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Research Article

Some $\alpha-\phi$-Fuzzy Cone Contraction Results with Integral Type Application

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In this paper, we define α-admissible and α-ϕ-fuzzy cone contraction in fuzzy cone metric space to prove some fixed point theorems. Some related sequences with contraction mappings have been discussed. Ultimately, our theoretical results have been utilized to show the existence of the solution to a nonlinear integral equation. This application is also illustrative of how fuzzy metric spaces can be used in other integral type operators.

1. Introduction

The concept of fuzzy metric space (FM space) was first introduced by Kramosil and Michale [1] while George and Veeramani [2] illustrated some well-known FM space properties. In the sense of Kramosil and Michale [1], George and Sapena [3] and Grabice [4] introduced the idea of fuzzy contraction of complete FM spaces and developed some fixed point (FP) results. Some more related results can be found in [5–8]. Samet et al. [9] proposed the concept of α-ψ-contraction in complete metric spaces in 2012. Later, Gopal and Vetro [10] presented the concepts of $\alpha-\phi$ and $\beta-\psi$ fuzzy contractive mappings, as well as several novel FP theorems in FM spaces. More FP results in the context of FM spaces can be found in references (see, for example, [11–15]). Mohammadi et al. [16] proved some generalized contraction results in FM spaces with application in integral equations. Recently, the rational type fuzzy contraction concept in complete FM space is given by Rehman et al. [17], and they proved some FP results with an application.

Oner et al. [18] introduced the idea of fuzzy cone metric space (FCM space), proved some basic properties, and developed the first version of “Banach contraction principle for fixed point” in FCM spaces which is stated as follows: “let $(\mathscr{U}, \ddot{M}_{\alpha}, *)$ be a complete FCM space in which fuzzy cone contractive sequences are Cauchy and let $h\colon \mathscr{U} \longrightarrow \mathscr{U}$ be a fuzzy cone contractive mapping being $a \in (0,1)$ the contractive constant. Then, h has a unique fixed point.” Ur Rehman et al. [19] presented some extended “fuzzy cone Banach contraction results” in FCM spaces for some weaker conditions. In topology and analysis, the definition of FCM spaces with different contractive conditions has been commonly used. For further reading, refer to [20–28].

The definition of α admissibility has been applied to certain directions by several writers. For a pair of functions, some authors expanded the concept of α admissibility. We may advise readers to look for more work in the field of α admissibility, as well as references (see [29–33]). Recently, Islam et al. [34] established some FP results in cone b_2-metric space by using generalized a-admissible Hardy–Rogers’ contractions over Banachs algebras with application.

In this paper, we define that a mapping h is α-admissible with respect to η and $\alpha-\phi$ fuzzy cone contraction in FCM space. By using the concept of α-admissibility with respect to η under mapping h, we establish some FP theorems under the $\alpha - \phi$ fuzzy cone contraction conditions in FCM space with an example. In support of our work, we present an integral type application. By using this concept, one can prove different contractive type FP results for nonlinear mappings with different types of applications in the context of FM spaces. The paper is organized as follows. In Section 2, we introduce the preliminary concepts to support our work. In Section 3, we present some FP results by using different types of contraction conditions in FCM spaces with an illustrative example. In Section 4, we present an integral equation application to validate the concept defined in the paper. Finally, in Section 5, we discuss the conclusion.

2. Preliminaries

The continuous t-norm is defined by by Schweizer and Sklar [35].

Definition 1 (see [35]). An operation $*$: $[0,1]^2 \longrightarrow [0,1]$ is called continuous t'-norm if it satisfies the following conditions:

(i) $*$ is commutative, associative, and continuous

(ii) $1 * \varrho_1 = \varrho_1$ and $\varrho_1 * \varrho_2 \leq \varrho_3 * \varrho_4$, whenever, $\varrho_1 \leq \varrho_3$ and $\varrho_2 \leq \varrho_4$, for each $\varrho_1, \varrho_2, \varrho_3, \varrho_4 \in [0,1]$

The basic t'-norms, i.e., product, minimum, and L Lukasiewicz continuous t'-norms are defined as follows (see [35]):

(i) $\varrho_1 * \varrho_2 = \varrho_1 \varrho_2$

(ii) $\varrho_1 * \varrho_2 = \min\{\varrho_1, \varrho_2\}$

(iii) $\varrho_1 * \varrho_2 = \max\{\varrho_1 + \varrho_2 - 1, 0\}$

Definition 2 (see [36]). A subset $\mathscr{C}$ of a real Banach space $\mathbf{E}$ is called cone if

(i) $\mathscr{C} \neq \varnothing$, closed, and $\mathscr{C} \neq \{\theta\}$

(ii) $0 \leq \varrho_1, \varrho_2 < \infty$ and $\mu, \omega \in \mathscr{C}$, then $\varrho_1 \mu + \varrho_2 \omega \in \mathscr{C}$

(iii) $-\mu, \mu \in \mathscr{C}$, then $\mu = \theta$

A cone $\mathscr{C} \subset \mathbf{E}$ and $\preceq$ is a partial ordering on $\mathbf{E}$ via $\mathscr{C}$ which is defined by $\mu \preceq \omega$ iff $\omega - \mu \in \mathscr{C}$. $\mu \prec \omega$ stands for $\mu \preceq \omega$ and $\mu \neq \omega$, while $\mu \ll \omega$ stands for $\omega - \mu \in \text{int}(\mathscr{C})$. All the cones in this paper have a nonempty interior.

Definition 3 (see [18]). A 3-tuple $(\mathscr{U}, \ddot{M}_\alpha, *)$ is called FCM space if $\mathscr{C}$ is a cone of $\mathbf{E}$, $\mathscr{U}$ is an arbitrary set, $*$ is a continuous t-norm, and a mapping $\ddot{M}_\alpha$: $\mathscr{U} \times \mathscr{U} \times \text{int}(\mathscr{C}) \longrightarrow [0,1]$ satisfies the following axioms:

(i) $\ddot{M}_\alpha(\mu, \omega, t') > 0$ and $\ddot{M}_\alpha(\mu, \omega, t') = 1$ iff $\mu = \omega$

(ii) $\ddot{M}_\alpha(\mu, \omega, t') = \ddot{M}_\alpha(\omega, \mu, t')$

(iii) $\ddot{M}_\alpha(\mu, \rho, s') * \ddot{M}_\alpha(\rho, \omega, t') \leq \ddot{M}_\alpha(\mu, \omega, s' + t')$

(iv) $\ddot{M}_\alpha(\mu, \omega, .)$: $\text{int}(\mathscr{C}) \longrightarrow [0,1]$ is continuous

$\forall \mu, \omega, \rho \in \mathscr{U}$ and $s', t' \in \text{int}(\mathscr{C})$.

Definition 4 (see [18]). Let $(\mathscr{U}, \ddot{M}_\alpha, *)$ be a FCM space, and $\mu \in \mathscr{U}$ and (μ_ℓ) be a sequence in U. Then,

(i) The sequence (μ_i) converges to μ, if $t' \gg \theta$, $0 < r' < 1$ and $\exists i_1 \in N$ so that $\ddot{M}_\alpha(\mu_\ell, \mu, t') > 1 - r$, $\forall \ell \geq \ell_1$. We denote this by $\lim_{\ell \longrightarrow \infty} \mu_\ell = \mu$ or $\mu_\ell \longrightarrow \mu$ as $\ell \longrightarrow \infty$.

(ii) The sequence (μ_ℓ) is Cauchy sequence if $0 < r' < 1$, $t' \gg \theta$, and $\exists \ell_1 \in N$ so that $\ddot{M}_\alpha(\mu_\ell, \mu_j, t') > 1 - r$, $\forall \ell, j \geq \ell_1$.

(iii) The, sequence (μ_ℓ) is $\mathscr{G}$-C auchy sequence if $0 < r < 1$, $t' \gg \theta$ and $\exists \ell_1 \in N$ so that $\ddot{M}_\alpha(\mu_\ell, \mu_j, t') = 1$, $\forall \ell, j \geq \ell_1$.

(iv) $(\mathscr{U}, \ddot{M}_\alpha, *)$ is $\mathscr{G}$-complete if every $\mathscr{G}$-C auchy sequence is convergent in $\mathscr{U}$.

