Koopman Operator Approximation Under Negative Imaginary Constraints

Mohamed A. Mabrok, Ilyasse Aksikas, and Nader Meskin, *Senior Member, IEEE*

***Abstract*—Nonlinear Negative Imaginary (NI) systems arise in various engineering applications, such as controlling flexible structures and air vehicles. However, unlike linear NI systems, their theory is not well-developed. In this letter, we propose a data-driven method for learning a lifted linear NI dynamics that approximates a nonlinear dynamical system using the Koopman theory, which is an operator that captures the evolution of nonlinear systems in a lifted high-dimensional space. The linear matrix inequality that characterizes the NI property is embedded in the Koopman framework, which results in a non-convex optimization problem. To overcome the numerical challenges of solving a non-convex optimization problem with nonlinear constraints, the optimization variables are reformatted in order to convert the optimization problem into a convex one with the new variables. We compare our method with local linearization techniques and show that our method can accurately capture the nonlinear dynamics and achieve better control performance. Our method provides a numerically tractable solution for learning the Koopman operator under NI constraints for nonlinear NI systems and opens up new possibilities for applying linear control techniques to nonlinear NI systems without linearization approximations.**

***Index Terms*—Negative imaginary systems, Koopman operator, system identification.**

I. Introduction

NEGATIVE imaginary (NI) systems are systems that have a negative imaginary frequency response [1], [2] and their theory has been widely used for analysis and design of linear-time-invariant (LTI) control systems, especially for flexible structures and air vehicles [1], [2], [3]. NI systems have several desirable properties, such as robust stability under positive feedback, disspativity with respect to collocated inputs and outputs, and existence of optimal controllers [4]. A central result in the NI theory is that the positive feedback interconnection between an NI system and a strictly NI system results in a robustly stable feedback interconnection [1], [2]. This implies that if a given system is NI, then, it is a great advantage to synthesis a strictly NI controller to guarantee robust stability.

There have been various approaches to address the NI control synthesis problem [5], [6], [7]. In [5], a negative imaginary and strict negative imaginary lemma is used to construct a static controller that ensures robust stability against strict negative imaginary uncertainty. In [6], a data-driven controller synthesis methodology is proposed that uses measured frequency response data to construct the controller response and transfer function. A synthesis methodology for non-linear systems is proposed in [7] using a library of controllers parametrized and strictly negative imaginary, optimized through Sequential Quadratic Programming. This methodology can be applied to different non-linear systems. The data driven linear NI system identification problem is addressed in [8], where a modified subspace system identification algorithm that guarantees the negative imaginary property in the identified model by imposing constraints to ensure stability and negative imaginariness was proposed.

Negative imaginary systems' theory provides a way to analyze robustness and design robust controllers. In other words, the theory can be used to design controllers that are able to maintain stability and performance even in the presence of uncertainties in the system, such as unmodeled spillover dynamics or variations in resonant frequencies and damping levels [9].

However, many real-world systems are nonlinear in nature and cannot be adequately modeled by LTI systems. For example, the dynamics of a mass-spring-damper system with a nonlinear spring or a nonlinear damper are nonlinear NI systems [10]. Nonlinear NI systems pose significant challenges for analysis and control, as the existing NI systems theory does not directly apply to them. Therefore, there is a need to develop a general framework for nonlinear NI systems that can capture their essential features and enable their effective control.

One possible approach to deal with nonlinear systems is to use the Koopman operator, which is a linear operator that describes the evolution of scalar observables of nonlinear systems in an infinite-dimensional Hilbert space [11]. The Koopman operator has attracted considerable attention in recent years as a powerful tool for nonlinear dynamics

Manuscript received 6 May 2023; revised 21 June 2023; accepted 25 June 2023. Date of publication 30 June 2023; date of current version 12 July 2023. Recommended by Senior Editor A. P. Aguiar. *(Corresponding author: Mohamed A. Mabrok.)*

Mohamed A. Mabrok and Ilyasse Aksikas are with the Mathematics Program, Department of Mathematics, Statistics and Physics, College of Arts and Sciences, Qatar University, Doha, Qatar (e-mail: m.a.mabrok@gmail.com; aksikas@qu.edu.qa).

Nader Meskin is with the Electircal Engineering Department, College of Engineering, Qatar University, Doha, Qatar (e-mail: nader.meskin@qu.edu.qa).

Digital Object Identifier 10.1109/LCSYS.2023.3290195

modeling and control [11], [12], [13]. By using the Koopman operator, one can lift the nonlinear dynamics to a high-dimensional space where they become linear, and then apply linear techniques to analyze and control them. However, learning the Koopman operator from data is generally a challenging task, as it involves solving a non-convex optimization problem with nonlinear constraints [11].

