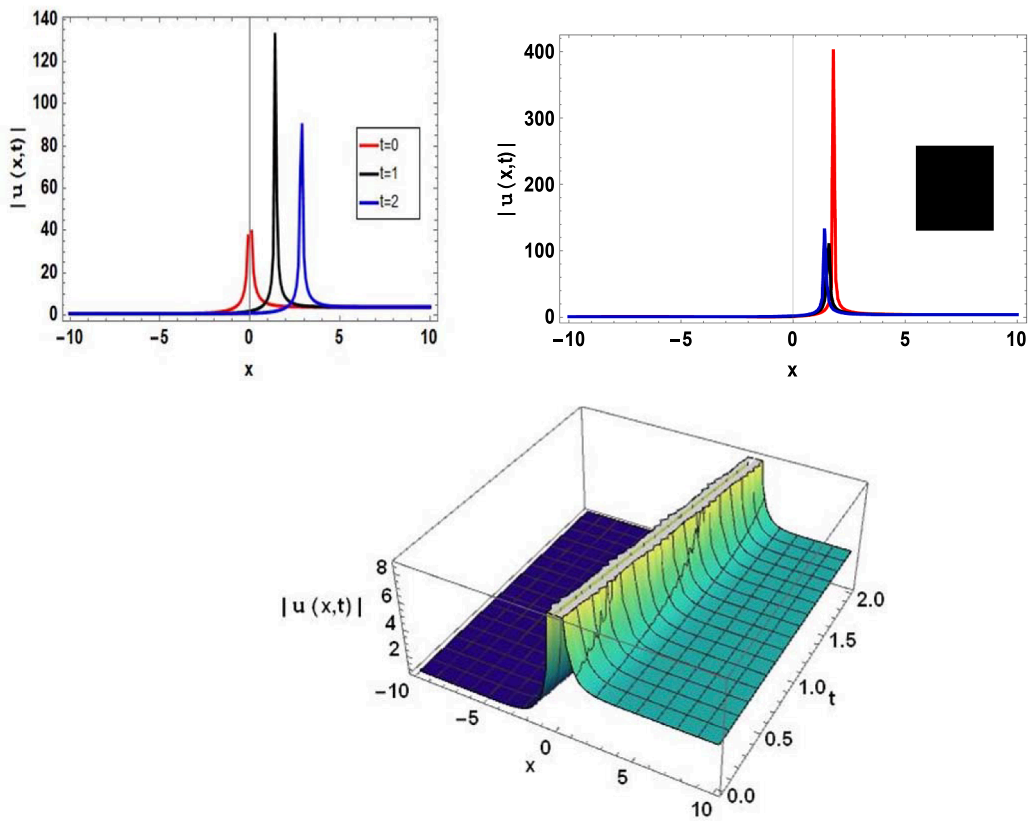

Fig. 2. 2D and 3D graphics of case 1 for hyperbolic traveling wave solution (31) at.$\{k = 0.7, p = -0.05, r = 1, \lambda = -1.5, A_1 = 1, \beta = 2, c = 1\}$

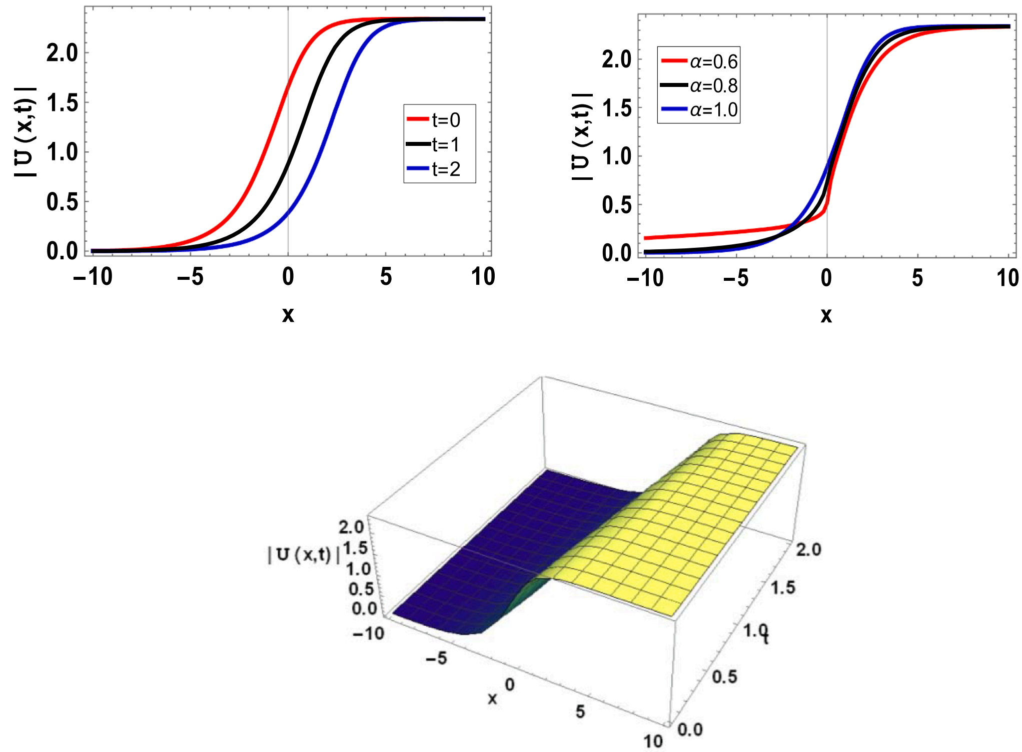

Fig. 3. 2D and 3D graphics of case 2 for trignometric traveling wave solution (34) at.$\{k = 0.7, p = -0.05, r = 1, \lambda = -0.8, A_2 = 1, \beta = 2\}$

$$U(\eta) = \frac{-3(c^2\lambda + k^2p\lambda)}{2r} \pm \frac{3i(c^2+k^2p)\sqrt{\lambda}}{2r}\left(\frac{A_1\sqrt{-\lambda}\,cosh\left(\sqrt{-\lambda}\eta\right) + A_2\sqrt{-\lambda}\,sinh\left(\sqrt{-\lambda}\eta\right)}{A_1 sinh\left(\sqrt{-\lambda}\eta\right) + A_2 cosh\left(\sqrt{-\lambda}\eta\right) + \frac{\mu}{\lambda}}\right)$$
$$\pm \frac{3(c^2+k^2p)\sqrt{\mu^2-\lambda^2\sigma}}{2r}\left(\frac{1}{A_1 sinh\left(\sqrt{-\lambda}\eta\right) + A_2 cosh\left(\sqrt{-\lambda}\eta\right) + \frac{\mu}{\lambda}}\right), \tag{29}$$

where$\sigma = A_1^2 - A_2^2$, $\eta = \frac{\Gamma(\beta+1)}{\alpha}(kx^\alpha - ct^\alpha)$

Family 1.1: If $A_1 = 0$, $A_2 \neq 0$ and $\mu = 0$ in Eq. (29), we obtain the following type of the hyperbolic traveling wave solution: (Sees Fig. 1)

$$U(\eta) = \frac{-3(c^2\lambda + k^2p\lambda)}{2r} \mp \frac{3(c^2+k^2p)}{2r}\lambda tanh\left(\sqrt{-\lambda}\eta\right)$$
$$\pm \frac{3(c^2+k^2p)}{2r}\sqrt{-\lambda^2\sigma}\,\frac{1}{A_2}\text{sech}\left(\sqrt{-\lambda}\eta\right). \tag{30}$$

Family 1.2: If $A_1 \neq 0$, $A_2 = 0$ and $\mu = 0$ in Eq. (29), we obtain the following hyperbolic traveling wave solution: (Sees Fig. 2)

$$U(\eta) = \frac{-3(c^2\lambda + k^2p\lambda)}{2r} \mp \frac{3(c^2+k^2p)}{2r}\lambda coth\left(\sqrt{-\lambda}\eta\right)$$
$$\pm \frac{3(c^2+k^2p)}{2r}\sqrt{-\lambda\sigma}\,\frac{1}{A_1}\text{cosech}\left(\sqrt{-\lambda}\eta\right). \tag{31}$$

Case 2. For $\lambda > 0$, substituting Eq. (27) into Eq. (26) along with Eqs. (5) and (9) yields a polynomial equation and setting each coefficient polynomial to zero gives a set of algebraic equations for a_0, a_1, b_1, μ, σ, λ, p, q, s, c and k. Solving the system of algebraic equations with the help of software MATHEMATICA, we reach the following results:

$$a_0 = \frac{-3(c^2\lambda + k^2p\lambda)}{2r},\ a_1 = \pm\frac{3i(c^2+k^2p)\sqrt{\lambda}}{2r},$$
$$b_1 = \pm\frac{3(c^2+k^2p)\sqrt{\mu^2-\lambda^2\sigma}}{2r},\ q = (c^2+k^2p), s = \frac{2r^2}{9(c^2+k^2p)\lambda}. \tag{32}$$

Substituting Eq. (32) into Eq. (27), we have the following periodic trigonometric traveling wave solution of Eq. (1) as follows:

$$U(\eta) = \frac{-3(c^2\lambda + k^2p\lambda)}{2r} \pm \frac{3i(c^2+k^2p)\sqrt{\lambda}}{2r}\left(\frac{A_1\sqrt{\lambda}\cos\left(\sqrt{\lambda}\eta\right) - A_2\sqrt{\lambda}sin\left(\sqrt{\lambda}\eta\right)}{A_1 sin\left(\sqrt{\lambda}\eta\right) + A_2 cos\left(\sqrt{\lambda}\eta\right) + \frac{\mu}{\lambda}}\right)$$
$$\pm \frac{3(c^2+k^2p)\sqrt{\mu^2-\lambda^2\sigma}}{2r}\left(\frac{1}{A_1 sin\left(\sqrt{\lambda}\eta\right) + A_2 cos\left(\sqrt{\lambda}\eta\right) + \frac{\mu}{\lambda}}\right), \tag{33}$$

where$\sigma = A_1^2 + A_2^2$.