(v) (μ_ℓ) is said to be fuzzy cone contractive if $\exists\, 0 < \varrho < 1$ so that

$$\frac{1}{\ddot{M}_\alpha(\mu_\ell, \mu_{\ell+1}, t')} - 1 \leq \varrho\left(\frac{1}{\ddot{M}_\alpha(\mu_{\ell-1}, \mu_\ell, t')} - 1\right), \quad \text{for } t' \gg \theta\, \ell \geq 1. \tag{1}$$

Definition 5 (see [19]). Let a FCM $\ddot{M}_\alpha$ be triangular in FCM space $(\mathscr{U}, \ddot{M}_\alpha, *)$ if

$$\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1 \leq \left(\frac{1}{\ddot{M}_\alpha(\mu, \rho, t')} - 1\right) + \left(\frac{1}{\ddot{M}_\alpha(\rho, \omega, t')} - 1\right), \tag{2}$$

$\forall \mu, \omega, \rho \in \mathscr{U}$ and $t' \gg \theta$.

Lemma 1 (see [18]). *Let $\mu \in \mathscr{U}$ in a FCM space $(\mathscr{U}, \ddot{M}_\alpha, *)$ and (μ_ℓ) be a sequence in $\mathscr{U}$. Then, $\mu_\ell \longrightarrow \mu \Leftrightarrow \ddot{M}_\alpha(\mu_\ell, \mu, t') \longrightarrow 1$ as $\ell \longrightarrow \infty$, for $t' \gg \theta$.*

Definition 6 (see [18]). Let $(\mathscr{U}, \ddot{M}_\alpha, *)$ be a FCM space and A: $\mathscr{U} \longrightarrow \mathscr{U}$. Then, T is called fuzzy cone contractive if $\exists\, 0 < \varrho < 1$, so that

$$\frac{1}{\ddot{M}_\alpha(A\mu, A\omega, t')} - 1 \le \varrho\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right), \quad \forall \mu, \omega \in \mathcal{U},\ t' \gg \theta. \tag{3}$$

3. Main Result

Definition 7. Let $(\mathcal{U}, \ddot{M}_\alpha, *)$ be a FCM space, and let $\alpha, \eta\colon \mathcal{U} \times \mathcal{U} \times \operatorname{int}(\mathscr{C}) \longrightarrow [0, \infty)$ be two functions. We say that $h\colon \mathcal{U} \longrightarrow \mathcal{U}$ is α-admissible w.r.t η if

$$\begin{aligned}\alpha(\mu, \omega, t') \ge \eta(\mu, \omega, t') &\Rightarrow \alpha(h\mu, h\omega, t') \\ &\ge \eta(h\mu, h\omega, t'), \quad \forall \mu, \omega \in \mathcal{U}.\end{aligned} \tag{4}$$

Note that in a special case, if we take $\eta(\mu, \omega, t') = 1$, then Definition 7 is reduced to the α-admissible mapping (i.e., Definition 3.4 in the study by Gopal and Vetro [10]), and also if we take $\alpha(\mu, \omega, t') = 1$, then we can say that h is a η-subadmissible mapping.

In the following, Φ denotes the family of all right continuous functions $\phi\colon [0, +\infty) \longrightarrow [0, +\infty)$ with $\phi(\tau) < \tau$, $\forall \tau > 0$.

Definition 8. Let $(\mathcal{U}, \ddot{M}_\alpha, *)$ be a FCM space, and the mapping $h\colon \mathcal{U} \longrightarrow \mathcal{U}$ is called α-ϕ-fuzzy cone contractive if there exist three functions $\alpha, \eta\colon \mathcal{U} \times \mathcal{U} \times \operatorname{int}(\mathscr{C}) \longrightarrow [0, \infty)$ and $\phi \in \Phi$, so that

$$\alpha(\mu, \omega, t') \ge \eta(\mu, \omega, t') \Rightarrow \frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1 \le \phi\left(\frac{1}{\Omega(h, \mu, \omega, t')} - 1\right), \tag{5}$$

where

$$\Omega(h, \mu, \omega, t') = \min\left\{\begin{array}{c}\ddot{M}_\alpha(\mu, \omega, t'), \ddot{M}_\alpha(\mu, h\mu, t'), \\ \ddot{M}_\alpha(\omega, h\mu, t'), \ddot{M}_\alpha(\omega, h\omega, t')\end{array}\right\}, \quad \forall \mu, \omega \in \mathcal{U},\ t' \gg \theta. \tag{6}$$

Theorem 1. *Let a FM $\ddot{M}_\alpha$ be triangular in a $\mathscr{G}$-complete FCM space $(\mathcal{U}, \ddot{M}_\alpha, *)$ and let $h\colon \mathcal{U} \longrightarrow \mathcal{U}$ be an α-ϕ-fuzzy cone contractive if the following axioms hold:*

(1) h is an α-admissible w.r.t η

(2) $\exists \mu_0 \in \mathcal{U}$ such that $\alpha(\mu_0, h\mu_0, t') \ge \eta(\mu_0, h\mu_0, t')$, for $t' \gg \theta$

(3) The sequence (μ_ℓ) in $\mathcal{U}$ with $\alpha(\mu_\ell, \mu_{\ell+1}, t') \ge \eta(\mu_\ell, \mu_{\ell+1}, t'), \forall \ell \in N$, $t' \gg \theta$ and $\mu_\ell \longrightarrow z$ as $\ell \longrightarrow +\infty$, then $\alpha(\mu_\ell, z, t') \ge \eta(\mu_\ell, z, t')$, $\forall \ell \in N$ and, $t' \gg \theta$

Then, h has a FP $z \in \mathcal{U}$ such that $hz = z$.

Proof. Let $\mu_0 \in \mathcal{U}$, such that

$$\alpha(\mu_0, h\mu_0, t') \ge \eta(\mu_0, h\mu_0, t'), \quad \text{for } t' \gg \theta. \tag{7}$$

We choose a sequence

$$(\mu_\ell) \in \mathcal{U}, \text{ i.e. } \quad \mu_\ell = h\mu_{\ell-1} = h^\ell \mu_0, \forall \ell \in N. \tag{8}$$

If $\mu_{\ell+1} = u_\ell$ for some $\ell \in N$, then $\mu_\ell = \mu$ is a FP of h in $\mathcal{U}$. Otherwise, we assume that $\mu_{\ell+1} \ne \mu_\ell$, $\forall \ell \in N$. Since the mapping h is an α-admissible w.r.t η and $\alpha(\mu_0, h\mu_0, t') \ge \eta(\mu_0, h\mu_0, t')$, for $t' \gg \theta$, we have to find that

$$\begin{aligned}\alpha(\mu_1, \mu_2, t') &= \alpha\left(h\mu_0, h^2\mu_0, t'\right) \\ &\ge \eta\left(h\mu_0, h^2\mu_0, t'\right) = \eta(\mu_1, \mu_2, t'), \\ &\Rightarrow \alpha(\mu_1, \mu_2, t') \ge \eta(\mu_1, \mu_2, t'), \quad \text{for } t' \gg \theta.\end{aligned} \tag{9}$$

Continuing this process $\forall \ell \in N$, we get

$$\alpha(\mu_\ell, \mu_{\ell+1}, t') \ge \eta(\mu_\ell, \mu_{\ell+1}, t'), \quad \text{for } t' \gg \theta. \tag{10}$$

From (5), we have

$$\begin{aligned}\frac{1}{\ddot{M}_\alpha(\mu_\ell, \mu_{\ell+1}, t')} - 1 &= \frac{1}{\ddot{M}_\alpha(h\mu_{\ell-1}, h\mu_\ell, t')} - 1, \\ &\le \phi\left(\frac{1}{\Omega(h, \mu_{\ell-1}, \mu_\ell, t')} - 1\right),\end{aligned} \tag{11}$$

where

$$\frac{1}{\Omega(h,\mu_{\ell-1},\mu_\ell,t')}-1=\frac{1}{\min\left\{\begin{array}{c}\ddot{M}_\alpha(\mu_{\ell-1},\mu_\ell,t'),\ddot{M}_\alpha(\mu_{\ell-1},h\mu_{\ell-1},t'),\\ \ddot{M}_\alpha(\mu_\ell,h\mu_{\ell-1},t'),\ddot{M}_\alpha(\mu_\ell,h\mu_\ell,t')\end{array}\right\}}-1,$$
$$=\frac{1}{\min\left\{\begin{array}{c}\ddot{M}_\alpha(\mu_{\ell-1},\mu_\ell,t'),\ddot{M}_\alpha(\mu_{\ell-1},\mu_\ell,t'),\\ \ddot{M}_\alpha(\mu_\ell,\mu_{\ell+1},t')\end{array}\right\}}-1 \tag{12}$$
$$=\frac{1}{\min\left\{\ddot{M}_\alpha(\mu_{\ell-1},\mu_\ell,t'),\ddot{M}_\alpha(\mu_\ell,\mu_{\ell+1},t')\right\}}-1.$$

