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Weak-coupling, strong-coupling and large-order parametrization of the hypergeometric-Meijer approximants

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ABSTRACT

Without Borel or Padé techniques, we show that for a divergent series with $n!$ large-order growth factor, the set of hypergeometric series $_pF_{p-2}$ represents suitable approximants. The divergent $_pF_{p-2}$ series are then resummed via their representation in terms of the Meijer-G function. The choice of $_pF_{p-2}$ accelerates the convergence even with only weak-coupling information as input. For more acceleration of the convergence, we employ the strong-coupling and large-order information. We obtained a new constraint that relates the difference between the sum of the numerator and the sum of denominator parameters in the hypergeometric approximant to one of the large-order parameters. To test the validity of that constraint, we employed it to obtain the exact partition function of the zero-dimensional ϕ^4 scalar field theory. The algorithm is also applied for the resummation of the ground state energies of ϕ^4_{0+1} and $i\phi^3_{0+1}$ scalar field theories. We get accurate results for the whole coupling space and the precision is improved systematically in using higher orders. Precise results for the critical exponents of the $O(4)$-symmetric field model in three dimensions have been obtained from resummation of the recent six-loops order of the corresponding perturbation series. The recent seven-loops order for the β-function of the ϕ^4_{3+1} field theory has been resummed which shows non-existence of fixed points. The first resummation result of the seven-loop series representing the fractal dimension of the two-dimensional self-avoiding polymer is presented here where we get a very accurate value of $d_f = 1.3307$ compared to its exact value ($4/3 \approx 1.3333$).

1. Introduction

In many situations in quantum field theory, perturbative calculations are producing divergent series with zero radius of convergence [1–6]. Being divergent, one can face a situation for which the predictions are not accurate even for small values of the argument. The point is that divergent series are possessing factorial growth factors and thus the ignored terms might contribute more than the ones taken into account. To get reliable results from divergent series, resummation techniques are introduced. The most famous one is Borel [3,1] resummation technique and its extension Borel-Padé [4]. Recently, a hypergeometric resummation technique has been introduced and applied to various examples [7–13]. Although it gives accurate predictions from resumming divergent series, the algorithm has some limitations [8,14]. As reported in Refs. [8,14], one might not be able to get aimed precision for small coupling values because of the use of hypergeometric function of finite radius of convergence ($_2F_1$) to resum a divergent series with zero radius of convergence. This issue has been solved (by the same authors) for $_2F_1$ resummation in Ref. [8] by brute-force disposition of the branch-cut (make it running from 0 to ∞). Another resummation algorithm (Borel-hypergeometric) has been employed in Ref. [7] too which has been extended to the Meijer-G approximant algorithm in Ref. [14]. In fact, the algorithm in Ref. [14] is shown to have precise predictions from relatively low orders of perturbation series. In Ref. [15], a closely related algorithm has been used where the authors match a series by a linear combination of asymptotic series of confluent hypergeometric functions. These algorithms can overcome the problem of precision at small coupling values. For instance, the series expansion of the used Meijer-G functions [14,16,17] has zero-radius of convergence while for the work in [15] they are matching a series with confluent hypergeometric functions which are in turn is equivalent to a series with zero radius of convergence.

The hypergeometric-Borel algorithm in Ref.[14] used Padé as well as Borel techniques to accomplish final approximants in terms of the Meijer-G function. To apply Borel transformation to a divergent series, one needs to know the large-order growth factor ($n!$ for instance) of the given perturbation series. As long as the large-order behavior is indispensable for the application of Borel transformation, one might wonder if the Borel transformation is really needed to achieve the Meijer-G function approximants. Besides, it is traditionally known that the incorporation of parameters from asymptotic behaviors (strong-coupling and large-order) of the perturbation series accelerates the convergence

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of resummation algorithms [4] and one might in a need to suggest a way to incorporate them in the Meijer-G function parametrization. In this work we aim to introduce a resummation algorithm that incorporates information from asymptotic behaviors without using Borel or padé techniques. The suggested algorithm has the same level of simplicity as the first algorithm in Ref. [7] (hypergeometric resummation one) but on the other hand can give precise results for the whole coupling space. By simple we mean no usage of Borel or Padé techniques but rather using hypergeometric functions that have zero-radius of convergence to approximate the given series and then resum them using a representation in terms of Mellin-Barnes integrals. The suggested algorithm will not only stress simplicity but also can guarantee faster convergence as it will be able to accommodate available information from asymptotic large-order and strong coupling data for the first time in such type of algorithms.