Family 2.1: If $A_1 = 0$, $A_2 \neq 0$ and $\mu = 0$ in Eq. (33), we obtain the following trigonometric traveling wave solution: (Sees Fig. 3)

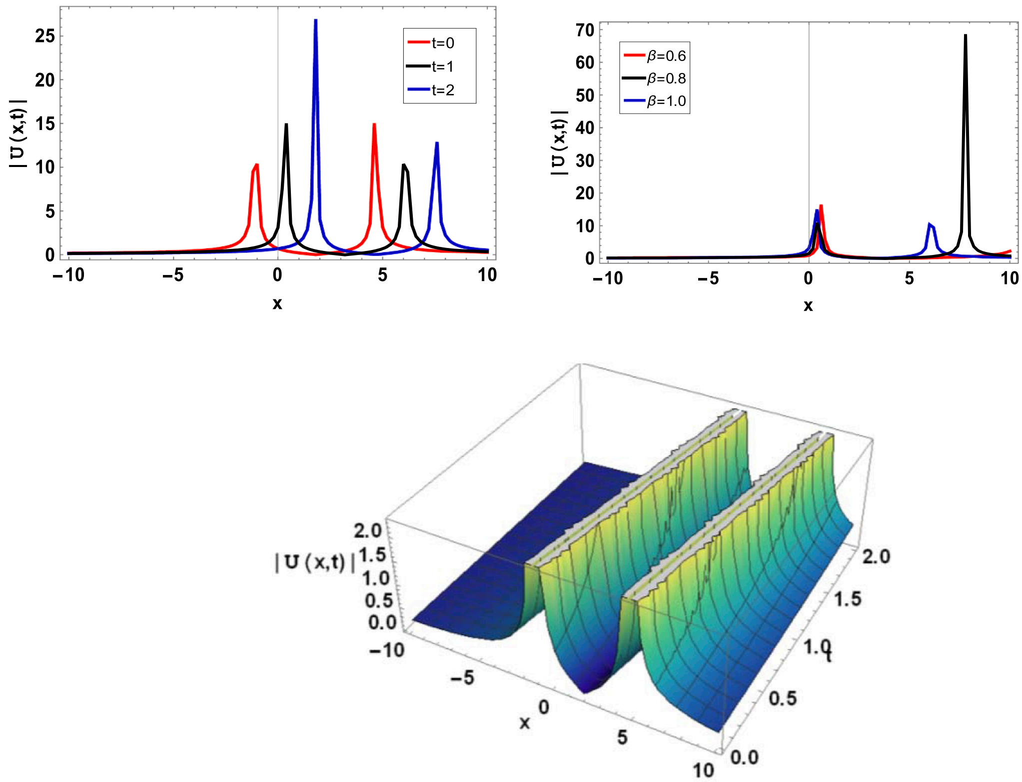

Fig. 4. 2D and 3D graphics of case 3 for rational solution (39) at$\{k = 0.7,\ \mu\ =\ \text{- } 0.8,\ A_2 = 1, A_1 = 1,\ \beta = 2,\ a_2 = 0.3\}$.

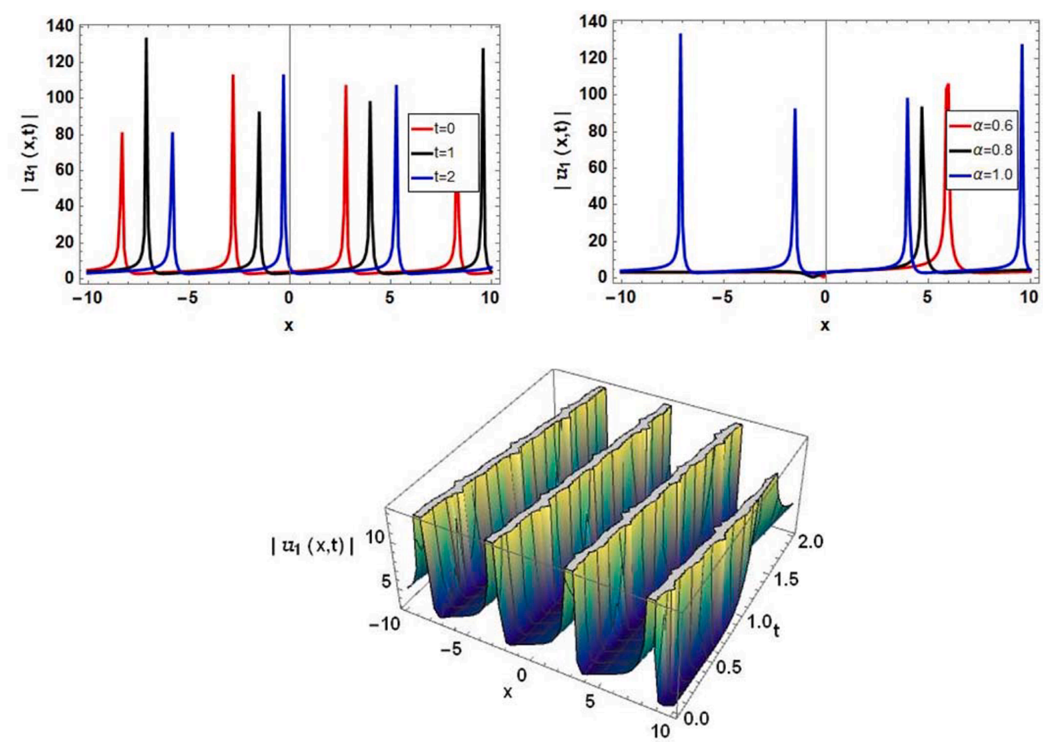

Fig. 5. 2D and 3D graphics of hyperbolic periodic traveling wave solution for (42) at$\{k = 0.8, p = -0.05, r = 1, \lambda_0 = 0.5, \lambda_1 = -1, \beta = 2, C_1 = 1, C_2 = 0\}$

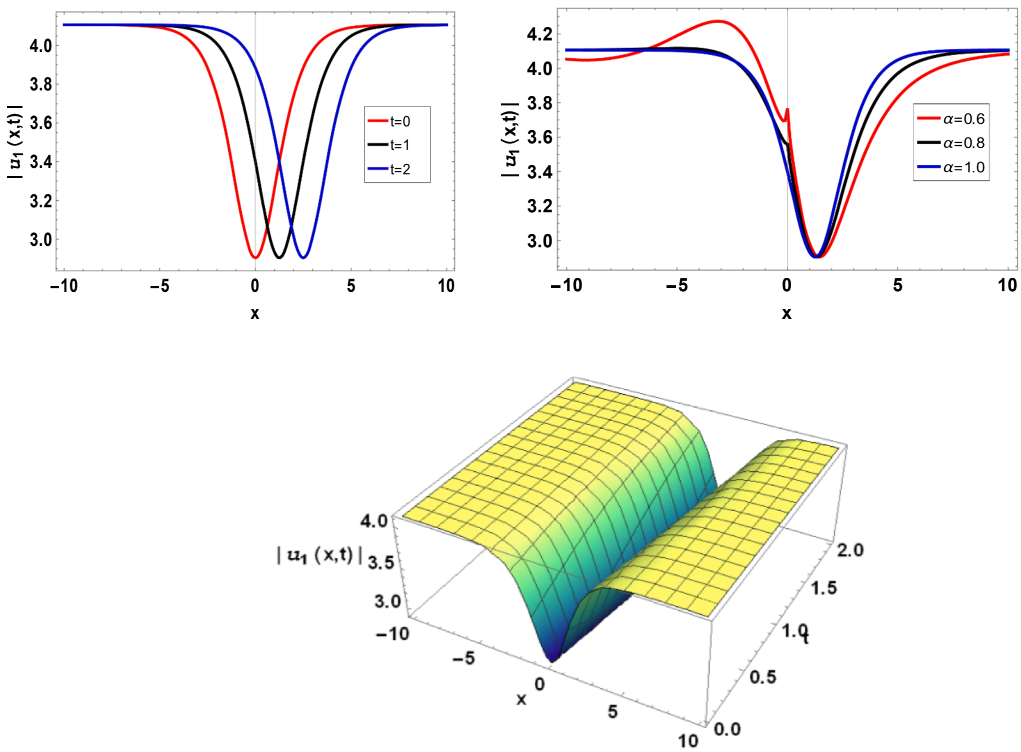

Fig. 6. 2D and 3D graphics of trignometric traveling wave solution for (43) at$\{k = 0.8, p = -0.05, r = 1, \lambda_0 = 0.5, \lambda_1 = -1, \beta = 2, C_2 = 1, C_1 = 0\}$