Now, from (11), for $t'\gg\theta$, we have

$$\frac{1}{\ddot{M}_\alpha(\mu_\ell,\mu_{\ell+1},t')}-1\le\phi\left(\frac{1}{\min\left\{\ddot{M}_\alpha(\mu_{\ell-1},\mu_\ell,t'),\ddot{M}_\alpha(\mu_\ell,\mu_{\ell+1},t')\right\}}-1\right). \tag{13}$$

If $\ddot{M}_\alpha(\mu_\ell,\mu_{\ell+1},t')$ is minimum for some $\ell\in N$, then by (13), we obtain

$$\frac{1}{\ddot{M}_\alpha(\mu_\ell,\mu_{\ell+1},t')}-1\le\phi\left(\frac{1}{\ddot{M}_\alpha(\mu_\ell,\mu_{\ell+1},t')}-1\right),$$
$$<\frac{1}{\ddot{M}_\alpha(\mu_\ell,\mu_{\ell+1},t')}-1, \tag{14}$$

which is not possible. Hence, $\forall\,\ell\in N$ and $t'\gg\theta$, we get

$$\frac{1}{\ddot{M}_\alpha(\mu_\ell,\mu_{\ell+1},t')}-1\le\phi\left(\frac{1}{\ddot{M}_\alpha(\mu_{\ell-1},\mu_\ell,t')}-1\right)$$
$$<\frac{1}{\ddot{M}_\alpha(\mu_{\ell-1},\mu_\ell,t')}-1. \tag{15}$$

This implies that

$$\ddot{M}_\alpha(\mu_\ell,\mu_{\ell+1},t')>\ddot{M}_\alpha(\mu_{\ell-1},\mu_\ell,t'). \tag{16}$$

Thus, $(\ddot{M}_\alpha(\mu_{\ell-1}, \mu_\ell, t'))$ is an increasing sequence in $[0,1]$. Let $m_1(t') = \lim_{\ell \longrightarrow \infty} \ddot{M}_\alpha(\mu_{\ell-1}, \mu_\ell, t')$, we have to show that $m_1(t') = 1$, for $t' \gg \theta$. Let $\exists t'^* \gg \theta \ni m(t'^*) < 1$.

$$\frac{1}{\ddot{M}_\alpha(\mu_\ell, \mu_{\ell+1}, t'^*)} - 1 < \frac{1}{\ddot{M}_\alpha(\mu_{\ell-1}, \mu_\ell, t'^*)} - 1. \tag{17}$$

Using the right side continuity of a function ϕ and let the limit $\ell \longrightarrow +\infty$, we obtained the contradiction as follows:

$$\frac{1}{m(t'^*)} - 1 \le \phi\left(\frac{1}{m(t'^*)} - 1\right) < \frac{1}{m(t'^*)} - 1. \tag{18}$$

This implies that $\lim_{\ell \longrightarrow \infty} \ddot{M}_\alpha(\mu_{\ell-1}, \mu_\ell, t') = 1$, for $t' \gg \theta$. Let $j > \ell$, where $\ell, j \in N$ and $j = \ell + p$, for a fixed $p \in N$,

$$\begin{aligned}\frac{1}{\ddot{M}_\alpha(\mu_\ell, \mu_j, t')} - 1 &= \frac{1}{\ddot{M}_\alpha(\mu_\ell, \mu_{\ell+p}, t')} - 1,\\ &\le \frac{1}{\left\{\ddot{M}_\alpha(\mu_\ell, \mu_{\ell+1}, (t'/p)) * \ddot{M}_\alpha(\mu_{\ell+1}, \mu_{\ell+2}, (t'/p)) * \cdots * \ddot{M}_\alpha(\mu_{\ell+p-1}, \mu_{\ell+p}, (t'/p))\right\}} - 1 \longrightarrow \theta, \quad \text{as } \ell \longrightarrow \infty.\end{aligned} \tag{19}$$

This implies that $\lim_{\ell \longrightarrow \infty} \ddot{M}_\alpha(\mu_\ell, \mu_j, t') = 1$. It is proved that the sequence (μ_ℓ) is $\mathcal{G}$-C auchy. Since $(\mathcal{U}, \ddot{M}_\alpha, *)$ is $\mathcal{G}$-complete, then $\exists z \in \mathcal{U}$ such that $\mu_\ell \longrightarrow z$ as $\ell \longrightarrow +\infty$, i.e.,

$$\lim_{\ell \longrightarrow \infty} \ddot{M}_\alpha(\mu_\ell, z, t') = 1, \quad \text{for } t' \gg \theta. \tag{20}$$

Now, by the view of Definition 4 (iii),

$$\alpha(\mu_\ell, z, t') \ge \eta(\mu_\ell, z, t'), \quad \forall \ell \in N, t' \gg \theta. \tag{21}$$

If $hz \ne z$, i.e., $\ddot{M}_\alpha(z, hz, t') < 1$, then $t' \gg \theta$. Since $\ddot{M}_\alpha$ is triangular, we have that

$$\frac{1}{\ddot{M}_\alpha(z, hz, t')} - 1 \le \left(\frac{1}{\ddot{M}_\alpha(z, \mu_{\ell+1}, t')} - 1\right) + \left(\frac{1}{\ddot{M}_\alpha(\mu_{\ell+1}, hz, t')} - 1\right), \quad \text{for } t' \gg \theta. \tag{22}$$

Then, from (5) and (21), we have

$$\frac{1}{\ddot{M}_\alpha(\mu_{\ell+1}, hz, t')} - 1 = \frac{1}{\ddot{M}_\alpha(h\mu_\ell, hz, t')} - 1 \le \phi\left(\frac{1}{\Omega(h, \mu_\ell, z, t')} - 1\right), \tag{23}$$

where

$$\frac{1}{\Omega(h,\mu_\ell,z,t')}-1\frac{1}{\min\{\ddot{M}_\alpha(\mu_\ell,z,t'),\ddot{M}_\alpha(\mu_\ell,h\mu_\ell,t'),\ddot{M}_\alpha(z,h\mu_\ell,t'),\ddot{M}_\alpha(z,hz,t')\}}-1,$$
$$=\frac{1}{\min\{\ddot{M}_\alpha(\mu_\ell,z,t'),\ddot{M}_\alpha(\mu_\ell,\mu_{\ell+1},t'),\ddot{M}_\alpha(z,\mu_{\ell+1},t'),\ddot{M}_\alpha(z,hz,t')\}}-1$$
$$-1. \tag{24}$$
$$\longrightarrow\frac{1}{\min\{1,1,1,\ddot{M}_\alpha(z,hz,t')\}}-1,\quad \text{as } \ell\longrightarrow\infty,$$
$$=\frac{1}{\ddot{M}_\alpha(z,hz,t')}$$

Thus, to avoid the contradiction with $\phi(c)<c$, for $c>0$,

$$\limsup_{\ell\longrightarrow\infty}\left(\frac{1}{\ddot{M}_\alpha(\mu_{\ell+1},hz,t')}-1\right)\leq\left(\frac{1}{\ddot{M}_\alpha(z,hz,t')}-1\right), \tag{25}$$

Thus, together with (20) and (22), we conclude that $\ddot{M}_\alpha(z,hz,t')=1\Rightarrow hz=z$. □