The key point to achieve our goal is to approximate the given divergent series by the set of hypergeometric functions ${}_pF_q$ which have zero-radius of convergence ($p \geqslant q+2$) [18]. Note that ${}_pF_{p-1}$ approximants with finite radius of convergence are still suitable in resumming divergent series of finite radius of convergence like the strong coupling expansion series of the Yang-Lee model in Ref. [2]. However, when the series under consideration has a zero radius of convergence, it would be more suitable to use the ${}_pF_q$ series with $p \geqslant q+2$ to approximate the divergent series under investigation. For $p \geqslant q+2$, the series expansion of ${}_pF_q$ is divergent but it can be analytically continued via use of the Meijer-G function [18] where we have the representation:

$$ {}_pF_q\left(a_1,\ldots a_p;b_1\ldots b_q;z\right) = \frac{\prod_{k=1}^{q}\Gamma(b_k)}{\prod_{k=1}^{p}\Gamma(a_k)} G^{1,p}_{p,q+1}\left(\begin{matrix}1-a_1,\ldots,1-a_p\\ 0,1-b_1,\ldots,1-b_q\end{matrix}\middle| z\right). \tag{1}$$

The Meijer-G function, on the other hand, has the integral representation of the form [18]:

$$ G^{m,n}_{p,q}\left(\begin{matrix}c_1,\ldots,c_p\\ d_1,\ldots,d_q\end{matrix}\middle| z\right) = \frac{1}{2\pi i}\int_C \frac{\prod_{k=1}^{n}\Gamma(s-c_k+1)\prod_{k=1}^{m}\Gamma(d_k-s)}{\prod_{k=n+1}^{p}\Gamma(-s+c_k)\prod_{k=m+1}^{q}\Gamma(s-d_k+1)} z^s ds. \tag{2}$$

A suitable choice of the contour C enables one to get an analytic continuation for ${}_pF_q(a_1,\ldots a_p;b_1\ldots b_q;z)$. For instance when C is taken from $-i\infty$ to $+i\infty$ [18], the integral above converges for $p+q<2(m+n)$. For reasons that will be clearer later, we are interested in the functions ${}_pF_{p-2}(a_1,\ldots a_p;b_1\ldots b_{p-2};z)$ in our work. So in using Eq. (2), we have and thus we have $p+q=2p-2$ which is smaller than $2(m+n)=2p+2$. So the resummation of the series of ${}_pF_{p-2}(a_1,\ldots a_p;b_1\ldots b_{p-2};z)$ is possible. Although here no Borel transform is used, the Mellin-Barnes transform defining the G-function might suffer from Stokes phenomena [19] which is then equivalent to Non-Borel summability. There exits algorithms in literature [19] to smooth them out but it is out of the scope of this work. Instead when facing such problems, we will apply the hypergeometric-Meijer resummation algorithm (introduced in this work) to resum the resurgent transseries [14,19–23] associated with that problem. The example of the resummation of the non-Borel summable series representing the zero-dimensional partition function of the degenerate-vcua ϕ^4 scalar field theory will be given.

The structure of this paper will be as follows. In Section 2, we stress the strong-coupling and the large-order asymptotic behaviors of the expansion of the hypergeometric function ${}_pF_{p-2}(a_1,\ldots a_p;b_1\ldots b_{p-2};z)$. In Section 3, the hypergeometric-Meijer resummation algorithm is presented. Resummation of the divergent series of the zero-dimensional partition function of the single-vacuum (Borel-summable) and the double vacua (non-Borel summable) ϕ^4 theory is presented in Section 4. In Section 5 and Section 6, we apply the resummation algorithm to the series of vacuum energies of the ϕ^4_{0+1} and the $\mathscr{PT}$-symmetric $i\phi^3_{0+1}$ field theories. The resummation results for the recent six-loops order of the renormalization group functions of the $O(4)$-symmetric quantum field model in three dimensions is introduced in Section 7 while the application of the algorithm to resum the recent seven-loops order of the β-function of the ϕ^4_{3+1} theory is included in Section 8. In Section 9, we present the first resummation result of the seven-loop (ε-expansion) for the fractal dimension of the self-avoiding polymer. Summary and conclusions will follow in Section 10.

2. Large-order and strong-coupling asymptotic behaviors of the hypergeometric ${}_pF_{p-2}$

We mentioned above that toward the resummation of a divergent series with zero radius of convergence, the functions ${}_pF_{p-2}(a_1,\ldots a_p;b_1\ldots b_q;z)$ are suitable when the weak-coupling information are available up to some order. It is well known that employing strong-coupling as well as large-order data can accelerate the convergence of a resummation technique [4]. Now we need to show that the set of ${}_pF_{p-2}(a_1,\ldots a_p;b_1\ldots b_{p-2};z)$ functions are able to incorporate both strong-coupling as well as the large-order data of the perturbation series to be resummed. To do that, consider the divergent series of the expansion of a physical quantity $Q(g)$ such that:

$$ Q(g) = \sum_{n=0}^{\infty}\beta_n g^n. \tag{3}$$

In fact, for divergent series of the renormalization group functions in quantum field theory (for instance), the large-order asymptotic behavior of the perturbation series takes the form: [4]

$$ \beta_n \sim \gamma n!(-\sigma)^n n^b\left(1+O\left(\frac{1}{n}\right)\right),\ n\to\infty. \tag{4}$$