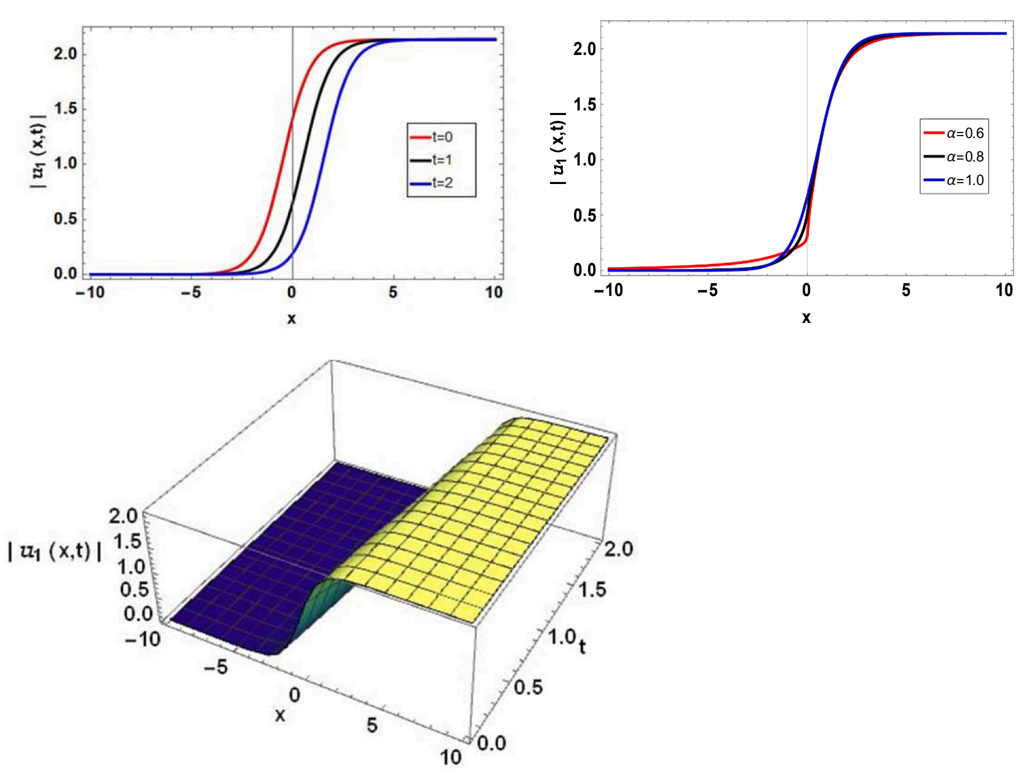

Fig. 7. 2D and 3D graphics of hyperbolic traveling wave solution for (46) at$\{k = 3, p = -0.05, r = 3, \lambda_0 = 0.5, \lambda_1 = -1, \beta = 2, c = 1, C_1 = 1\}$

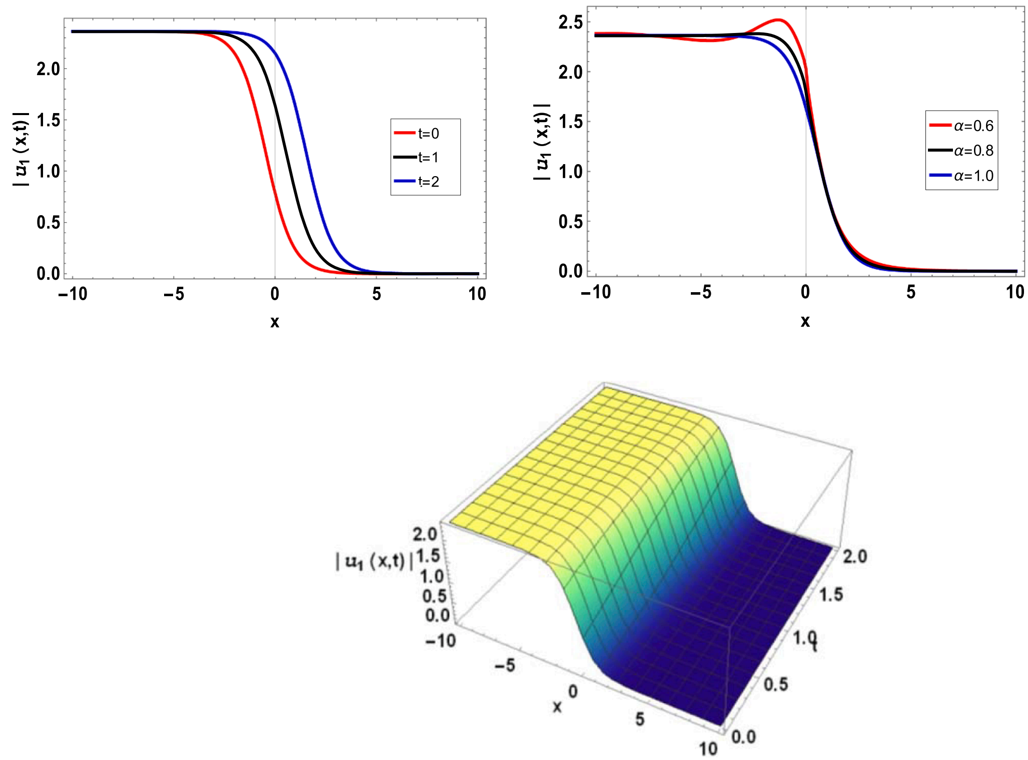

Fig. 8. 2D and 3D graphics of hyperbolic traveling wave solution for (51) at$\{k=3, p=-0.05, r=3, \lambda_0=0.5, \lambda_1=-1, \beta=2, c=1, C_1=1\}$

$$U(\eta)=\frac{-3\left(c^2\lambda+k^2p\lambda\right)}{2r}\mp\frac{3i\left(c^2+k^2p\right)\lambda}{2r}tan\left(\sqrt{\lambda}\,\eta\right)\pm\frac{3\left(c^2+k^2p\right)\sqrt{-\lambda^2\sigma}}{2r}\left(\frac{1}{A_2}sec\left(\sqrt{\lambda}\,\eta\right)\right) \tag{34}$$

Family 2.2: If $A_1\neq0,\ A_2=0$ and $\mu=0$ in Eq. (33), we obtain the following trigonometric traveling wave solution

Case 3. For $\lambda=0$, substituting Eq. (27) into Eq. (26) along with Eqs. (5) and (11) yields a set of algebraic equations for $a_0,\ a_1,\ a_2,\ b_1,\ \mu$ and q. Solving the obtained system of algebraic with the add of MATHEMATICA, we reach the following results:

$$a_0=0,\ a_1=\pm\sqrt{2a_2\mu},\ b_1=0,\ q=0. \tag{38}$$

Substituting Eq. (38) into Eq. (27), we have the following rational solution for Eq. (1): (Sees Fig. 4)

$$U(\eta)=\pm\sqrt{2a_2\mu}\left(\frac{\mu\eta+A_1}{\frac{\mu}{2}\eta^2+A_1\eta+A_2}\right) \tag{39}$$

Solutions with the modified $(G^{'}/G^2)$ −expansion method

By applying the homogenous balance technique into Eq. (26), we get $m=1$. For $m=1$, Eq. (16) reduces in the form

$$U(\eta)=a_0+a_1\left(\frac{G^{'}}{G^2}\right), \tag{40}$$

where a_0 and a_1 are unknown parameters. By using Eq. (40) with Eq. (17) into Eq. (26) and summing up all the coefficients of same order of $(G^{'}/G^2)$, we get the set of algebraic equations involving $a_0,\ a_1$ and other parameters. Solving the obtained set of algebraic equations with MATHEMATICA, we reach the following results:

$$a_0=\frac{-6\left(c^2\lambda_0\lambda_1+k^2p\lambda_0\lambda_1\right)}{r},\ a_1=\frac{\pm6i\left(c^2+k^2p\right)\sqrt{\lambda_0}\lambda_1^{3/2}}{r}, q=4\left(c^2\lambda_0\lambda_1+k^2\lambda_0\lambda_1\right),\ \ s=\frac{r^2}{18\left(c^2+k^2p\right)\lambda_0\lambda_1}. \tag{41}$$

Now we use the Eqs. (41), (18)–(20) into Eq. (40) and set the below cases.

If $\lambda_0\lambda_1<0$, then, we have hyperbolic traveling wave solution of Eq. (1): (Sees Fig. 5)

$$U(\eta)=\frac{-6\left(c^2\lambda_0\lambda_1+k^2p\lambda_0\lambda_1\right)}{r}\pm\frac{6\left(c^2+k^2p\right)\sqrt{\lambda_0}\lambda_1^{3/2}}{r}\left(\frac{-\sqrt{|\lambda_0\lambda_1|}}{\lambda_1}+\frac{\sqrt{|\lambda_0\lambda_1|}}{2}\left[\frac{C_1\ sinh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)+C_2\ cosh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}{C_1\ cosh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)+C_2\ sinh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}\right]\right) \tag{42}$$

If $\lambda_0\lambda_1>0$, we have trignometric traveling wave solution of Eq. (1): (Sees Fig. 6)

$$U(\eta)=\frac{-6\left(c^2\lambda_0\lambda_1+k^2p\lambda_0\lambda_1\right)}{r}\pm\frac{6\left(c^2+k^2p\right)\sqrt{\lambda_0}\lambda_1^{3/2}}{r}\left(\sqrt{\frac{\lambda_0}{\lambda_1}}\left[\frac{C_1\,cos\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)+C_2\,sin\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}{C_1\,sin\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)-C_2\,cos\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}\right]\right) \tag{43}$$

Solutions with the $(1/G^{'})$-expansion method

By applying the homogenous balance technique into Eq. (26), we get $m=1$. For $m=1$, Eq. (21) reduces into:

$$U(\eta)=a_0+a_1\left(\frac{1}{G^{'}}\right), \tag{44}$$

where a_0 and a_1 are unknown parameters. By substituting Eq. (44) with Eq. (24) into Eq. (26) and summing up all the coefficients of same order