Corollary 1. *Let $(\mathscr{U},\ddot{M}_\alpha,*)$ be a $\mathscr{G}$-complete FCM space in which $\ddot{M}_\alpha$ is triangular, and let $h\colon\mathscr{U}\longrightarrow\mathscr{U}$ be an α-admissible. Assume that $\exists\,\phi\in\Phi$, so that*

$$\alpha(\mu,\omega,t')\geq 1\Rightarrow\frac{1}{\ddot{M}_\alpha(h\mu,h\omega,t')}-1\leq\phi\left(\frac{1}{\Omega(h,\mu,\omega,t')}-1\right), \tag{26}$$

where

$$\Omega(h,\mu,\omega,t')=\min\{\ddot{M}_\alpha(\mu,\omega,t'),\ddot{M}_\alpha(\mu,h\mu,t'),\ddot{M}_\alpha(\omega,h\mu,t'),\ddot{M}_\alpha(\omega,h\omega,t')\},\quad\forall\mu,\omega\in\mathscr{U},\ t'\gg\theta. \tag{27}$$

Assume that the following assertions hold:

(i) $\exists\,\mu_0\in\mathscr{U}$ such that $\alpha(\mu_0,h\mu_0,t')\geq 1$, for $t'\gg\theta$

(ii) Any sequence (μ_ℓ) in $\mathscr{U}$ with $\alpha(\mu_\ell,\mu_{\ell+1},t')\geq 1$, $\forall\ell\in N$, $t'\gg\theta$ and $\mu_\ell\longrightarrow z$ as $\ell\longrightarrow+\infty$, then $\alpha(\mu_\ell,z,t')\geq 1$, $\forall\ell\in N$ and, $t'\gg\theta$

Then, h has a FP in $\mathscr{U}$.

Corollary 2. *Let a FM $\ddot{M}_\alpha$ be triangular in a $\mathscr{G}$-complete FCM space $(\mathscr{U},\ddot{M}_\alpha,*)$, and let $h\colon\mathscr{U}\longrightarrow\mathscr{U}$ be an η-admissible. Assume that $\exists\,\phi\in\Phi$, so that*

$$\eta(\mu,\omega,t')\leq 1\Rightarrow\frac{1}{\ddot{M}_\alpha(h\mu,h\omega,t')}-1\leq\phi\left(\frac{1}{\Omega(h,\mu,\omega,t')}-1\right), \tag{28}$$

where

$$\Omega(h,\mu,\omega,t')=\min\{\ddot{M}_\alpha(\mu,\omega,t'),\ddot{M}_\alpha(\mu,h\mu,t'),\ddot{M}_\alpha(\omega,h\mu,t'),\ddot{M}_\alpha(\omega,h\omega,t')\},\quad\forall\mu,\omega\in\mathscr{U},\ t'\gg\theta. \tag{29}$$

Suppose that the following axioms hold:

(i) $\exists\,\mu_0\in\mathscr{U}$ such that $\eta(\mu_0,h\mu_0,t')\leq 1$, for $t'\gg\theta$

(ii) *Any sequence* (μ_ℓ) *in* $\mathcal{U}$ *with* $\alpha(\mu_\ell, \mu_{\ell+1}, t') \geq 1$, $\forall \ell \in N$, $t' \gg \theta$ *and* $\mu_\ell \longrightarrow z$ *as* $\ell \longrightarrow +\infty$, *then* $\alpha(\mu_\ell, z, t') \geq 1$, $\forall \ell \in N$ *and* $t' \gg \theta$

Then. h has a FP in $\mathcal{U}$.

Now, to establish the unique FP of an α-ϕ-*fuzzy cone contraction map, let the hypothesis* (***H***) *is given as follows:*

(H) *For all* $\mu, \omega \in \mathcal{U}$, $t' \gg \theta$, $\exists \rho \in \mathcal{U}$ *such that*

$$\alpha(\mu, \rho, t') \geq \eta(\mu, \rho, t'), \ \alpha(\omega, \rho, t') \geq \eta(\omega, \rho, t'),$$
$$\lim_{\ell \longrightarrow \infty} \ddot{M}_\alpha\left(h^{\ell-1}\rho, h^\ell \rho, t'\right) = 1. \tag{30}$$

Theorem 2. *Adding the hypothesis (H) in Theorem 1, we obtain the uniqueness of a FP of h provided the function* $\phi \in \Phi$ *is nondecreasing.*

Proof. Assume that z and ρ be the two FPs of h in $\mathcal{U}$. If $\alpha(z, \rho, t') \geq \eta(z, \rho, t')$, then by (5), we get $z = \rho$. Suppose $\alpha(z, \rho, t') < \eta(z, \rho, t')$, then from (H), $\exists\, y \in \mathcal{U}$, so that

$$\alpha(z, y, t') \geq \eta(z, y, t'), \quad \text{and } \alpha(\rho, y, t') \geq \eta(\rho, y, t'), \quad \text{for } t' \gg \theta. \tag{31}$$

Since h is an α-admissible w.r.t η, then we get

$$\alpha\left(z, h^\ell y, t'\right) \geq \eta\left(z, h^\ell y, t'\right), \quad \forall\, \ell \in N \text{ and}, t' \gg \theta. \tag{32}$$

Now, we have to show that $\ddot{M}_\alpha(z, h^\ell y, t') \longrightarrow 1$, as $\ell \longrightarrow \infty$, for $t' \gg \theta$. Since $\ddot{M}_\alpha$ is triangular, then from (5) and (31),

$$\begin{aligned}\frac{1}{\ddot{M}_\alpha\left(z, h^\ell y, t'\right)} - 1 &\leq \left(\frac{1}{\ddot{M}_\alpha(z, hz, t')} - 1\right) + \left(\frac{1}{\ddot{M}_\alpha\left(hz, h^\ell y, t'\right)} - 1\right) \\ &= \left(\frac{1}{\ddot{M}_\alpha(z, hz, t')} - 1\right) + \left(\frac{1}{\ddot{M}_\alpha\left(hz, h\left(h^{\ell-1} y\right), t'\right)} - 1\right) \leq \phi\left(\frac{1}{\Omega\left(h, z, h^{\ell-1} y, t'\right)} - 1\right),\end{aligned} \tag{33}$$

where z is the FP of h and for $t' \gg \theta$,

$$\begin{aligned}\frac{1}{\Omega\left(h, z, h^{\ell-1} y, t'\right)} - 1 &= \left(\frac{1}{\min\left\{\ddot{M}_\alpha\left(z, h^{\ell-1} y, t'\right), \ddot{M}_\alpha(z, hz, t'), \ddot{M}_\alpha\left(h^{\ell-1} y, hz, t'\right), \ddot{M}_\alpha\left(h^{\ell-1} y, h^\ell y, t'\right)\right\}} - 1\right), \\ &= \left(\frac{1}{\min\left\{\ddot{M}_\alpha\left(z, h^{\ell-1} y, t'\right), 1, \ddot{M}_\alpha\left(h^{\ell-1} y, z, t'\right), \ddot{M}_\alpha\left(h^{\ell-1} y, h^\ell y, t'\right)\right\}} - 1\right) \\ &= \left(\frac{1}{\min\left\{\ddot{M}_\alpha\left(z, h^{\ell-1} y, t'\right), \ddot{M}_\alpha\left(h^{\ell-1} y, h^\ell y, t'\right)\right\}} - 1\right).\end{aligned} \tag{34}$$

Let $\ell_0 \in N$ such that $\ddot{M}_\alpha (z, h^{\ell-1} y, t') \leq \ddot{M}_\alpha (h^{\ell-1} y, h^\ell y, t')$, $\forall \ell \geq \ell_0$ and by the hypothesis (**H**). Thus,

$$\frac{1}{\ddot{M}_\alpha\left(z, h^\ell y, t'\right)} - 1 \leq \phi\left(\frac{1}{\ddot{M}_\alpha\left(z, h^{\ell-1} y, t'\right)} - 1\right), \quad \text{for } t' \gg \theta. \tag{35}$$