The nonlinear NI systems possess unique characteristics that make their analysis and control challenging compared to linear systems. Our choice of the Koopman theory is motivated by its ability to capture the evolution of nonlinear systems in a lifted high-dimensional space, providing a powerful framework for modeling and control. Moreover, the Koopman operator provides a data-driven approach to system identification. By analyzing observed system trajectories, the Koopman operator captures the underlying dynamics and generates a linear model that approximates the behavior of the nonlinear system. This data-driven nature of the Koopman-based technique is particularly advantageous for NI systems, as it does not rely on explicit knowledge of the system dynamics, which can be challenging to obtain in practice.

In this letter, we propose a data-driven method for learning the Koopman operator under NI constraints for nonlinear NI systems. We design a data-driven method for learning a lifted linear NI dynamics that approximates a nonlinear dynamical system using the Koopman theory. To overcome the numerical challenges of solving a non-convex optimization problem with nonlinear constraints, we use a change of variable technique, which allow us to reformulate the problem as a convex optimization that can be solved efficiently. We compare our method with local linearization techniques and show that our method can accurately capture the nonlinear dynamics and achieve better control performance. Our method provides a numerically tractable solution for learning the Koopman operator under NI constraints for nonlinear NI systems and opens up new possibilities for applying linear control techniques to nonlinear systems without linearization approximations.

II. Preliminaries and Notation

A. Negative Imaginary Systems

Consider the following LTI system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

$$y(t) = Cx(t) + Du(t), \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, and with the square transfer function matrix $G(s) = C(sI - A)^{-1}B + D$. (1), (2). A NI system is defined as follows:

Definition 1 [2]: A square transfer function matrix $G(s)$ is NI if the following conditions are satisfied:

1) $G(s)$ has no pole in $\mathrm{Re}[s] > 0$.
2) For all $\omega > 0$ such that $s = j\omega$ is not a pole of $G(s)$,

$$j\big(G(j\omega) - G(j\omega)^*\big) \geq 0. \tag{3}$$

3) If $s = j\omega_0$ with $\omega_0 > 0$ is a pole of $G(s)$, then it is a simple pole and the residue matrix $K = \lim_{s\longrightarrow j\omega_0} (s - j\omega_0)jG(s)$ is Hermitian and positive semidefinite.
4) If $s = 0$ is a pole of $G(s)$, then $\lim_{s\longrightarrow 0} s^k G(s) = 0$ for all $k \geq 3$ and $\lim_{s\longrightarrow 0} s^2 G(s)$ is Hermitian and positive semidefinite.

Definition 2 [14]: A square transfer function matrix $G(s)$ is SNI if the following conditions are satisfied:

1) $G(s)$ has no pole in $\mathrm{Re}[s] \geq 0$.
2) For all $\omega > 0$, $j(G(j\omega) - G(j\omega)^*) > 0$.

The negative imaginary lemma is a result that describes NI systems using a pair of LMIs, similar to how the positive-real lemma does [15], [16]. This result was presented in [1], [17] and it also covers the case where there are poles on the imaginary axis except at zero [14].

Lemma 1 (See [14]): Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a minimal state space realization of a transfer function matrix $G(s)$. Then $G(s)$ is NI if and only if $\det(A) \neq 0$, $D = D^{\mathrm{T}}$ and there exists a real matrix $P > 0$ such that

$$AP + PA^* \leq 0, \quad B = -APC^*. \tag{4}$$

The theory of nonlinear NI is not yet well investigated. Preliminary work has been carried out in [10], [18], [19]. Consider the following nonlinear dynamical system

$$\dot{x}(t) = f(x(t), u(t)), \quad y(t) = h(x(t)), \tag{5}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^m$ are the system state, control input, and output, respectively.

Definition 3: The nonlinear system (5) is said to be NI if there exists a positive storage function $S_f : \mathbb{R}^n \to \mathbb{R}$ of a class C^1 such that

$$S_f(x(t)) \leq S_f(x(0)) + \int_0^t u^{\mathrm{T}}(\tau)\dot{y}(\tau)\mathrm{d}\tau, \forall t > 0. \tag{6}$$

It is important to note that the way output measurement is defined in NI systems is different from passive systems. NI systems use position or acceleration measurements, while passive systems use velocity sensors. NI systems also allow for transfer functions with a relative degree of up to two, while passive systems permit relative degree zero or one. This is seen in systems that use force actuators and position sensors, like robotics and nano-positioning systems. Stability results for negative feedback interconnections between passive and strictly passive systems do not apply to positive position feedback interconnections between negative imaginary and strictly negative imaginary systems. This was shown in [20] for physical systems like mass-spring-damper systems and RLC electrical networks by comparing stability results for feedback interconnections of negative imaginary systems with those for passive systems.