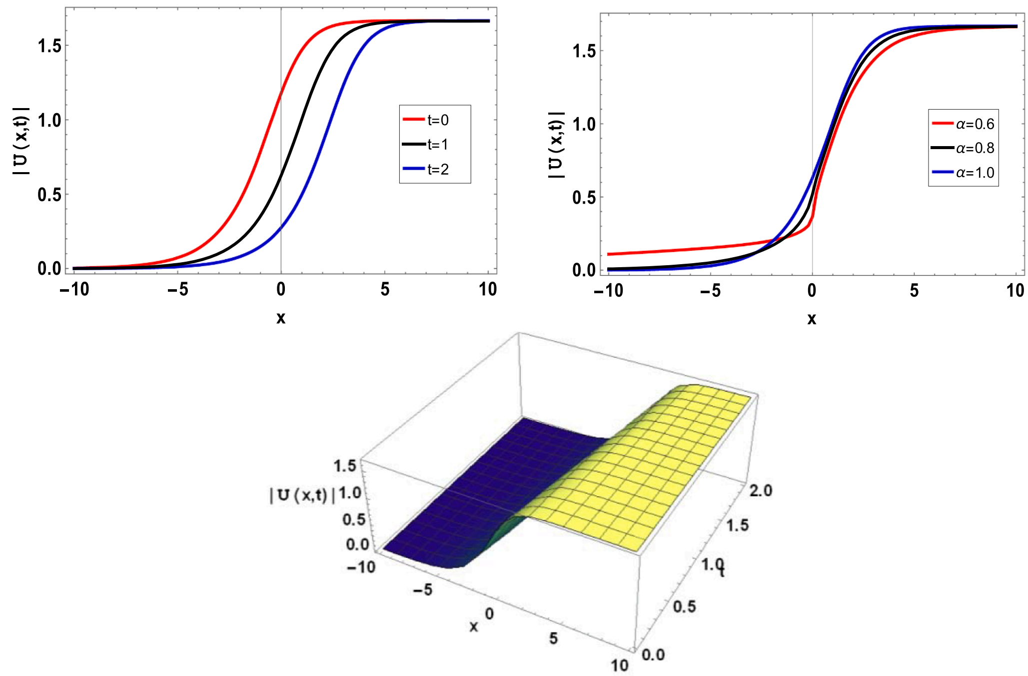

Fig. 9. 2D and 3D graphics of case 1 for hyperbolic traveling wave solution (56) at$\{s = 0.7, a = 0.5, b = 0.3, \lambda = -0.8, A_2 = 1, \beta = 2, c = 1\}$

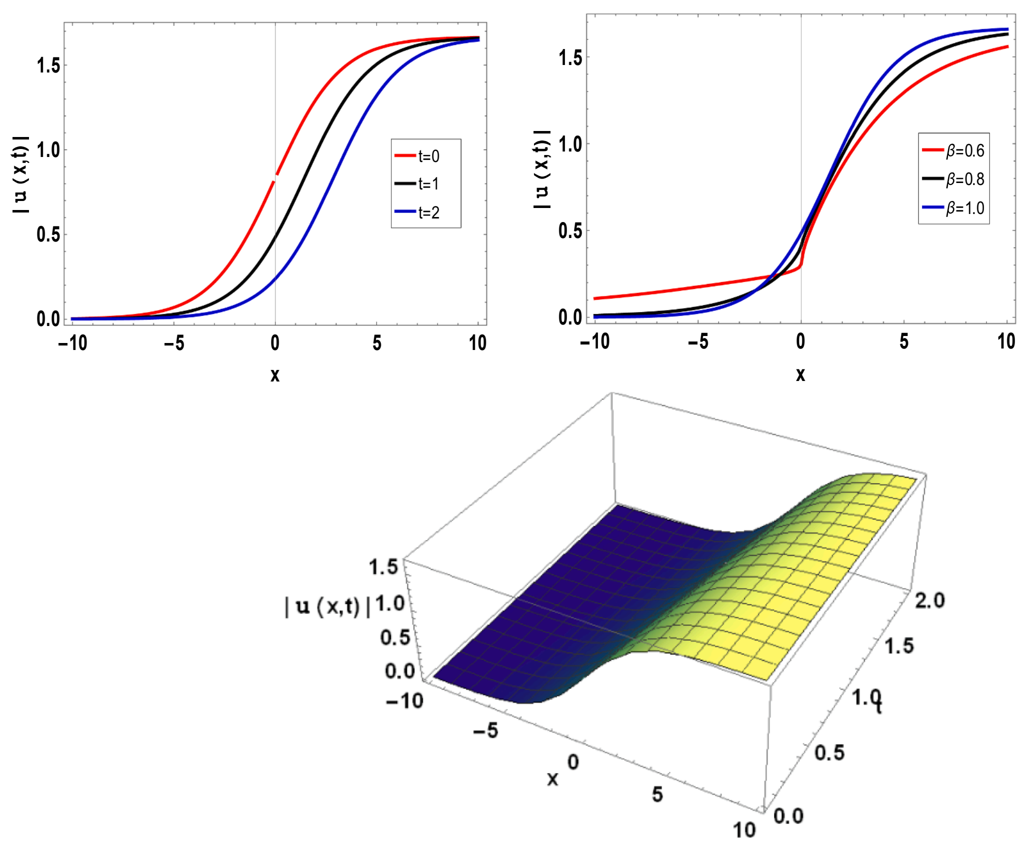

Fig. 10. 2D and 3D graphics of case 1 for hyperbolic traveling wave solution (57) at$\{s = 0.7, a = 0.5, b = 0.3, \lambda = -0.8, A_1 = 1, \beta = 2, c = 1\}$

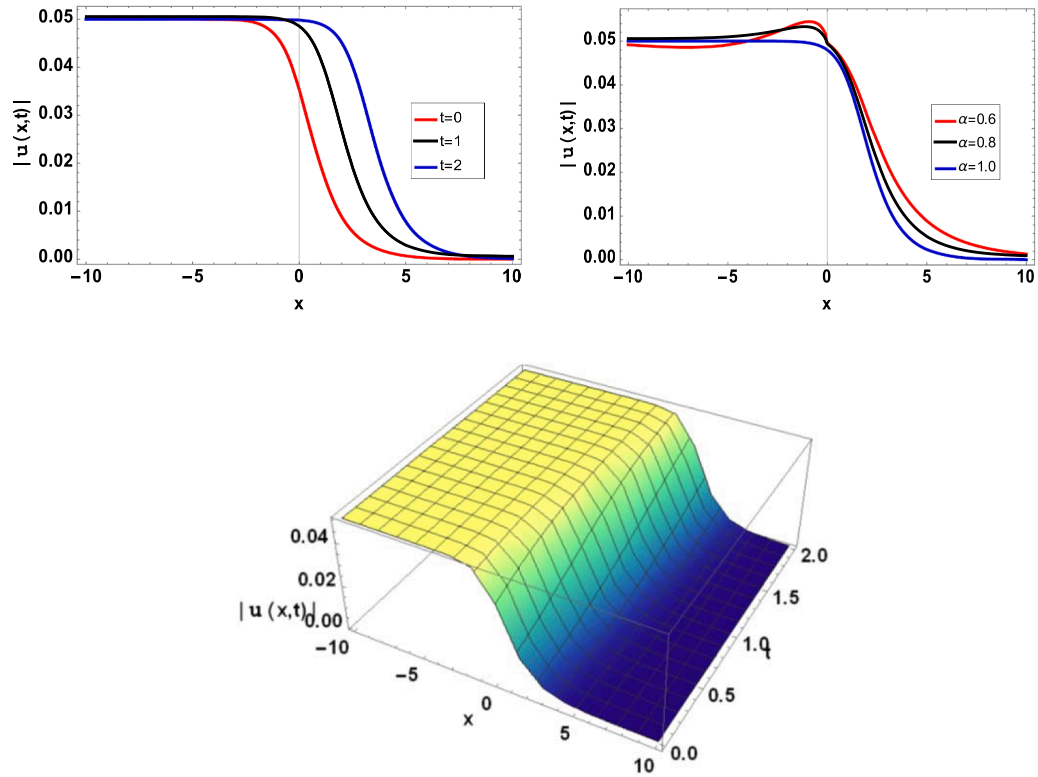

Fig. 11. 2D and 3D graphics of case 2 for trignometric traveling wave solution (60) at$\{s=0.7, a=0.5, b=0.3, \lambda=-0.8, A_1=1, \beta=2, c=1\}$

of $(1/G')$, we get the algebraic equations involving a_0, a_1 and other parameters. Then by solving the obtained set of algebraic equations by MATHEMATICA, we reach the following results:

Set 1:

$$a_0=0, a_1=\frac{-3\left(c^2\lambda\mu+k^2p\lambda\mu\right)}{r}, q=-c^2\lambda^2-k^2p\lambda, s=\frac{-2r^2}{9\lambda^2\left(c^2+k^2p\right)}. \tag{45}$$

Replacing values of Eq. (45) into Eq. (44), we have the following different type hyperbolic traveling wave solution of Eq. (1): (Sees Fig. 7)

$$U(\eta)=\frac{-3\left(c^2\lambda\mu+k^2p\lambda\mu\right)}{r}\left(\frac{\lambda}{-\mu+\lambda C_1[cosh(\lambda\eta)-sinh(\lambda\eta)]}\right) \tag{46}$$

Set 2:

$$a_0=\frac{3\left(c^2\lambda^2+k^2p\lambda^2\right)}{r}, a_1=\frac{3\left(c^2\lambda\mu+k^2p\lambda\mu\right)}{r}, q=-c^2\lambda^2-k^2p\lambda^2,\ s=\frac{-2r^2}{9\lambda^2\left(c^2+k^2p\right)}, \tag{50}$$

Replacing values of Eq. (50) into Eq. (44), we have the following different type hyperbolic traveling wave solution of Eq. (1): (Sees Fig. 8)

$$u_2(x,t)=\frac{3\left(c^2\lambda^2+k^2p\lambda^2\right)}{r}+\frac{3\left(c^2\lambda\mu+k^2p\lambda\mu\right)}{r}\left(\frac{\lambda}{-\mu+\lambda C_1[cosh(\lambda\eta)-sinh(\lambda\eta)]}\right) \tag{51}$$

For Density-Dependent fractional Diffusion-Reaction equation

Adopting the similar procedure as in sec. 4.1. Let's assume the transformation:

$$u(x,t)=U(\eta),\ \eta=\frac{\Gamma(\beta+1)}{\alpha}\left(sx^\alpha-ct^\alpha\right), \tag{52}$$

where s and c are constants. By using Eq. (52) into Eq. (2), we get the following ODE

$$Ds^2U''-cU'-ksUU'+aU-bU^2=0. \tag{53}$$

In the following subsections, the proposed methods are applied to extract the required solutions:

Solutions with the$(G'/G, 1/G)$-expansion method

By applying the homogenous balance technique between the terms U'' and UU' into Eq. (53), we get $m=1$.For $m=1$, Eq. (15) reduces in the form of Eq. (27).