Again by the triangular property of $\ddot{M}_\alpha$ and the view of (5) and (31),

$$\begin{aligned}
\frac{1}{\ddot{M}_\alpha\left(z, h^\ell y, t'\right)} - 1 &\leq \phi\left(\frac{1}{\ddot{M}_\alpha\left(z, h^{\ell-1} y, t'\right)} - 1\right), \\
&\leq \phi\left(\left(\frac{1}{\ddot{M}_\alpha (z, hz, t')} - 1\right) + \left(\frac{1}{\ddot{M}_\alpha\left(hz, h^{\ell-1} y, t'\right)} - 1\right)\right) \\
&= \phi\left(\left(\frac{1}{\ddot{M}_\alpha (z, hz, t')} - 1\right) + \left(\frac{1}{\ddot{M}_\alpha\left(hz, h\left(h^{\ell-2} y\right), t'\right)} - 1\right)\right) \\
&\leq \phi^2\left(\frac{1}{\Omega\left(h, z, h^{\ell-1} y, t'\right)} - 1\right),
\end{aligned} \tag{36}$$

where z is the FP of h and for $t' \gg \theta$,

$$\begin{aligned}
\frac{1}{\Omega\left(h, z, h^{\ell-2} y, t'\right)} - 1 &= \left(\frac{1}{\min\left\{\ddot{M}_\alpha\left(z, h^{\ell-2} y, t'\right), \ddot{M}_\alpha (z, hz, t'), \ddot{M}_\alpha\left(h^{\ell-2} y, hz, t'\right), \ddot{M}_\alpha\left(h^{\ell-2} y, h^{\ell-1} y, t'\right)\right\}} - 1\right), \\
&= \left(\frac{1}{\min\left\{\ddot{M}_\alpha\left(z, h^{\ell-2} y, t'\right), 1, \ddot{M}_\alpha\left(h^{\ell-2} y, z, t'\right), \ddot{M}_\alpha\left(h^{\ell-2} y, h^{\ell-1} y, t'\right)\right\}} - 1\right) \\
&= \left(\frac{1}{\min\left\{\ddot{M}_\alpha\left(z, h^{\ell-2} y, t'\right), \ddot{M}_\alpha\left(h^{\ell-2} y, h^{\ell-1} y, t'\right)\right\}} - 1\right).
\end{aligned} \tag{37}$$

Let $\ell_0 \in N$ such that $\ddot{M}_\alpha(z, h^{\ell-2}y, t') \leq \ddot{M}_\alpha(h^{\ell-2}y, h^{\ell-1}y, t')$, $\forall \ell \geq \ell_0$, and by the hypothesis (**H**). Thus,

$$\frac{1}{\ddot{M}_\alpha(z, h^{\ell}y, t')} - 1 \leq \phi^2\left(\frac{1}{\ddot{M}_\alpha(z, h^{\ell-2}y, t')} - 1\right), \quad \text{for } t' \gg \theta. \tag{38}$$

By continuing the same argument, we obtain

$$\frac{1}{\ddot{M}_\alpha(z, h^{\ell}y, t')} - 1 \leq \phi^{\ell-\ell_0}\left(\frac{1}{\ddot{M}_\alpha(z, h^{\ell_0}y, t')} - 1\right), \quad \text{for } t' \gg \theta. \tag{39}$$

Then, by taking the limit $\ell \longrightarrow \infty$, we get that

$$h^{\ell}y \longrightarrow z. \tag{40}$$

Similarly, we can prove that

$$h^{\ell}y \longrightarrow \rho, \quad \text{as } \ell \longrightarrow \infty. \tag{41}$$

Now, from (40) and (41), we get the uniqueness, i.e., $z = \rho$. Subsequently, we use the following classes of functions in our results without $\ddot{M}_\alpha$ triangularity condition. Suppose that

$$\Psi = \{\psi\colon [0, +\infty) \longrightarrow [0, +\infty)\}, \tag{42}$$

where ψ is nondecreasing and continuous, and

$$\Phi_0 = \{\phi\colon [0, +\infty) \longrightarrow [0, +\infty)\}, \tag{43}$$

where ϕ is lower semicontinuous, where $\psi(c) = \phi(c) = 0 \Leftrightarrow c = 0$. □

Theorem 3. *Let $(\mathscr{U}, \ddot{M}_\alpha, *)$ be a $\mathscr{G}$-complete FCM space, and let $h\colon \mathscr{U} \longrightarrow \mathscr{U}$ be an α-admissible w.r.t η. Suppose that $\exists\, \psi \in \Psi$ and $\phi \in \Phi_0$, so that*

$$\begin{aligned} &\alpha(\mu, h\mu, t')\alpha(\omega, h\omega, t') \geq \eta(\mu, h\mu, t')\eta(\omega, h\omega, t'), \\ &\Rightarrow \psi\left(\frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1\right) \leq \psi\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right) - \phi\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right), \end{aligned} \tag{44}$$

$\forall \mu, \omega \in \mathscr{U}$ and $t' \gg \theta$. Let the following axioms hold:

(i) $\exists \mu_0 \in \mathscr{U}$ such that $\alpha(\mu_0, h\mu_0, t') \geq \eta(\mu_0, h\mu_0, t')$, for $t' \gg \theta$

(ii) Any sequence (μ_ℓ) in $\mathscr{U}$ with $\alpha(\mu_\ell, \mu_{\ell+1}, t') \geq \eta(\mu_\ell, \mu_{\ell+1}, t')$, $\forall \ell \in N$, $t' \gg \theta$ and $\mu_\ell \longrightarrow z \in \mathscr{U}$, as $\ell \longrightarrow +\infty$, then $\alpha(\mu_\ell, z, t') \geq \eta(\mu_\ell, z, t')$ and $\alpha(z, hz, t') \geq \eta(z, hz, t')$, $\forall \ell \in N$ and $t' \gg \theta$

Then, h has an FP in $\mathscr{U}$.

Proof. Let $\mu_0 \in \mathscr{U}$ such that

$$\alpha(\mu_0, h\mu_0, t') \geq \eta(\mu_0, h\mu_0, t'), \quad \text{for } t' \gg \theta. \tag{45}$$

We define a sequence (μ_ℓ) in $\mathscr{U}$ such that $\mu_\ell = h\mu_{\ell-1} = h^{\ell}\mu_0$, $\forall \ell \in N$. If $\mu_{\ell+1} = u_\ell$ for some $\ell \in N$, then $\mu_\ell = \mu$ is an FP of h in $\mathscr{U}$.

Otherwise, we assume that $\mu_{\ell+1} \neq \mu_\ell\ \forall \ell \in N$. However, the mapping h is an α-admissible w.r.t η and $\alpha(\mu_0, h\mu_0, t') \geq \eta(\mu_0, h\mu_0, t')$. Now, we have to deduce that

$$\begin{aligned} &\alpha(\mu_1, \mu_2, t') = \alpha(h\mu_0, h^2\mu_0, t') \geq \eta(h\mu_0, h^2\mu_0, t') = \eta(\mu_1, \mu_2, t'), \\ &\Rightarrow \alpha(\mu_1, \mu_2, t') \geq \eta(\mu_1, \mu_2, t'), \quad \text{for } t' \gg \theta. \end{aligned} \tag{46}$$

Continuing this process $\forall \ell \in N$, we may obtain

$$\alpha(\mu_\ell, \mu_{\ell+1}, t') \geq \eta(\mu_\ell, \mu_{\ell+1}, t'), \quad \text{for } t' \gg \theta. \tag{47}$$

Clearly,

$$\alpha(\mu_{\ell-1}, h\mu_{\ell-1}, t')\alpha(\mu_\ell, h\mu_\ell, t') \geq \eta(\mu_{\ell-1}, h\mu_{\ell-1}, t')\eta(\mu_\ell, h\mu_\ell, t'), \quad \text{for } t' \gg \theta. \tag{48}$$

Now, by (44), for $t' \gg \theta$,

$$\psi\left(\frac{1}{\ddot{M}_\alpha(\mu_\ell,\mu_{\ell+1},t')}-1\right)=\psi\left(\frac{1}{\ddot{M}_\alpha(h\mu_{\ell-1},h\mu_\ell,t')}-1\right)\leq\psi\left(\frac{1}{\ddot{M}_\alpha(\mu_{\ell-1},\mu_\ell,t')}-1\right)-\phi\left(\frac{1}{\ddot{M}_\alpha(\mu_{\ell-1},\mu_\ell,t')}-1\right). \quad (49)$$

If $\ddot{M}_\alpha(\mu_{\ell-1},\mu_\ell,t')=1$, then $\ddot{M}_\alpha(\mu_\ell,\mu_{\ell+1},t')=1$. Otherwise, if $\ddot{M}_\alpha(\mu_{\ell-1},\mu_\ell,t')<1$, then

$$\psi\left(\frac{1}{\ddot{M}_\alpha(\mu_\ell,\mu_{\ell+1},t')}-1\right)<\psi\left(\frac{1}{\ddot{M}_\alpha(\mu_{\ell-1},\mu_\ell,t')}-1\right),\quad \text{for } t'\gg\theta. \quad (50)$$