There are several important mechanical systems that satisfy the nonlinear NI property. For instance, a mass-spring-damper system with positive nonlinear damping coefficient is a nonlinear NI from force input and displacement output. A more general class of nonlinear NI systems is the class of systems that are governed by Euler-Lagrangian dynamics,

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = F. \tag{7}$$

It has been shown in [10] that the dynamical systems governed by (7) are indeed nonlinear NI systems.

B. Koopman Theory

In this section, a brief introduction for the Koopman operator given in [21] is presented. The Koopman operator is a mathematical tool used in the analysis of dynamical systems. In particular, in the context of dynamical systems, the Koopman operator provides a powerful tool for studying the evolution of the system over time. By mapping the system's state space to a higher dimensional space, the operator allows for the analysis of nonlinear dynamics through linear methods. This is particularly useful in cases where traditional nonlinear analysis techniques, such as numerical simulations or bifurcation analysis, become computationally expensive or infeasible. One key application of the Koopman operator is in data-driven modeling of complex systems. By using data from the system, the Koopman operator can be estimated and used to approximate the system's behavior.

Consider the following nonlinear difference equation

$$\begin{cases} x(j+1) &= f(x(j), u(j)), \\ y(j) &= g(x(j)), \end{cases} \tag{8}$$

where j represents the discrete time, $u \in \mathbb{R}^m$ represents the input, $x \in \mathbb{R}^n$ represents the state of the dynamical system, $y \in \mathbb{R}^l$ represents the output, and $f(x, u) : \mathbb{R}^{n+m} \to \mathbb{R}^n$ and $g(x) : \mathbb{R}^n \to \mathbb{R}^l$ are the nonlinear functions. Let ξ to be defined as, $\xi := \begin{bmatrix} x \\ u \end{bmatrix} \in \mathbb{R}^{n+m}$. Also, define $\mathbb{F}$ to be the nonlinear operator given by $\mathbb{F}(\xi) := \begin{bmatrix} f(x,u) \\ \Xi(u) \end{bmatrix}$, where Ξ is the time-shift operator, i.e., $\Xi(u(j)) := u(j+1)$. This implies that the time evolution of ξ can be given as $\xi(j+1) = \mathbb{F}(\xi(j))$. Also, define a set of scalar valued observable function $\phi : \mathbb{R}^{n+m} \to \mathbb{R}$, which forms an infinite dimensional Hilbert space $\mathcal{H}$ consists of the Lebesque square-integrable functions. Now, we can define the Koopman operator $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ as

$$\mathcal{K}(\phi(\xi)) := \phi(\mathbb{F}(\xi)) \tag{9}$$

where the time evolution of the observable lifting function $\phi(\xi)$ is given as follows;

$$\phi(\xi(j+1)) = \mathcal{K}(\phi(\xi(j))). \tag{10}$$

which is a linear mapping defined on the infinite-dimensional state space [26].

In order to approximate the infinite-dimensional Koopman operator by a finite-dimensional approximation, we define the N_ϕ-dimensional lifting function $\phi(\xi) : \mathbb{R}^{n+m} \to \mathbb{R}^{N_\phi}$ as

$$\phi(\xi) = \begin{bmatrix} \phi_1(\xi) \\ \vdots \\ \phi_{N_\phi}(\xi) \end{bmatrix} \in \mathbb{R}^{N_\phi}.$$

Now, define $U \in \mathbb{R}^{N_\phi \times N_\phi}$ to be a finite-dimensional matrix that approximates the Koopman operator $\mathcal{K}$, i.e., U minimizes the following norm $\|U\phi(\xi) - \phi(\mathcal{F}(\xi))\|$. This implies that

$$\phi(\xi(j+1)) \approx U\phi(\xi(j)), \tag{11}$$

which describes the behavior of $\phi_{\inf}(\xi)$, defined in (10). Similar to the approach in [21], we consider a class of the lifting function $\phi(\xi) = [\psi^{\mathrm{T}}(x), u^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{N_\phi + m}$, where, $\psi(x) = [\,\psi_1(x), \ldots, \psi_N(x)\,]^{\mathrm{T}} \in \mathbb{R}^N$, and $N + m = N_\phi$ holds. The matrix U can be given as

$$U = \begin{bmatrix} A_d & B_d \end{bmatrix} \in \mathbb{R}^{(N)\times(N+m)},$$

where $A_d \in \mathbb{R}^{N\times N}$ and $B_d \in \mathbb{R}^{N\times m}$. This implies that (11) can be rewritten as

$$\psi(x(j+1)) \approx A_d\psi(x(j)) + B_d u(j). \tag{12}$$

In order to formulate the output equation in Koopman operator, the approximation of the output equation in (8) can be approximated as $y(j) \approx C_d\psi(x(j))$ where $C_d \in \mathbb{R}^{l\times N}$. The final linear state-space model can be defined as follows:

$$\begin{cases} \psi(j+1) &= A_d\psi(j) + B_d u(j), \\ y(j) &= C_d\psi(j). \end{cases} \tag{13}$$