Case 1. For $\lambda<0$, substituting Eq. (27) into Eq. (53) along with Eqs. (5) and (7) yields a polynomial equation and setting each coefficient polynomial to zero gives a set of algebraic equations fora_0, a_1, b_1, μ, σ, λ, s, c and D. Solving the obtained system of algebraic equations with MATHEMATICA, we reach the following results:

$$a_0=\frac{a}{2b},\ a_1=\pm\frac{ia}{2b\sqrt{\lambda}},\ b_1=\pm\frac{a\sqrt{\mu^2+\lambda^2\sigma}}{2b\lambda},\ c=\pm\frac{i\left(4ab^2D+a^2k^2\right)}{4b^2D\sqrt{\lambda}},\ s=\pm\frac{iak}{2bD\sqrt{\lambda}}. \tag{54}$$

Substituting Eq. (54) into Eq. (27), we get the hyperbolic traveling wave solution of Eq. (2) as follows:

$$U(\eta)=\frac{a}{2b}\pm\frac{ia}{2b\sqrt{\lambda}}\left(\frac{A_1\sqrt{-\lambda}\,cosh\left(\sqrt{-\lambda}\eta\right)+A_2\sqrt{-\lambda}\,sinh\left(\sqrt{-\lambda}\eta\right)}{A_1sinh\left(\sqrt{-\lambda}\eta\right)+A_2cosh\left(\sqrt{-\lambda}\eta\right)+\frac{\mu}{\lambda}}\right)\pm\frac{a\sqrt{\mu^2+\lambda^2\sigma}}{2b\lambda}\left(\frac{1}{A_1sinh\left(\sqrt{-\lambda}\eta\right)+A_2cosh\left(\sqrt{-\lambda}\eta\right)+\frac{\mu}{\lambda}}\right), \tag{55}$$

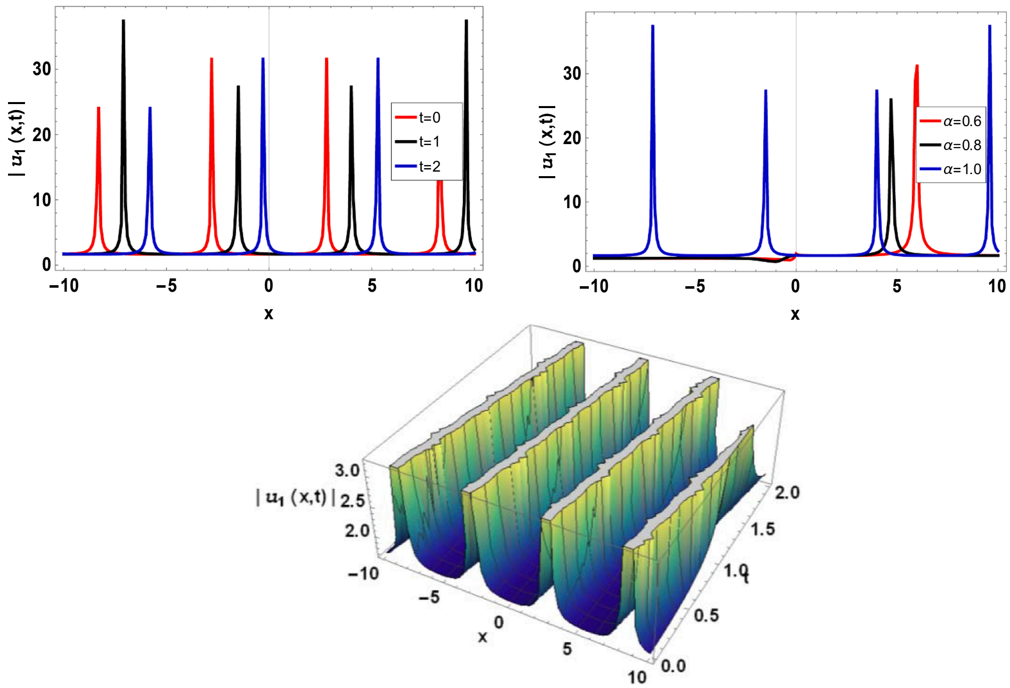

Fig. 12. 2D and 3D graphics of hyperbolic periodic traveling wave solution for (65) at$\{s = 0.8, a = 0.5, b = 0.3, \lambda_0 = 0.5, \lambda_1 = -1, \beta = 2, C_1 = 1, C_2 = 0, c = 1\}$

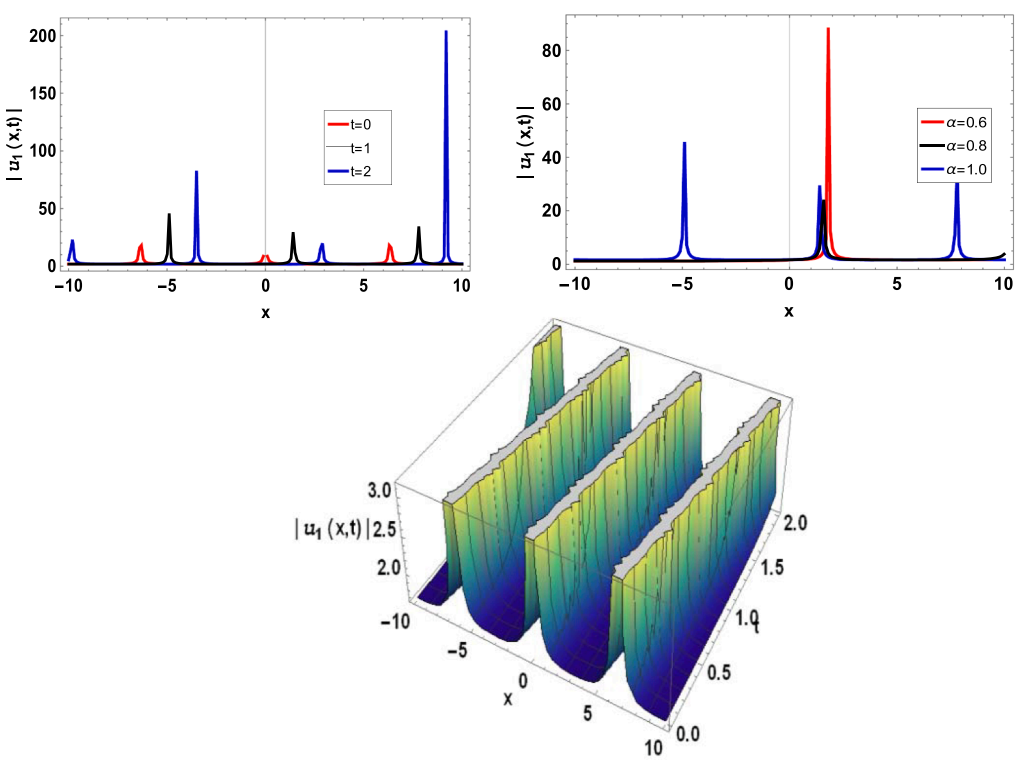

Fig. 13. 2D and 3D graphics of hyperbolic perionic traveling wave solution for (68) at$\{s = 0.7, a = 0.5, b = 0.3, \lambda_0 = 0.5, \lambda_1 = -1, \beta = 2, C_1 = 0, C_2 = 1, c = 1\}$

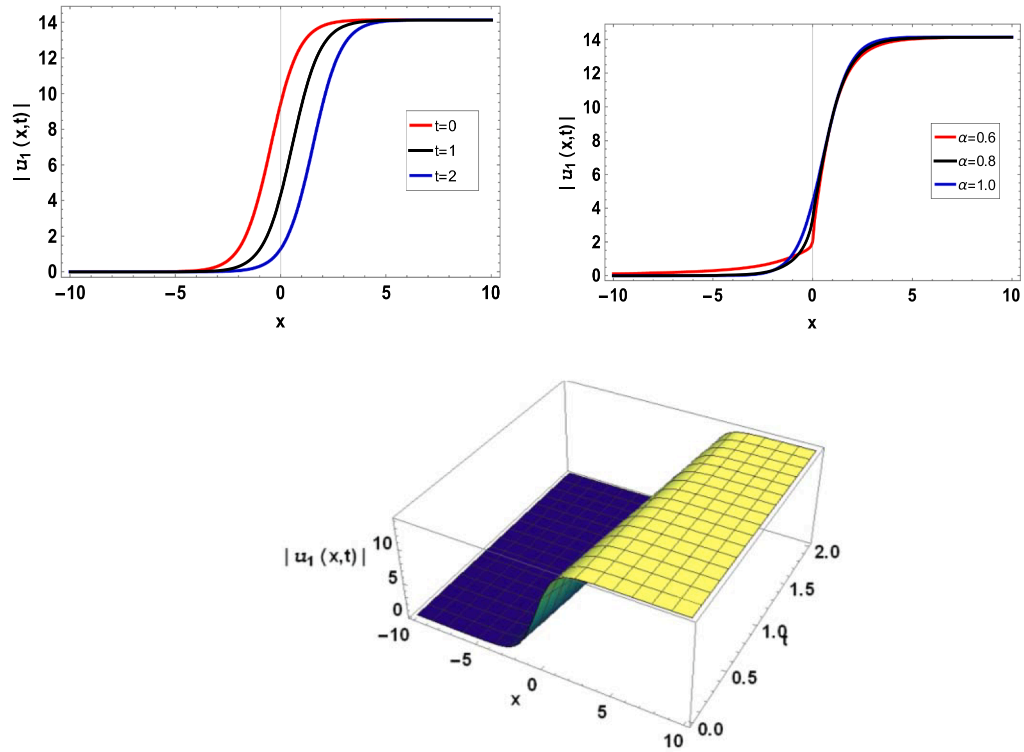

Fig. 14. 2D and 3D graphics of hyperbolic rational solution for (71) at$\{s = 3, a = 0.5, b = 1, \lambda_0 = 0.5, \lambda_1 = -1, \beta = 2, C_1 = 1, \mathrm{k} = 0.2, c = 3\}$

where $\sigma = A_1^2 - A_2^2$, $\eta = \frac{\Gamma(\beta+1)}{\alpha}(sx^\alpha - ct^\alpha)$