Since ψ is nondecreasing, we may obtain that $\ddot{M}_\alpha(\mu_\ell,\mu_{\ell+1},t')\geq\ddot{M}_\alpha(\mu_{\ell-1},\mu_\ell,t')$, $\forall\ell\in N$ and, $t'\gg\theta$. Thus, $(\ddot{M}_\alpha(\mu_{\ell-1},\mu_\ell,t'))$ is nondecreasing sequence in $[0,1]$. Let $m_1(t')=\lim_{\ell\longrightarrow\infty}\ddot{M}_\alpha(\mu_{\ell-1},\mu_\ell,t')$, $t'\gg\theta$. Now, we have to show that $m_1(t')=t'$, for $t'\gg\theta$. If not, then $\exists t'\gg\theta$ such that $m_1(t')<1$. Therefore, by taking the limit $\ell\longrightarrow\infty$ on (50), we get

$$\psi\left(\frac{1}{m_1(t')}-1\right)\leq\psi\left(\frac{1}{m_1(t')}-1\right)-\phi\left(\frac{1}{m_1(t')}-1\right), \quad (51)$$

which is a contradiction. Thus, we get that

$$\lim_{\ell\longrightarrow\infty}\ddot{M}_\alpha(\mu_{\ell-1},\mu_\ell,t')=1,\quad \text{for } t'\gg\theta. \quad (52)$$

Let $\ell, j\in N$ such that $j>\ell$ and $j=\ell+p$, where $p\in N$. We have that

$$\ddot{M}_\alpha(\mu_\ell,\mu_{\ell+p},t')\geq\ddot{M}_\alpha\left(\mu_\ell,\mu_{\ell+1},\frac{t'}{p}\right)*\ddot{M}_\alpha\left(\mu_{\ell+1},\mu_{\ell+2},\frac{t'}{p}\right)*\cdots*\ddot{M}_\alpha\left(\mu_{\ell+p-1},\mu_{\ell+p},\frac{t'}{p}\right)\longrightarrow 1*1*\cdots*1=1, \text{ as } \ell\longrightarrow\infty,\quad \text{for } t'\gg\theta. \quad (53)$$

Thus, (μ_ℓ) is a $\mathcal{G}$-C auchy sequence, and since the space $(\mathcal{U},\ddot{M}_\alpha,*)$ is $\mathcal{G}$-complete, therefore, $\mu_\ell\longrightarrow z\in\mathcal{U}$. Now, for any sequence (μ_ℓ) in $\mathcal{U}$ with $\alpha(\mu_\ell,\mu_{\ell+1},t')\geq\eta(\mu_\ell,\mu_{\ell+1},t')$, $\forall\ell\in N$ and $t'\gg\theta$. By the completeness of $\mathcal{U}$, $\mu_\ell\longrightarrow z$ as $\ell\longrightarrow\infty$, $\alpha(\mu_\ell,z,t')\geq\eta(\mu_\ell,z,t')$ and $\alpha(z,hz,t')\geq\eta(z,hz,t')$, $\forall\ell\in N$ and $t'\gg\theta$. Then, easily we may obtain

$$\alpha(\mu_\ell,\mu_{\ell+1},t')\alpha(z,hz,t')\geq\eta(\mu_\ell,\mu_{\ell+1},t')\eta(z,hz,t'),\quad \text{for } t'\gg\theta. \quad (54)$$

Now, from (44),

$$\psi\left(\frac{1}{\ddot{M}_\alpha(\mu_{\ell+1},hz,t')}-1\right)=\psi\left(\frac{1}{\ddot{M}_\alpha(h\mu_\ell,hz,t')}-1\right)\leq\psi\left(\frac{1}{\ddot{M}_\alpha(\mu_\ell,z,t')}-1\right)-\phi\left(\frac{1}{\ddot{M}_\alpha(\mu_\ell,z,t')}-1\right),\quad \text{for } t'\gg\theta. \quad (55)$$

If $\ddot{M}_\alpha(\mu_\ell,z,t')=1$, then $\ddot{M}_\alpha(\mu_{\ell+1},hz,t')=1$, for $t'\gg\theta$. If $\ddot{M}_\alpha(\mu_\ell,z,t')<1$, for $t'\gg\theta$, then we have

$$\psi\left(\frac{1}{\ddot{M}_\alpha(\mu_{\ell+1},hz,t')}-1\right)<\psi\left(\frac{1}{\ddot{M}_\alpha(\mu_\ell,z,t')}-1\right)\Rightarrow\ddot{M}_\alpha(\mu_{\ell+1},hz,t')\geq\ddot{M}_\alpha(\mu_\ell,z,t')\longrightarrow 1,\quad \text{as } \ell\longrightarrow\infty, \quad (56)$$

for $t' \gg \theta$. Thus, we get that $hz = z$. □

Example 1. Let $\mathcal{U} = [0, \infty)$ and t'-norm is a continuous norm. Let a FM $\ddot{M}_\alpha$: $\mathcal{U} \times \mathcal{U} \times (0, \infty) \longrightarrow [0, 1]$ be defined as

$$\ddot{M}_\alpha(\mu, \omega, t') = \frac{t'}{t' + |\mu - \omega|}, \quad \forall \mu, \omega \in \mathcal{U}, \text{ and } t' > 0. \quad (57)$$

Then, $(\mathcal{U}, \ddot{M}_\alpha, *)$ is a $\mathcal{G}$-complete FCM space and a FCM $\ddot{M}_\alpha$ is triangular. Now, we define a mapping h: $\mathcal{U} \longrightarrow \mathcal{U}$ by

$$h(\mu) = \begin{cases} \frac{\mu}{3}, & \text{if } \mu \in [0, 1], \\ 3\mu, & \text{if } \mu \in (1, \infty). \end{cases} \quad (58)$$

Next, we define α, η: $\mathcal{U} \times \mathcal{U} \times \operatorname{int}(\mathcal{C}) \longrightarrow [0, \infty)$ and ϕ, ψ: $[0, +\infty) \longrightarrow [0, +\infty)$, and we have

$$\alpha(\mu, \omega, t') = \begin{cases} 3, & \text{if } \mu, \omega \in [0, 1], \text{ and } t' \gg \theta, \\ \frac{3}{4}, & \text{otherwise}, \end{cases} \quad (59)$$

$$\eta(\mu, \omega, t') = \begin{cases} 2, & \text{if } \mu, \omega \in [0, 1], \text{ and } t' \gg \theta, \\ \frac{2}{5}, & \text{otherwise}, \end{cases} \quad (60)$$

$$\psi(\xi) = \frac{2\xi}{3}, \quad \text{and } \phi(\xi) = \frac{\xi}{3}, \forall \xi \in [0, +\infty). \quad (61)$$

However, $(\mathcal{U}, \ddot{M}_\alpha, *)$ is a $\mathcal{G}$-complete FCM space. Now, first, we have to show that h is α-admissible w.r.t η. Since in (59) and (60), $\alpha(\mu, \omega, t') \geq 1$ and $\eta(\mu, \omega, t') \geq 1$ for all $\mu, \omega \in [0, 1]$ and $t' \gg \theta$ and $\alpha(\mu, \omega, t') \geq \eta(\mu, \omega, t')$, which shows that α is admissible w.r.t η and h is an α-admissible w.r.t η, that is, $\alpha(h\mu, h\omega, t') \geq \eta(h\mu, h\omega, t')$ for all $\mu, \omega \in [0, 1]$ and $t' \gg \theta$. On the other hand, $h(\mu) \leq 1$ for all $\mu \in [0, 1]$. If $\mu \notin [0, 1]$, then, again by using (59) and (60), we have that $\alpha(\mu, h\mu, t') = 3/4$ and $\eta(\mu, h\mu, t') = 2/5$ for $t' \gg \theta$ and so that $\alpha(\mu, h\mu, t')\alpha(\omega, h\omega, t') = 9/16 < 1$ and $\eta(\mu, h\mu, t')\eta \ (\omega, h\omega, t') = 4/25 < 1$, for $t' \gg \theta$, which is contradiction. Similarly, if $\omega \notin [0, 1]$, then again we get that $\alpha(\mu, h\mu, t')\alpha(\omega, h\omega, t') < 1$ and $\eta(\mu, h\mu, t')\eta(\omega, h\omega, t') < 1$, for $t' \gg \theta$, which is contradiction. Hence, $\alpha(\mu, h\mu, t')\alpha(\omega, h\omega, t') \geq 1$ and $\eta(\mu, h\mu, t')\eta(\omega, h\omega, t') \geq 1$, for $t' \gg \theta$. It follows that the mapping h is both α-admissible and η-admissible and $\alpha(\mu, h\mu, t')\alpha(\omega, h\omega, t') \geq \eta(\mu, h\mu, t')\eta(\omega, h\omega, t')$, for $t' \gg \theta$.