III. Formulating the Koopman NI Constraints

In this section, the NI constraints for discrete dynamical systems are formulated and then imposed into the learning problem for obtaining the approximated Koopman operator given in (11). Consider the following discrete time system

$$x(j+1) = A_d x(j) + B_d u(j), \tag{14}$$
$$y(j) = C_d x(j) + D_d u(j), \tag{15}$$

where $A_d \in \mathbb{R}^{n\times n}$, $B_d \in \mathbb{R}^{n\times m}$, $C_d \in \mathbb{R}^{m\times n}$ and $D_d \in \mathbb{R}^{m\times m}$. The NI lemma given in Lemma 1, which is formulated in continuous-time and therefore, a bilinear transformation, as presented in [8], in the following form,

$$A = \tfrac{1}{T}(I + A_d)^{-1}(A_d - I), \quad B = \tfrac{1}{\sqrt{T}}(I + A_d)^{-1}B_d,$$
$$C = \tfrac{1}{\sqrt{T}}C_d(I + A_d)^{-1}, \quad D = D_d - C_d(I + A_d)^{-1}B_d,$$

with T denoting a sampling time is used to transform the conditions (4) into corresponding discrete-time conditions. The Lemma 1 conditions are transformed from a continuous-time to a discrete-time form as follows:

$$AP + PA^{\mathrm{T}} \le 0 \Leftrightarrow A_d P A_d^{\mathrm{T}} - P \le 0, \tag{16}$$
$$B = -APC^{\mathrm{T}} \Leftrightarrow B_d = -\frac{1}{T}(A_d - I)P(I + A_d^{\mathrm{T}})^{-1}C_d^{\mathrm{T}}. \tag{17}$$

A. Koopman Operator Learning Problem

Similar approach in [21], define the following data matrices, which are generated from L measurements,

$$\Omega := [u(j), u(j+1), \ldots, u(j+L-1)] \in \mathbb{R}^{m\times L}, \tag{18}$$
$$Y := [y(j), y(j+1), \ldots, y(j+L-1)] \in \mathbb{R}^{m\times L}, \tag{19}$$
$$\Theta := [\psi(j), \psi(j+1), \ldots, \psi(j+L-1)] \in \mathbb{R}^{N\times L}, \tag{20}$$
$$\Theta_+ := [\psi(j+1), \psi(j+2), \ldots, \psi(j+L)] \in \mathbb{R}^{N\times L} \tag{21}$$

This implies that the Koopman operator learning problem can be formulated as follows:

$$\begin{aligned} \min_{A_d, B_d, C_d} \quad & J_1(A_d, B_d) + J_2(C_d) \\ \text{s.t.} \quad & \\ P \ge 0, (16), (17) & \end{aligned} \tag{22}$$

where

$$J_1(A_d, B_d) := \left\| \Theta_+ - \begin{bmatrix} A_d & B_d \end{bmatrix} \begin{bmatrix} \Theta \\ \Omega \end{bmatrix} \right\|_F^2, \tag{23}$$

$$J_2(C_d) := \left\| Y_j - C_d \Theta \right\|_F^2. \tag{24}$$

It should be noted that the condition (16) assure the stability of the obtain NI model. The optimization problem given in (22) is non-convex due to the non-convexity of the constraints. In order to overcome this issue, we replace the matrix inequality (16) by a strict inequality with a positive $\alpha > 0$ such that the following holds,

$$P \geq \alpha I \tag{25}$$

$$A_d P A_d^{\mathrm{T}} - P \leq -\alpha I. \tag{26}$$

Now, using Schur complements, the LMIs given in (25) and (26) can be written as follows;

$$\begin{bmatrix} P - \alpha I & A_d P \\ P A_d^{\mathrm{T}} & P \end{bmatrix} \geq 0 \quad \Rightarrow \begin{bmatrix} P - \alpha I & Q \\ Q^{\mathrm{T}} & P \end{bmatrix} \geq 0, \tag{27}$$

where $Q = A_d P$.