Family 1.1: If $A_1 = 0$, $A_2 \neq 0$ and $\mu = 0$ in Eq. (55), we obtain the following hyperbolic traveling wave solution: (Sees Fig. 9)

$$U(\eta) = \frac{a}{2b} \pm \frac{ia}{2b\sqrt{\lambda}}\sqrt{-\lambda}tanh\left(\sqrt{-\lambda}\eta\right) \pm \frac{a\sqrt{\lambda^2\sigma}}{2b\lambda}\frac{1}{A_2}\mathrm{sech}\left(\sqrt{-\lambda}\eta\right). \tag{56}$$

Family 1.2: If $A_1 \neq 0$, $A_2 = 0$ and $\mu = 0$ in Eq. (55), we obtain the following hyperbolic traveling wave solution: (Sees Fig. 10)

$$U(\eta) = \frac{a}{2b} \pm \frac{ia}{2b\sqrt{\lambda}}\sqrt{-\lambda}coth\left(\sqrt{-\lambda}\eta\right) \pm \frac{a\sqrt{\lambda^2\sigma}}{2b\lambda}\frac{1}{A_1}\mathrm{cosech}\left(\sqrt{-\lambda}\eta\right). \tag{57}$$

Case 2. For $\lambda > 0$, substituting Eq. (27) into Eq. (53) along with Eqs. (5) and (9) yields a polynomial equation and setting each coefficient polynomial to zero gives a set of algebraic equations for a_0, a_1, b_1, μ, σ, λ, s, c and D. Then by solving the obtained system of algebraic equations with MATHEMATICA, we reach the following results:

$$a_0 = \frac{a}{2b},\ a_1 = \pm\frac{ia}{2b\sqrt{\lambda}},\ b_1 = \pm\frac{a\sqrt{\mu^2 - \lambda^2\sigma}}{2b\lambda},\ c = \pm\frac{i\left(4ab^2D + a^2k^2\right)}{4b^2D\sqrt{\lambda}},\ s = \pm\frac{iak}{2bD\sqrt{\lambda}}. \tag{58}$$

Substituting Eq. (58) into Eq. (53), we have the following trigonometric traveling wave solution for Eq. (2).

$$U(\eta) = \frac{a}{2b} \pm \frac{ia}{2b\sqrt{\lambda}}\left(\frac{A_1\sqrt{\lambda}\cos\left(\sqrt{\lambda}\eta\right) - A_2\sqrt{\lambda}\sin\left(\sqrt{\lambda}\eta\right)}{A_1 sin\left(\sqrt{\lambda}\eta\right) + A_2 cos\left(\sqrt{\lambda}\eta\right) + \frac{\mu}{\lambda}}\right) \pm \frac{a\sqrt{\mu^2 - \lambda^2\sigma}}{2b\lambda}\left(\frac{1}{A_1 sin\left(\sqrt{\lambda}\eta\right) + A_2 cos\left(\sqrt{\lambda}\eta\right) + \frac{\mu}{\lambda}}\right), \tag{59}$$

where $\sigma = A_1^2 + A_2^2$.

Family 2.1: If $A_1 = 0$, $A_2 \neq 0$ and $\mu = 0$ in Eq. (59), we have the following trigonometric traveling wave solution: (Sees Fig. 11)

$$U(\eta) = \frac{a}{2b} \pm \frac{ia}{2b}tan\left(\sqrt{\lambda}\eta\right) \pm \frac{a\sqrt{-\lambda^2\sigma}}{2b\lambda}\left(\frac{1}{A_2}sec\left(\sqrt{\lambda}\eta\right)\right) \tag{60}$$

Family 2.2: If $A_1 \neq 0$, $A_2 = 0$ and $\mu = 0$ in Eq. (59), we have the following trigonometric traveling wave solution:

$$U(\eta) = \frac{a}{2b} \pm \frac{ia}{2b}cot\left(\sqrt{\lambda}\eta\right) \pm \frac{a\sqrt{-\lambda^2\sigma}}{2b\lambda}\left(\frac{1}{A_1}cosec\left(\sqrt{\lambda}\eta\right)\right) \tag{61}$$

Case 3. For $\lambda = 0$, substituting Eq. (27) into Eq. (53) along with Eqs. (5) and (11) yields a set of algebraic equations for a_0, a_1, b_1, μ, σ, λ, s, c and D. Then by solving the obtained system of algebraic equations with software MATHEMATICA, we reach the following results:

$$a_0 = 0,\ a_1 = \pm\sqrt{2a_2\mu},\ b_1 = 0,\ a = 0. \tag{62}$$

Substituting Eq. (62) into Eq. (53), we have the following different

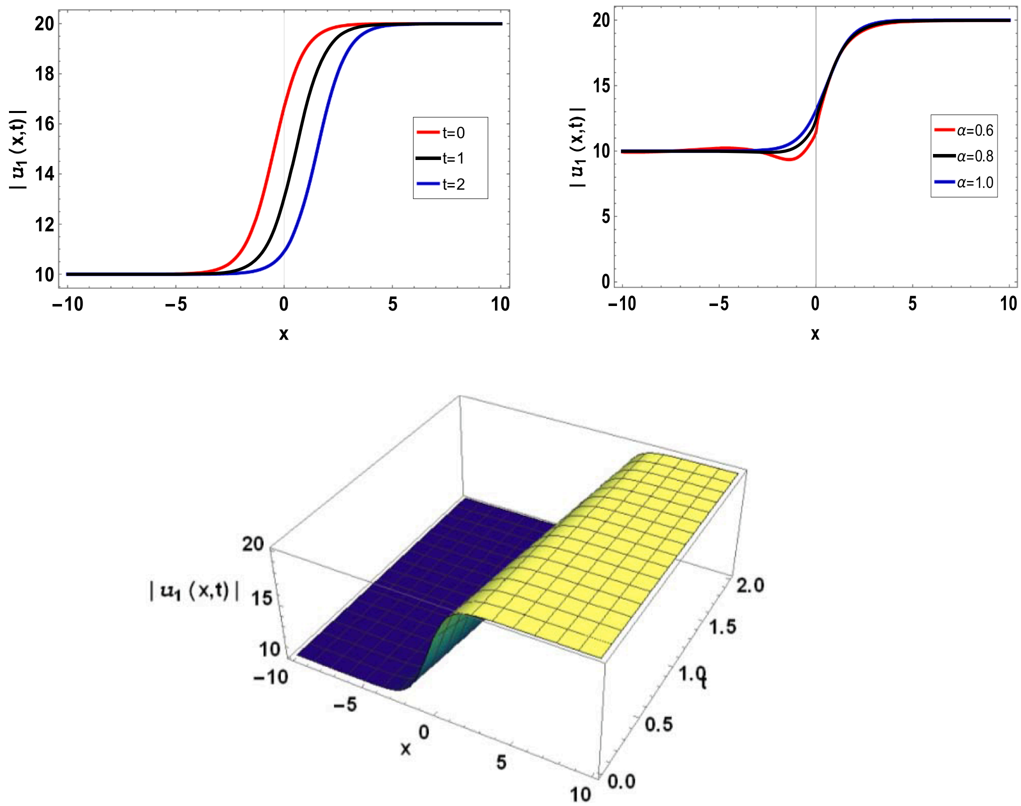

Fig. 15. 2D and 3D graphics of hyperbolic traveling wave solution for (73) at$\{s = 3, a = 0.5, b = 1, \lambda_0 = 0.5, \lambda_1 = -1, \beta = 2, C_1 = 1, \mathrm{k} = 0.2, c = 3\}$

type hyperbolic traveling wave solution for Eq. (2):

$$U(\eta) = \pm\sqrt{2a_2\mu}\left(\frac{\mu\eta + A_1}{\frac{\mu}{2}\eta^2 + A_1\eta + A_2}\right) \tag{63}$$

Solutions with the modified (G'/G^2) *−expansion method*

By applying the homogenous balance technique into Eq. (53), we get $m = 1$. For $m = 1$, Eq. (16) reduces in the form of Eq. (40)

By using Eq. (40) with Eq. (17) into Eq. (53) and summing up all the coefficients of same order of (G'/G^2), we get the set of algebraic equations involving a_0, a_1 and other parameters, then by solving the algebraic equations with software MATHEMATICA, we get the following solutions (Sees Figs. 9 and 10):

Set 1

$$a_0 = \frac{a}{2b}, a_1 = \pm\frac{ia\sqrt{\lambda_1}}{2b\sqrt{\lambda_0}}, s = 0, c = \pm\frac{ia}{2\sqrt{\lambda_0}\sqrt{\lambda_1}}. \tag{64}$$

We now using Eqs. (64), (18)–(20) into Eq. (40) and set to the below cases.