Now, if (μ_ℓ) in a sequence in $\mathcal{U}$ such that $\alpha(\mu_\ell, \mu_{\ell+1}, t') \geq 1$ and $\eta(\mu_\ell, \mu_{\ell+1}, t') \geq 1$ for all $\ell \in \mathbb{N} \cup \{0\}$ and $\mu_\ell \longrightarrow z$ as $\ell \longrightarrow +\infty$, then $(\mu_\ell) \subset [0, 1]$ and hence $z \in [0, 1]$. This implies that $\alpha(\mu_\ell, z, t') \geq 1$ and $\eta(\mu_\ell, z, t') \geq 1$ for all $\ell \in \mathbb{N}$ and $t' \gg \theta$. Next, we prove that inequality (44) of Theorem 3 is satisfied by using (59)–(61), for all $\mu, \omega \in [0, 1]$ and $t' \gg \theta$:

$$\begin{aligned} \psi\left(\frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1\right) &= \psi\left(\frac{|h\mu - h\omega|}{t'}\right), \\ &= \frac{2|h\mu - h\omega|}{3t'} = \frac{2|\mu - \omega|}{9t'} \\ &\leq \frac{2|\mu - \omega|}{3t'} - \frac{|\mu - \omega|}{3t'} \\ &= \psi\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right) - \phi\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right). \end{aligned} \quad (62)$$

That is,

$$\begin{aligned} &\alpha(\mu, h\mu, t')\alpha(\omega, h\omega, t') \geq \eta(\mu, h\mu, t')\eta(\omega, h\omega, t') \quad \text{for } t' \gg \theta, \\ &\Rightarrow \psi\left(\frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1\right) \\ &\leq \psi\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right) - \phi\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right) \quad \text{for } t' \gg \theta. \end{aligned} \quad (63)$$

Hence, the conditions of Theorem 3 are satisfied, and the mapping h has a FP in $\mathcal{U}$, i.e., 0.

Note. In special case, by using metric space, the main result of Dutta and Choudhury [37] for the mapping h is not applicable; if we put $\mu = 3$ and $\omega = 4$, then we have

$$\psi(d(h\mu, h\omega)) = 2 > \frac{1}{3} = \psi(d(\mu, \omega)) - \phi(d(\mu, \omega)). \quad (64)$$

Note. if we take $\eta(\mu, \omega, t') = 1$ in Theorem 3, then we obtain the following two corollaries.

Corollary 3. *Let $(\mathcal{U}, \ddot{M}_\alpha, *)$ be a $\mathcal{G}$-complete FCM space, and let h: $\mathcal{U} \longrightarrow \mathcal{U}$ be an α-admissible. Suppose that $\exists\, \psi \in \Psi$ and $\phi \in \Phi_0$, so that*

$$\begin{aligned} &\alpha(\mu, h\mu, t')\alpha(\omega, h\omega, t') \geq 1 \\ &\Rightarrow \psi\left(\frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1\right) \leq \psi\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right) - \phi\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right), \end{aligned} \quad (65)$$

$\forall \mu, \omega \in \mathcal{U}$ and $t' \gg \theta$. Let the following axioms hold:

(i) $\exists \mu_0 \in \mathcal{U}$ such that $\alpha(\mu_0, h\mu_0, t') \geq 1$, for $t' \gg \theta$

(ii) Any sequence (μ_ℓ) in U with $\alpha(\mu_\ell, \mu_{\ell+1}, t') \geq 1$, $\forall \ell \in N$, $t' \gg \theta$ and $\mu_\ell \longrightarrow z \in \mathcal{U}$ as $\ell \longrightarrow +\infty$, then $\alpha(\mu_\ell, z, t') \geq 1$ and $\alpha(z, hz, t') \geq 1$, $\forall \ell \in N\, t' \gg \theta$

Then, h has a FP in $\mathcal{U}$.

Corollary 4. *Let $(\mathcal{U}, \ddot{M}_\alpha, *)$ be a $\mathcal{G}$-complete FCM space, and let h: $\mathcal{U} \longrightarrow \mathcal{U}$ be an α-admissible. Suppose that $\exists \psi \in \Psi$ and $\phi \in \Phi_0$, so that*

$$\alpha(\mu, h\mu, t')\alpha(\omega, h\omega, t') \leq 1$$
$$\Rightarrow \psi\left(\frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1\right) \leq \psi\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right) - \phi\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right), \tag{66}$$

$\forall \mu, \omega \in \mathcal{U}$ and $t' \gg \theta$. Let the following conditions hold:

(i) $\exists \mu_0 \in \mathcal{U}$ such that $\alpha(\mu_0, h\mu_0, t') \leq 1$, for $t' \gg \theta$

(ii) Any sequence (μ_ℓ) in $\mathcal{U}$ with $\alpha(\mu_\ell, \mu_{\ell+1}, t') \leq 1$, $\forall \ell \in N$, $t' \gg \theta$ and $\mu_\ell \longrightarrow z \in \mathcal{U}$ as $\ell \longrightarrow +\infty$, then $\alpha(\mu_\ell, z, t') \leq 1$ and $\alpha(z, hz, t') \leq 1$, $\forall \ell \in N\, t' \gg \theta$

Then, h has a FP in $\mathcal{U}$.

4. Supportive Application

In this section, we present an integral type application for FP to support our result.

Let

$$\mu(r) = f(\tau) + \int_0^r \kappa(\tau, s', \mu(s'))ds', \quad \forall \tau \in [0, r] = I_0, \tag{67}$$

where $r > 0$. Let $\mathcal{U} = C(I_0, \mathcal{R})$ be a Banach space of all continuous functions defined on I_0. The induced metric is defined by

$$\ddot{d}(\mu, \omega) = \sup_{\tau \in I_0} |\mu(\tau) - \omega(\tau)|, \quad \mu, \omega \in C(I_0, \mathcal{R}). \tag{68}$$

Let $\varrho_1 * \varrho_2 = \varrho_1\varrho_2$, $\forall \varrho_1, \varrho_2 \in [0, 1]$ and consider the fuzzy metric be defined as follows:

$$\ddot{M}_\alpha(\mu, \omega, t') = \frac{t'}{t' + \ddot{d}(\mu, \omega)}, \tag{69}$$

for $t' > 0$ and $\mu, \omega \in C(I, \mathcal{R})$. The space $(C(I_0, \mathcal{R}), \ddot{M}_\alpha, *_3)$ is $\mathcal{G}$-complete FM space indeed by the Banach space $C(I_0, \mathcal{R})$. Now, here we discuss an integral type application for FCM space and prove the existing solution for the integral equation (67).