The LMI (27) is now convex in the variables Q and P. However, the overall optimization problem is still non-convex because the variables in the cost function still A_d, B_d, C_d. To tackle this issue, the identification of the matrix C_d is considered as a separate optimization problem by minimizing the cost function given in (24), which has an optimal solution given as

$$C_d = Y \Theta^{\dagger}, \tag{28}$$

where $(\cdot)^{\dagger}$ donates the Moore-Penrose pseudoinverse of Θ.

Second, two weighting matrices W and $\hat{W}$ are introduced into the cost function (23) as follows:

$$J_1(A_d, B_d) := \left\| W \left(\Theta_+ - \begin{bmatrix} A_d & B_d \end{bmatrix} \begin{bmatrix} \Theta \\ \Omega \end{bmatrix} \right) \hat{W} \right\|_F^2, \tag{29}$$

where $\hat{W}$ is selected as:

$$\hat{W} = \begin{bmatrix} \Theta \\ \Omega \end{bmatrix}^{\mathrm{T}} \times \left(\begin{bmatrix} \Theta \\ \Omega \end{bmatrix} \begin{bmatrix} \Theta \\ \Omega \end{bmatrix}^{\mathrm{T}} \right)^{\dagger} \times \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}. \tag{30}$$

It is assumed that there exists sufficient data to be collected, which allows us to assume that $\begin{bmatrix} \Theta \\ \Omega \end{bmatrix}$ to be full row rank. This implies that the cost function in (29) can be written as follows:

$$J_1(P, Q, B_d) := \left\| W \Theta_+ \hat{W} - W \begin{bmatrix} Q & B_d \end{bmatrix} \right\|_F^2. \tag{31}$$

Therefore, the optimization problem can now be written as follows;

Problem 1:

$$\begin{aligned} \min_{P, Q, B_d} \quad & \left\| W \Theta_+ \hat{W} - W \begin{bmatrix} Q & B_d \end{bmatrix} \right\|_F^2. \\ \text{s.t.} \quad & \\ & \begin{bmatrix} P - \alpha I & Q \\ Q^{\mathrm{T}} & P \end{bmatrix} \geq 0. \end{aligned} \tag{32}$$

where $\hat{W}$ is defined as in (30), and the matrix W is a weighting matrix, which can be treated as a design parameter to tune the optimization problem.

The optimization problem in (32) is now convex in the variables P, Q, B_d. Using any convex optimization solver, such as CVX, one can accurately compute the variables P, Q, B_d, and consequently the matrix A_d from the fact that $Q = A_d P$ as $A_d = Q P^{-1}$. Note that the matrix C_d is computed separately in (28).

In the event of mismatches between the true system and the Koopman models, one suggested approach to reduce this mismatches is presented in [22]. The concept of reformatting necessary nonlinear constraints/variables within the Koopman framework has been previously studied in [23], [24]

IV. Example: Mass-Spring-Damper System

Consider a nonlinear mass-spring-damper system

$$\ddot{z}(t) + \beta(z(t), \dot{z}(t))\dot{z}(t) + K(z(t)) = u(t), \quad y(t) = z(t). \tag{33}$$

where $\beta(z(t), \dot{z}(t))$ and $k(z(t))$ donate the friction coefficient and the spring stiffness, respectively. It can be easily shown that the above system is a nonlinear NI system by choosing a storage function as follows;

$$V(z(t), \dot{z}(t)) = \frac{1}{2} m \dot{z}^2(t) + \int_0^{z(t)} k(\xi) d\xi.$$

The time derivative of $V(z(t), \dot{z}(t))$ gives,

$$\dot{V}(z(t), \dot{z}(t)) = \dot{z}(t) u(t) - \beta(z(t), \dot{z}(t)) \dot{z}^2(t) \leq \dot{z}(t) u(t),$$

which implies that the mass-spring-damper system given in (33) is indeed NI system.

Now, by defining the states as $x_1(t) = z(t)$ and $x_2(t) = \dot{z}(t)$, the system (33) can be written as follows:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ \frac{-K(x_1(t))}{m} - \frac{\beta(x_1(t), x_2(t))}{m} x_2(t) + \frac{u(t)}{m} \end{bmatrix}, \tag{34}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \tag{35}$$

Next, the optimization problem in (32) is used to learn a linear system that represents the nonlinear dynamical system given in (34).

The lifting functions $\phi(x)$ is chosen as a class of radial basis functions as $\psi_i(x(j)) = \|x(j) - r_i\|_2^2 \ln \|x(j) - r_i\|_2$ where the center r_i is selected from the uniform distribution randomly.