If $\lambda_0\lambda_1 < 0$, then we have hyperbolic traveling wave solution of Eq. (2): (Sees Fig. 12)

$$U(\eta) = \frac{a}{2b}\left(1 \pm \frac{i\sqrt{\lambda_1}}{\sqrt{\lambda_0}}\left(\frac{-\sqrt{|\lambda_0\lambda_1|}}{\lambda_1} + \frac{\sqrt{|\lambda_0\lambda_1|}}{2}\left[\frac{C_1 sinh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right) + C_2 sinh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}{C_1 cosh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right) + C_2 sinh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}\right]\right)\right). \tag{65}$$

if $\lambda_0\lambda_1 > 0$, then we have trignimetric traveling wave solution of Eq. (2):

$$U(\eta) = \frac{a}{2b} \pm \frac{ia\sqrt{\lambda_1}}{2b\sqrt{\lambda_0}}\left(\sqrt{\frac{\lambda_0}{\lambda_1}}\left[\frac{C_1 cos\left(\sqrt{\lambda_0\lambda 1}\,\eta\right) + C_2 sin\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}{C_1 sin\left(\sqrt{\lambda_0\lambda_1}\,\eta\right) - C_2 sin\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}\right]\right). \tag{66}$$

Set2:

$$a_0 = \frac{a}{2b}, a_1 = \pm\frac{ia\sqrt{\lambda_1}}{2b\sqrt{\lambda_0}},\ \mathrm{s} = \pm\frac{iak}{4bD\sqrt{\lambda_0}\sqrt{\lambda_1}},\ \mathrm{c} = \pm\frac{i\left(4ab^2 + a^2k^2\right)}{8b^2D\sqrt{\lambda_0}\sqrt{\lambda_1}}. \tag{67}$$

If $\lambda_0\lambda_1 < 0$, then we have hyperbolic traveling wave solution of Eq. (2): (Sees Fig. 13)

$$U(\eta) = \frac{a}{2b}\left(1 \pm \frac{i\sqrt{\lambda_1}}{\sqrt{\lambda_0}}\left(\frac{-\sqrt{|\lambda_0\lambda_1|}}{\lambda_1} + \frac{\sqrt{|\lambda_0\lambda_1|}}{2}\left[\frac{C_1 sinh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right) + C_2 cosh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}{C_1 cosh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right) + C_2 sinh\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}\right]\right)\right) \tag{68}$$

If $\lambda_0\lambda_1 > 0$, then we have trignimetric traveling wave solution of Eq. (2):

$$U(\eta) = \frac{a}{2b} \pm \frac{ia\sqrt{\lambda_1}}{2b\sqrt{\lambda_0}}\left(\sqrt{\frac{\lambda_1}{\lambda_0}}\left[\frac{C_1 cos\left(\sqrt{\lambda_0\lambda_1}\,\eta\right) + C_2 sin\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}{C_1 sin\left(\sqrt{\lambda_0\lambda_1}\,\eta\right) - C_2 sin\left(\sqrt{\lambda_0\lambda_1}\,\eta\right)}\right]\right) \tag{69}$$

Solutions with the $(1/G')$ *−expansion method*

By applying the homogenous balance technique into Eq. (53), we get $m = 1$. For $m = 1$, Eq. (21) reduces into the form of Eq. (44):

By substituting Eq. (44) with Eq. (24) into Eq. (53) and summing up all the coefficients of same order of $(1/G')$, we get the algebraic equations involving a_0, $a_1$and other parameters. Then by solving the obtained system of algebraic equations with MATHEMATICA, we reach the following results:

Set 1:

$$a_0 = 0, a_1 = \pm\frac{2\sqrt{D}\sqrt{-a - c\lambda}\mu}{k\lambda}, b = \frac{ak}{2\sqrt{D}\sqrt{-a - c\lambda}}, s = \pm\frac{\sqrt{-a - c\lambda}}{\sqrt{D}\lambda}. \tag{70}$$

Replacing the values of Eq. (70) into Eq. (44), we have the following different type hyperbolic traveling wave solution of Eq. (2): (Sees

Fig. 14)

$$U(\eta) = \pm\frac{2\sqrt{D}\sqrt{-a-c\lambda}\mu}{k}\left(\frac{1}{-\mu+\lambda C_1[cosh(\lambda\eta)-sinh(\lambda\eta)]}\right) \tag{71}$$

Set 2:

$$a_0 = \frac{-2\sqrt{D}\sqrt{-a+c\lambda}}{k}, a_1 = \frac{-2\sqrt{D}\sqrt{-a+c\lambda}\mu}{k\lambda}, b = \frac{ak\sqrt{-a+c\lambda}}{\sqrt{D}(2a-2c\lambda)}, s = \frac{-\sqrt{-a+c\lambda}}{\sqrt{D}\lambda}. \tag{72}$$

Replacing the values of Eq. (72) into Eq. (44), we have the following different type hyperbolic traveling wave solution of Eq. (2) (see Fig. 15)

$$U(\eta) = \frac{-2\sqrt{D}\sqrt{-a+c\lambda}}{k} - \frac{2\sqrt{D}\sqrt{-a+c\lambda}\mu}{k}\left(\frac{1}{-\mu+\lambda C_1[cosh(\lambda\eta)-sinh(\lambda\eta)]}\right) \tag{73}$$

Set 3:

$$a_0 = \frac{2\sqrt{D}\sqrt{-a+c\lambda}}{k}, a_1 = \frac{2\sqrt{D}\sqrt{-a+c\lambda}\mu}{k\lambda}, b = \frac{ak}{2\sqrt{D}\sqrt{-a+c\lambda}}, s = \frac{\sqrt{-a+c\lambda}}{\sqrt{D}\lambda}. \tag{74}$$

Replacing the values of Eq. (74) into Eq. (44), we have the following different type hyperbolic traveling wave solution of Eq. (2)

$$U(\eta) = \frac{2\sqrt{D}\sqrt{-a+c\lambda}}{k} + \frac{2\sqrt{D}\sqrt{-a+c\lambda}\mu}{k}\left(\frac{1}{-\mu+\lambda C_1[cosh(\lambda\eta)-sinh(\lambda\eta)]}\right) \tag{75}$$

Conclusions

In this article, the three dependent expansion methods$(G'/G, 1/G)$, modified $(G'/G^2)$and $(1/G')$ have been applied to the M- fractional generalized reaction Duffing model and density dependent M-fractional diffusion reaction equation. M-fractional truncated derivative is used. A variety of new exact solutions in the form of hyperbolic and trigonometric functions have obtained. We have also depicted some of the obtained solutions graphically and concluded that the obtained results are accurate, efficient and versatile in mathematical physics to solve other NLEEs. Also, it has observed that the results obtained in this chapter have been presented for the first time.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

The publication of this article was funded by the Qatar National Library.