Theorem 4. . *Let h: $C(I_0, \mathcal{R}) \longrightarrow C(I_0, \mathcal{R})$ be an integral operator defined as follows:*

$$h(\mu(\tau)) = f(\tau) + \int_0^\tau \kappa(\tau, s', \mu(s'))ds', \tag{70}$$

where $\tau \in I_0$, $f \in C(I_0, \mathcal{R})$, and κ: $I_0 \times I_0 \times \mathcal{R} \longrightarrow \mathcal{R}$, that is, $\kappa \in C(I_0 \times I_0 \times \mathcal{R}, \mathcal{R})$ satisfies the following: $\exists g$: $I_0 * I_0 \longrightarrow [0, +\infty)$ such that , $\forall r \in I_0$, $g(\tau, .) \in L^1(I_0, \mathcal{R})$, $\forall \mu, \omega \in C(I_0, \mathcal{R})$, and $\forall \tau, s \in I_0$, and we have that

$$|\kappa(\tau, s', \mu(s')) - \kappa(\tau, s', \omega(s'))| \leq g(\tau, s')\mathbf{B}(h, \mu, \omega), \tag{71}$$

where

$$\mathbf{B}(h, \mu, \omega) = \min\left\{\begin{array}{ll} \sup_{s' \in I_0} |\mu(s') - \omega(s')|, & \sup_{s' \in I_0} |\mu(s') - h(\mu(s'))|, \\ \sup_{s' \in I_0} |\omega(s') - h(\mu(s'))|, & \sup_{s' \in I_0} |\omega(s') - h(\omega(s'))| \end{array}\right\}, \tag{72}$$

where $\int_0^\tau g(\tau, s')ds'$ is bounded on I_0 and the $\sup_{\tau \in I_0} \int_0^\tau g(\tau, s')ds' \leq \lambda < 1$. Then, the integral equation (67) has a solution $\mu^* \in C(I_0, \mathcal{R})$.

Proof. The integral operator h is defined in (70). Now, we have to apply Corollary 1, for all $\mu, \omega \in C(I_0, \mathcal{R})$, and by the view of (69)–(71), we have

$$\begin{aligned}
\frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1 &= \frac{\mathrm{d}(h\mu, h\omega)}{t'},\\
&= \frac{\sup_{\tau\in I_0} |h(\mu(\tau)) - h(\omega(\tau))|}{t'}\\
&= \frac{1}{t'} \sup_{\tau\in I_0} \int_0^{\tau} |\kappa(\tau, s', \mu(s')) - \kappa(\tau, s', \omega(s'))| \mathrm{d}s'\\
&\le \frac{1}{t'} \sup_{\tau\in I_0} \int_0^{\tau} g(\tau, s')\mathbf{B}(h, \mu, \omega) \mathrm{d}s'\\
&= \frac{1}{t'} \mathbf{B}(h, \mu, \omega) \sup_{\tau\in I_0} \int_0^{\tau} g(\tau, s') \mathrm{d}s'.
\end{aligned} \tag{73}$$

Then, we have the following four cases:

(1) If $\sup_{s'\in I_0} |\mu(s') - \omega(s')|$ is the minimum term in (72), then $\mathbf{B}(h, \mu, \omega) = \sup_{s'\in I_0} |\mu(s') - \omega(s')|$. Now, by the view of (69), (71), and (73), for $t' \gg \theta$, we have

$$\begin{aligned}
\frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1 &\le \frac{1}{t'} \sup_{s'\in I_0} |\mu(s') - \omega(s')| \sup_{\tau\in I_0} \int_0^{\tau} g(\tau, s') \mathrm{d}s',\\
&\le \frac{1}{t'} \ddot{\mathrm{d}}(\mu, \omega) \sup_{\tau\in I_0} \int_0^{\tau} g(\tau, s') \mathrm{d}s'\\
&\le \lambda\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right)\\
&= \lambda\left(\frac{1}{\Omega(h, \mu, \omega, t')} - 1\right),
\end{aligned} \tag{74}$$

where $\Omega(h, \mu, \omega, t') = \ddot{M}_\alpha(\mu, \omega, t')$, for $t' \gg \theta$.

(2) If $\sup_{s'\in I_0} |\mu(s') - h(\mu(s'))|$ is the minimum term in (72), then $\mathbf{B}(h, \mu, \omega) = \sup_{s'\in I_0} |\mu(s') - h(\mu(s'))|$. Now, by the view of (69), (71), and (73), for $t' \gg \theta$, we have

$$\begin{aligned}
\frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1 &\le \frac{1}{t'} \sup_{s'\in I_0} |\mu(s') - h(\mu(s'))| \sup_{\tau\in I_0} \int_0^{\tau} g(\tau, s') \mathrm{d}s',\\
&\le \frac{1}{t'} \ddot{\mathrm{d}}(\mu, h\mu) \sup_{\tau\in I_0} \int_0^{\tau} g(\tau, s') \mathrm{d}s'\\
&\le \lambda\left(\frac{1}{\ddot{M}_\alpha(\mu, h\mu, t')} - 1\right)\\
&= \lambda\left(\frac{1}{\Omega(h, \mu, \omega, t')} - 1\right),
\end{aligned} \tag{75}$$

where $\Omega(h, \mu, \omega, t') = \ddot{M}_\alpha(\mu, h\mu, t')$, for $t' \gg \theta$.

(3) If $\sup_{s'\in I_0} |\omega(s') - h(\mu(s'))|$ is the minimum term in (72), then $\mathbf{B}(h, \mu, \omega) = \sup_{s'\in I_0} |\omega(s') - h(\mu(s'))|$. Now, by the view of (69), (71), and (73), for $t' \gg \theta$, we have

$$\begin{aligned}
\frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1 &\le \frac{1}{t'} \sup_{s'\in I_0} |\omega(s') - h(\mu(s'))| \sup_{\tau\in I_0} \int_0^{\tau} g(\tau, s') \mathrm{d}s',\\
&\le \frac{1}{t'} \ddot{\mathrm{d}}(\omega, h\mu) \sup_{\tau\in I_0} \int_0^{\tau} (\tau, s') \mathrm{d}s'\\
&\le \lambda\left(\frac{1}{\ddot{M}_\alpha(\omega, h\mu, t')} - 1\right)\\
&= \lambda\left(\frac{1}{\Omega(h, \mu, \omega, t')} - 1\right),
\end{aligned} \tag{76}$$

where $\Omega(h, \mu, \omega, t') = \ddot{M}_\alpha(\omega, h\mu, t')$, for $t' \gg \theta$.

(4) If $\sup_{s'\in I_0} |\omega(s') - h(\omega(s'))|$ is the minimum term in (72), then $\mathbf{B}(h, \mu, \omega) = \sup_{s'\in I_0} |\omega(s') - h(\omega(s'))|$. Now, by the view of (69), (71), and (73), for $t' \gg \theta$, we have

$$\begin{aligned}
\frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1 &\le \frac{1}{t'} \sup_{s'\in I_0} |\omega(s') - h(\omega(s'))| \sup_{\tau\in I_0} \int_0^{\tau} (\tau, s') \mathrm{d}s',\\
&\le \frac{1}{t'} \ddot{\mathrm{d}}(\omega, h\omega) \sup_{\tau\in I_0} \int_0^{\tau} g(\tau, s') \mathrm{d}s'\\
&\le \lambda\left(\frac{1}{\ddot{M}_\alpha(\omega, h\omega, t')} - 1\right)\\
&= \lambda\left(\frac{1}{\Omega(h, \mu, \omega, t')} - 1\right),
\end{aligned} \tag{77}$$

where $\Omega(h, \mu, \omega, t') = \ddot{M}_\alpha(\omega, h\omega, t')$, for $\gg \theta$.

Hence, in all the cases, it is obvious that (26) holds with $\phi(c) = \lambda c, \forall c \ge 0$ and $\alpha(\mu, \omega, t') = 1$, $\forall \mu, \omega \in C(I_0, \mathscr{R})$ and $t' \gg \theta$. As we have mentioned above, $C(I_0, \mathscr{R})$ is complete and then the FCM space $(C(I_0, \mathscr{R}), \ddot{M}_\alpha, *_3)$ is $\mathscr{G}$-complete in which $\ddot{M}_\alpha$ is triangular. The other conditions of Corollary 1 are fulfilled immediately. We deduce that the operator h has a FP $\mu^* \in C(I_0, \mathscr{R})$ which is the required solution of the integral equation (67). □

5. Conclusion

In this paper, we have presented the mappings, and h is α-admissible w.r.t η and α-ϕ-fuzzy cone contraction in FCM spaces. Using this kind of contractions, we proved FP theorems for $\mathcal{G}$-complete FCM space in the sense of George and Veeramani with an illustrative example. Moreover, some extended results in the form of corollaries are discussed. An application is presented to support the concepts defined in the paper. This integral type application is also illustrative of how fuzzy metrics can be used in other integral type operators.

Data Availability

No data set were generated or analyzed during this current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors have equally contributed to the final manuscript. All the authors have read and approved the manuscript.

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