The new state space representation in terms of the lifting functions is given as follows:

$$\psi(x(j)) = \left[x_1(j), x_2(j), \psi_1(x(j)), \ldots, \psi_{N_{\mathrm{rbf}}}(x(j)) \right]^{\mathrm{T}} \tag{36}$$

We assumed that $K(x_1(t)) = (x_1(t) + x_1^3(t))$ and the nonlinearity in the damper is considered as $\beta(x_1(t), x_2(t)) = x_1^2(t) + x_2^2(t)$. Also, let the number of the lifted states to be 6 states, i.e., $N_{\mathrm{rbf}} = 6$. The resulting model is compared to the unconstrained Koopman model with the same number of the states, the linearized model around the initial point $x_0 = [0, 0]^{\mathrm{T}}$, and at the linearized model around the initial point $x_0 = [0.5, 0.5]^{\mathrm{T}}$. A random forcing input signal was used for generating data for the optimization problem. Figure 1 presents the state x_2 evolutions for different models compared with the true data generated from the nonlinear system. Also,

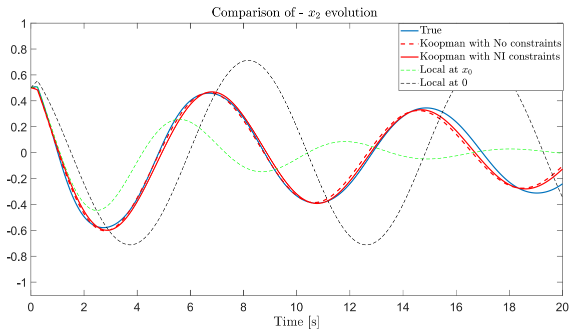

Fig. 1. The comparison of the state x_2 for different models compared with the true data generated from the nonlinear system.

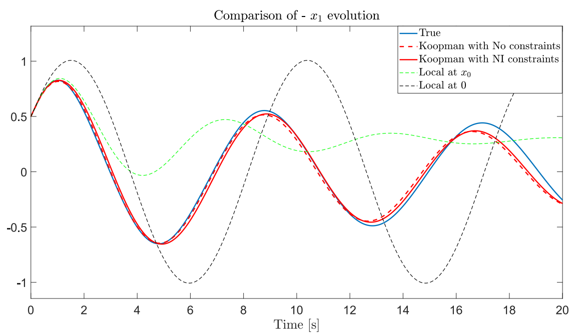

Fig. 2. The comparison of the state x_1 for different models compared with the true data generated from the nonlinear system.

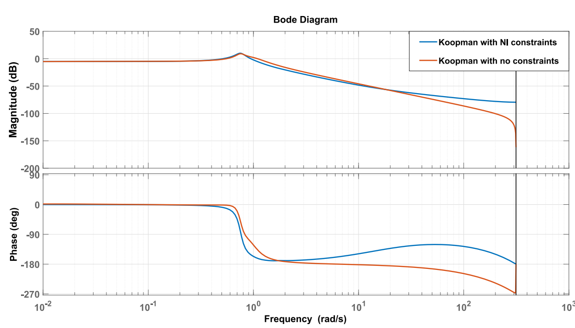

Fig. 3. Bode plots for the approximated linear system using Koopman operator. The blue lines represent the NI constrained model developed in this letter. While the orange lines represents the unconstrained model.

Figure 2 presents the state x_1 evolutions for different models compared with the true data generated from the nonlinear system. As it is shown in Figures 2 and 3, the linear models fail to approximate the nonlinear terms, whereas the Koopman model was able accurately to approximate the nonlinearity.

Figure 3 displays the Bode plots corresponding to the approximated linear system employing the Koopman operator. The blue lines depict the model constrained by NI that has been developed in this letter, whereas the unconstrained model is represented by the orange lines. It is clear from the phase plot that the NI constrained model satisfy the NI property, i.e., the phase between $(0, -\pi)$, however, the non-constrained model violates the NI property.

The mean squared error (MSE) between the true values of the states and the generated constrained model is presented in

TABLE I
THE MEAN SQUARED ERROR FOR DIFFERENT MODELS

MSE between:	x_1	x_2
True and the constrained model	0.0078	0.0042
True and the unconstrained model	0.0030	0.0021
Constrained and unconstrained model	0.0095	0.0059

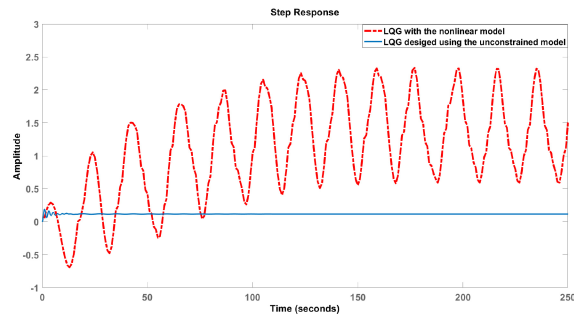

Fig. 4. The step response of one of the LQG controllers with the full nonlinear model as well as with the unconstrained linear model.

the Table I. The table shows that the raise in the MSE in the constrained model is very small.