References

[1] Podlubny I. Fractional differential equation. San Diego: Academic Press; 1999.
[2] Samko SG, Kilbas AA, Igorevich O. Fractional integrals and Derivatives. Yverdon, Gordon and Breach: Theory and Applications; 1993.
[3] Kallel W, Almusawa H, Mirhosseini-Alizamini SM, Eslami M, Rezazadeh H, Osman MS. Optical soliton solutions for the coupled conformable Fokas-Lenells equation with spatio-temporal dispersion. Results Phys 2021;26:104388.
[4] Djennadi S, Shawagfeh N, Inc M, Osman MS, Gómez-Aguilar JF, Abu Arqub O. The Tikhonov regularization method for the inverse source problem of time fractional heat equation in the view of ABC-fractional technique. Phys Scr 2021;96(9): 094006. https://doi.org/10.1088/1402-4896/ac0867.
[5] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. Amsterdam: Elsevier Science; 2006.
[6] Baleanu D, Diethelm K, Scalas E, Trujillo JJ. Fractional calculus models and numerical methods. series on complexity, nonlinearity and chaos. Boston: World Scientific; 2012.
[7] Cuahutenango-Barro B, Taneco-Hernández MA, Lv YP, Gómez-Aguilar JF, Osman MS, Jahanshahi H, et al. Analytical solutions of fractional wave equation with memory effect using the fractional derivative with exponential kernel. Results Phys 2021;25:104148.
[8] Dhawan S, Machado JAT, Brzeziński DW, Osman MS. A chebyshev wavelet collocation method for some types of differential problems. Symmetry 2021;13(4): 536.
[9] Hirota R. Exact solution of the Kortewegde Vries equation for multiple collisions of solitons. Phys Rev Lett 1971;27(18):1192–4.
[10] H. Bulut, T. Akturk. An application of the modified expansion method to nonlinear partial differential equation. Proceedings of International Conference on Mathematics and Mathematics Education. Turkish of Journal Mathematics and Computer Science. 10, 202-206 (2018).
[11] Yan C. A Simple transformation for nonlinear waves. Phys Lett A 1996;224(1-2): 77–84.
[12] Zhuo-Sheng Lü, Hong-Qing Z. On a new modified extended Tanh-function method. Commun Theor Phys 2003;39(4):405–8.
[13] Daftardar-Gejji V, Jafari H. Adomin decomposition: a tool for solving a system of fractional differential equations. J Mathemat Anal Appl 2005;301(2):508–18.
[14] Jafari H, Daftardar- Gejji V. Revised Adomin decomposition method for solving systems of ordinary and fractional differential equations. Appl Math Comput 2006; 181(1):598–608.
[15] Jafari H, Tajadodi H. He's variational iteration method for solving fractional riccati differential equation. Int J Diff Eqt 2010;2010:1–8.
[16] Jafari H, Kadem A, Baleanu D, Yilmaz T. Solutions of the fractional davey-stewartson equations with variational iterative method. Romonia Rep Phys 2012; 64(2):337–46.
[17] Jafari H, Kadkhoda N, Tajadodi H, Hosseini Matikolai SA. Int J Nonlin Sci Num 2010;11:271.
[18] Jafari H, Das S, Tajadodi H. Solving a multi-order fractional differential equation using homotopy analysis method. J King S Univers Sci 2011;23(2):151–5.
[19] Jafari H, Nazari M, Baleanu D, Khalique CM. A new approach for solving a system of fractional partial differential equations. Comput Math Appl 2013;66(5):838–43.
[20] Zafar A, Bekir A, Raheel M, Rezazadeh H. Investigation for optical soliton solutions of two nonlinear Schrödinger equations via two concrete finite series methods. Int J Appl Comput Math 2020;6(3):1–13.
[21] Ghanbari B, Gómez-Aguilar JF. New exact optical soliton solutions forn onlinear Schrödinger equation with second-order spatio-temporal dispersion involving M-derivative. Mod Phys Lett B 2019;33(20):1–9.
[22] Ghanbari B, Gómez-Aguilar JF. Optical soliton solutions for the nonlinear Radhakrishnan–Kundu– Lakshmanan equation. Mod Phys Lett B 2019;33(32):1–9.
[23] Rezazadehetal H. Hyperbolic rational solutions to a variety of conformable fractional Boussinesq-Like equations. Nonlinear Eng 2019;8(1):224–30.
[24] He J-H. Variational principle and periodic solution of the Kundu–Mukherjee–Naskar equation. Results Phys 2020;17:103031.
[25] Yıldırım Y, Mirzazadeh M. Optical pulses with Kundu–Mukherjee–Naskar model in fiber communication systems. Chin J Phys 2020;64:183–93.
[26] Zafar MA, Raheel AB. Exploring the dark and singular soliton solutions of Biswas-Arshed model with full nonlinear form. Optik 2020;204:164133.
[27] Rezazadeh H. New soliton solutions of the complex Ginzburg-Landau equation with Kerr law nonlinearity. Optik-Inter, J Light Elect Opt 2018;167:218–27.
[28] Jhangeer A, Seadawy AR, Ali F, Ahmed A. New complex waves of perturbed Shrodinger equation with Kerr law nonlinearity and Kundu-Mukherjee-Naskar equation. Results Phys 2020;16:102816.
[29] S. M M-Alizamini, H. Rezazadeh, New extended direct algebraic method for the Tzitzeica type evolution equations arising in nonlinear optics, 8, 28-53 (2020).
[30] Tozar A, Kurt A, Tasbozan O. New wave solutions of time fractional integrable dispersive wave equation arising in ocean engineering models. Kuwait J Sci 2020; 47(2):23–33.
[31] Kurt A, Tozar A, Tasbozan O. Applying the new extended direct algebraic method to solve the equation of obliquely interacting waves in shallow waters. J Ocean Univ China 2020;19(4):772–80.
[32] Hosseini K, Mirzazadeh M, I1ied M, Radmehr S. Dynamics of optical solitons in the perturbed gerdjikov-ivanov equation. Optik 2020;206:164350.
[33] Bekir A, Boz A. Exact solutions for nonlinear evolution equations using Exp-function method. Phys Lett A 2008;372(10):1619–25.
[34] Tozar A, Tasbozan O, Kurt A. Analytical solutions of Cahn-Hillard phase-field model for spinodal decomposition of a binary system. EPL (Europhysics Lett) 2020; 130(2):24001.
[35] Wang M, Li X. Extended F-expansion method and periodic wave solutions for the generalized Zakharov equations. Phys Lett A 2005;343(1-3):48–54.
[36] Tian B, Gao YT. Spherical nebulons and Backlund transformation for a space or laboratory unmagnetized dusty plasma with symbolic computation. Eur Phys J D-Atom Mol Opt Plasma Phys 2005;33:59–65.
[37] Tian L, Yin J. Stability of multi-compact on solutions and Backlund transformation in K (m, n, 1). Chaos Solitons Fractals 2005;23(1):159–69.
[38] Seadawy AR. Stability analysis solutions for nonlinear three-dimensional modified Korteweg-de Vries-Zakharov-Kuznetsov equation in magnetized electron-positron plasma. Phys A 2016;455:44–51.
[39] Fan E. Extended Tanh-function method and its applications to nonlinear equations. Phys Lett A 2000;277(4-5):212–8.

[40] Wazwaz A-M. The extended tanh method for new compact and non compact solutions for the KP-BBM and the ZK-BBM equations. Chaos Solitons Fractals 2008; 38(5):1505–16.
[41] Liu G-T, Fan T-Y. New applications of developed Jacobi elliptic function expansion methods. Phys Lett A 2005;345(1-3):161–6.
[42] Liu S, Fu Z, Liu S, Zhao Q. Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations. Phys Lett A 2001;289(1-2):69–74.
[43] Tasbozan O, Kurt A, Tozar A. New optical solutions of complex Ginzburg-Landau equation arising in semiconductor lasers. Appl Phys B 2019;125(6):1–12.
[44] Kurt A. New analytical and numerical results for fractional bogoyavlensky-konopelchenko equation arising in fluid dynamics. Appl Mathemat A J Chin Univers 2020;35(1):101–12.
[45] Kumar D, Manafian J, Hawlader F, Ranjbaran A. New closed form soliton and other solutions of the Kundu-Eckhaus equation via the extended sinh-Gordon equation expansion method. Optik 2018;160:159–67.
[46] Ali KK, Cattani C, Gómez-Aguilar JF, Baleanu D, Osman MS. Analytical and numerical study of the DNA dynamics arising in oscillator-chain of Peyrard-Bishop model. Chaos Soliton Fract 2020;139:110089. https://doi.org/10.1016/j.chaos.2020.110089.
[47] Hosseini K, Osman MS, Mirzazadeh M, Rabiei F. Investigation of different wave structures to the generalized third-order nonlinear Scrödinger equation. Optik 2020;206:164259. https://doi.org/10.1016/j.ijleo.2020.164259.
[48] Ding Y, Osman MS, Wazwaz A-M. Abundant complex wave solutions for the nonautonomous Fokas-Lenells equation in presence of perturbation terms. Optik 2019;181:503–13.
[49] Srivastava HM, Baleanu D, Machado JAT, Osman MS, Rezazadeh H, Arshed S, et al. Traveling wave solutions to nonlinear directional couplers by modified Kudryashov method. Phys Scr 2020;95(7):075217. https://doi.org/10.1088/1402-4896/ab95af.
[50] Osman MS, Ali KK. Optical soliton solutions of perturbing time-fractional nonlinear Schrödinger equations. Optik 2020;209:164589. https://doi.org/10.1016/j.ijleo.2020.164589.
[51] Jafari H, Tajadodi H, Baleanu D, Al-Zahrani AA, Alhamed YA, Zahid AH. Fractional sub-equation method for the fractional generalized reaction Duffing model and nonlinear fractional Sharma–Tasso–Olver equation. Cent European J Phys 2013;11 (10):1482–6.
[52] Sousa JVC, Oliveira EC. A new truncated M-fractional derivative type unifying some fractional derivative types with classical properties. Int J Anal Appl 2018;16: 83–96.
[53] Li LX, Li EQ, Wang ML. The (G′/G, 1/G)-expansion method and its application to travelling wave solutions of the Zakharov equation. Appl Mathemat B 2010;25(4): 454–62.
[54] Zayed EME, Abdelaziz MAM. The two variable (G′/G, 1/G)-expansion method for solving the nonlinear KdVmKdV equation. Math Prob Engr 2012;ID 725061:14.
[55] Zhang Y, Zhang L, Pang J. Application of (G'/G^2) expansion method for solving Schrodinger's equation with three-order dispersion. Adv Appl Math 2017;6(2): 212–7.
[56] S. Demiray, O. Unsal, A. Bekir, New exact solutions for Boussinesq type equations by using (G′/G,1/G) and (1/G′)expansion method, 125(5), 1093– 1098 (2014).
[57] Hafiz Uddin M. Close form solutions of the fractional generalized reaction duffing model and the density dependent fractional. Diff React Eqt 2017;6(4):177–84.
[58] Wazwaz AM. Coupling of a fifth-order nonlinear integrable equation: multiple kink solutions. Comput Fluids 2013;84:97–9.
[59] Wazwaz A-M. Kink solutions for three new fifth-order nonlinear equations. Appl Math Model 2014;38(1):110–8.
[60] Sirisubtawee S. Some applications of the expansion method for finding exact traveling wave solutions of nonlinear fractional evolution equations. Symmetry 2019;11:952.
[61] Mahak N, Akram G. Exact solitary wave solutions of the (1+1)-dimensional Fokas-Lenells equation. Optik 2020;208:164459.
[62] Noufe Aljahdaly H. Some applications of the modified expansion method in mathematical physics. Results Phys 2019;13:102272.
[63] Daghan D, Donmez O. Exact solutions of the gardener equation and their applications to the different physical plasmas. Braz J Phys 2016;46:321–33.
[64] Ali MN, Husnine SM, Saha A, Bhowmik SK, Dhawan S, Ak T. Exact solutions, conservation laws, bifurcation of nonlinear and super nonlinear traveling waves for Sharma-Tasso-Olver equation. Nonlinear Dyn 2018;94(3):1791–801.