In order to emphasize on the importance of imposing the NI constraints on the Koopman operator, we consider a feedback control design for the linearized model in both cases, the NI constrained model and the unconstrained model. We followed the following steps in order to illustrate the importance of the NI constraints: 1) We first used the identified unconstrained linear model to design an optimal controller using the LQG control design methodology for several noise covariance data and the weighting matrices. 2) The designed LQG controllers were then tested with the full nonlinear model. 3) For several noise covariance data and the weighting matrices, we obtained a limited performance, and in some cases, we obtained unstable behaviors. For instance, see Fig. 4 for the step response of one of the LQG controllers with the full nonlinear model as well as with the unconstrained linear model.

4) Then, we used the NI constrained model to design a list of controllers using different methodologies, such as the LQG-like methodology presented in our resent paper [25], the nonlinear optimization and H_2 performance measure presented in [7], and data driven controller presented in [6]. Although, some of the designed controllers had a limited performance, however, they never went unstable with the nonlinear full model. This is due to the NI property of the linearized model.

Finally, using the design methodology presented in [7], the following A position feedback controller (PPF) is designed:

$$C(s) = \frac{K}{s^2 + 2\zeta\omega s + \omega^2}. \quad (37)$$

When the controller given in (37) is connected in a positive feedback connection with both, the linear NI constrained model and the unconstrained model, a stable closed-loop is obtained in the case of the NI constrained model while in the case of the unconstrained model, the closed-loop system is unstable. This is shown in the step response of the closed-loop in Figure 5. It indeed shows the importance of obtaining

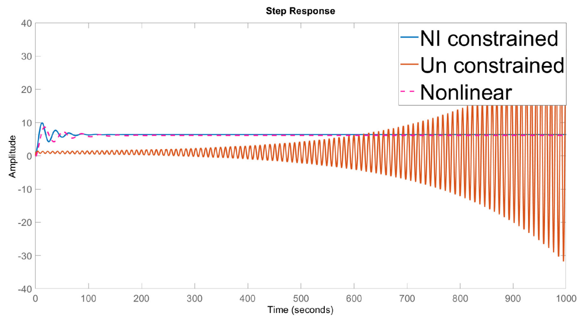

Fig. 5. A positive position feedback controller connection with the nonlinear model (dashed line), the linear NI constrained model (blue solid line) and the unconstrained model.

a NI linear model approximation of a NI nonlinear system for controller synthesis.

V. Conclusion

This letter proposes a data-driven approach to learning a lifted linear Negative Imaginary (NI) dynamics that approximate nonlinear dynamical systems using Koopman theory. The proposed method embeds the linear matrix inequality that characterizes the NI property in the Koopman framework, resulting in a non-convex optimization problem. To overcome the numerical challenges of solving the problem, the optimization variables are reformatted to convert the problem into a convex one. The comparison of the proposed method with local linearization techniques shows that the proposed method can accurately capture the nonlinear dynamics and achieve better control performance. This method provides a numerically tractable solution for learning the Koopman operator under NI constraints for nonlinear NI systems and opens up new possibilities for applying linear control techniques to nonlinear systems without linearization approximations. Overall, the proposed provides a promising approach for controlling nonlinear NI systems in various engineering applications.

References

[1] A. Lanzon and I. R. Petersen, "Stability robustness of a feedback interconnection of systems with negative imaginary frequency response," *IEEE Trans. Autom. Control*, vol. 53, no. 4, pp. 1042–1046, May 2008.

[2] M. A. Mabrok, A. G. Kallapur, I. R. Petersen, and A. Lanzon, "Generalizing negative imaginary systems theory to include free body dynamics: Control of highly resonant structures with free body motion," *IEEE Tran. Autom. Control*, vol. 59, no. 10, pp. 2692–2707, Oct. 2014.

[3] A. Preumont, *Vibration Control of Active Structures: An Introduction*, N. Jacobs, Ed. Cham, Switzerland: Springer, 2011.

[4] I. R. Petersen, "Negative imaginary systems theory and applications," *Annu. Rev. Control*, vol. 42, pp. 309–318, 2016.

[5] M. A. Mabrok, A. G. Kallapur, I. R. Petersen, and A. Lanzon, "Stabilization of conditional uncertain negative-imaginary systems using Riccati equation approach," in *Proc. 20th Int. Symp. MTNS*, 2012, pp. 9–13.

[6] M. A. Mabrok and I. R. Petersen, "Controller synthesis for negative imaginary systems: A data driven approach," *IET Control Theory Appl.*, vol. 10, no. 12, pp. 1480–1486, 2016.

[7] M. Mabrok, "Controller synthesis for negative imaginary systems using nonlinear optimisation and H_2 performance measure," *Int. J. Control*, vol. 94, no. 3, pp. 579–587, 2021.

[8] M. A. Mabrok, M. A. Haggag, and I. R. Petersen, "System identification algorithm for negative imaginary systems," *Appl. Comput. Math.*, vol. 14, no. 3, pp. 336–348, 2015.

[9] I. R. Petersen, A. Lanzon, and Z. Song, "Stabilization of uncertain negative-imaginary systems via state-feedback control," in *Proc. Eur. Control Conf.*, 2009, pp. 1605–1609.

[10] A. G. Ghallab and I. R. Petersen, "Negative imaginary systems theory for nonlinear systems: A dissipativity approach," 2022, *arXiv:2201.00144*.

[11] A. Mauroy, Y. Susuki, and I. Mezić, *Koopman Operator in Systems and Control*. Cham, Switzerland: Springer, 2020.

[12] M. Korda and I. Mezić, "Linear predictors for nonlinear dynamical systems: Koopman operator meets model predictive control," *Automatica*, vol. 93, pp. 149–160, Jul. 2018.

[13] P. Bevanda, S. Sosnowski, and S. Hirche, "Koopman operator dynamical models: Learning, analysis and control," *Annu. Rev. Control*, vol. 52, pp. 197–212, 2021.

[14] J. Xiong, I. R. Petersen, and A. Lanzon, "A negative imaginary lemma and the stability of interconnections of linear negative imaginary systems," *IEEE Trans. Autom. Control*, vol. 55, no. 10, pp. 2342–2347, Oct. 2010.

[15] B. D. O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis: A Modern Systems Approach*. Englewood Cliffs, NJ, USA: Prentice-Hall, 1973.

[16] B. Brogliato, R. Lozano, B. Maschke, and O. Egeland, *Dissipative Systems Analysis and Control: Theory and Applications*. London, U.K.: Springer, 2013.

[17] A. Lanzon and I. R. Petersen, "A modified positive-real type stability condition," in *Proc. Eur. Control Conf.*, 2007, pp. 3912–3918.

[18] A. G. Ghallab, M. A. Mabrok, and I. R. Petersen, "Extending negative imaginary systems theory to nonlinear systems," in *Proc. IEEE Conf. Decis. Control (CDC)*, 2018, pp. 2348–2353.

[19] M. A. Mabrok, M. A. Alyami, and E. E. Mahmoud, "On the dissipativity property of negative imaginary systems," *Alexandria Eng. J.*, vol. 60, no. 1, pp. 1403–1410, 2021.

[20] I. R. Petersen, "Physical interpretations of negative imaginary systems theory," in *Proc. 10th Asian Control Conf. (ASCC)*, May 2015, pp. 1–6.

[21] K. Hara, M. Inoue, and N. Sebe, "Learning Koopman operator under dissipativity constraints," *IFAC-PapersOnLine*, vol. 53, no. 2, pp. 1169–1174, 2020.

[22] S. H. Son, A. Narasingam, and J. S.-I. Kwon, "Development of offset-free Koopman Lyapunov-based model predictive control and mathematical analysis for zero steady-state offset condition considering influence of Lyapunov constraints on equilibrium point," *J. Process Control*, vol. 118, pp. 26–36, Oct. 2022.

[23] A. Narasingam and J. S.-I. Kwon, "Koopman Lyapunov-based model predictive control of nonlinear chemical process systems," *AIChE J.*, vol. 65, no. 11, 2019, Art. no. e16743.

[24] A. Narasingam, S. H. Son, and J. S.-I. Kwon, "Data-driven feedback stabilisation of nonlinear systems: Koopman-based model predictive control," *Int. J. Control*, vol. 96, no. 3, pp. 770–781, 2022.

[25] M. Mabrok, "Finding the nearest negative imaginary system with application to near-optimal controller design," 2022, *arXiv:2204.00952*.

[26] J. L. Proctor, S. L. Brunton, and J. N. Kutz, "Generalizing Koopman theory to allow for inputs and control," *SIAM J. Appl. Dynamical Syst.*, vol. 17, no. 1, pp. 909–930, 2018.