For the resummation of a divergent series that has such kind of large order behavior, we suggest the use of a hypergeometric function ${}_pF_q$ with a constraint on the relation between the number p of numerator parameters and the number q of denominator parameters such that it can reproduce the above large order behavior. To elucidate that point, consider the series expansion of the hypergeometric function ${}_pF_q$ of the form:

$$ \begin{aligned} {}_pF_q(a_1,\ldots\ldots a_p;b_1,\ldots\ldots b_q;-\sigma g) &= \sum_{n=0}^{\infty}\frac{\frac{\Gamma(a_1+n)}{\Gamma(a_1)}\cdots\frac{\Gamma(a_p+n)}{\Gamma(a_p)}}{n!\,\frac{\Gamma(b_1+n)}{\Gamma(b_1)}\cdots\frac{\Gamma(b_q+n)}{\Gamma(b_q)}}(-\sigma g)^n,\\ &= \alpha\sum_{n=0}^{\infty}\frac{\Gamma(a_1+n)\ldots\Gamma(a_p+n)}{\Gamma(n+1))\Gamma(b_1+n)\ldots\Gamma(b_q+n)}(-\sigma g)^n,\\ &= \alpha\sum_{n=0}^{\infty}c_n g^n \end{aligned} \tag{5}$$

where

$$ \alpha = \frac{\prod_{i=1}^{q}\Gamma(b_i)}{\prod_{i=1}^{p}\Gamma(a_i)}. \tag{6}$$

For large n, the asymptotic form of a ratio of two Γ functions is given by [29]:

$$ \frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} = n^{\alpha-\beta}\left(1+\frac{(\alpha-\beta)(-1+\alpha+\beta)}{n}+O\left(\frac{1}{n^2}\right)\right). \tag{7}$$

For the lowest order approximant (${}_2F_0(a_1,a_2;-\sigma x)$ we have $p=2$, $q=0$ and thus there are two Γ-functions in the numerator and one Γ-function in the denominator (coming from $n!=\Gamma(n+1)$). For that case and for $n\to\infty$ we have:

$$\frac{\Gamma(a_1+n)\Gamma(a_2+n)}{\Gamma(n+1)} = \frac{\Gamma(a_1+n)\Gamma(a_2+n)}{\Gamma(n+1)\Gamma(n+1)}\Gamma(n+1) \sim n!n^{a_1+a_2-2}\left(1+O\left(\frac{1}{n}\right)\right), \quad (8)$$

which resembles the large-order behavior of the given perturbation series in Eq. (4). To show that such large order behavior persists for any higher order approximant, we consider the function ${}_pF_{p-2}(a_1,...,a_p;b_1,...,b_{p-2};-\sigma x)$ which has p number of numerator Γ-functions and $p-1$ of denominator Γ-functions. Accordingly, the asymptotic behavior for large n takes the form:

$$\frac{\Gamma(a_1+n)\Gamma(a_2+n)\Gamma(a_3+n)...\Gamma(a_p+n)}{\Gamma(n+1)\Gamma(b_1+n)\Gamma(b_2+n)...\Gamma(b_{p-2}+n)} \sim n!n^{(a_1+a_2+...a_p)-(b_1-b_2...b_{p-2})-2}\left(1+O\left(\frac{1}{n}\right)\right). \quad (9)$$

Thus the large-order behavior in Eq. (4) can be reproduced from this form. To obtain the above large-order behavior, we used hypergeometric approximants for which $p=q+2$. In fact, any other relation between p and q can not account for the $n!$ growth factor in the large-order behavior of the given divergent series. Knowing this, the large-order information in Eq. (4) thus sets the constraint:

$$\sum_{i=1}^{p} a_i - \sum_{i=1}^{p-2} b_i - 2 = b, \quad (10)$$

on the numerator parameters (a_i) and the denominator parameters () of the hypergeometric approximant ${}_pF_{p-2}(a_1, ...a_p;b_1, ...b_{p-2};-\sigma g)$. In view of the above analysis, we conclude that out of the hypergeometric functions ${}_pF_q(a_1, ...a_p;b_1, ...b_q;-\sigma g)$, the suitable candidate to represent the perturbation series in Eq. (3) with the large order behavior in Eq. (4) is the function ${}_pF_{p-2}(a_1, ...a_p;b_1, ...b_{p-2};-\sigma g)$.

The strong coupling expansion of a physical quantity can also be obtained (for quantum field theory, it can only be obtained for some cases) using methods in Refs. [24,25]. The a_i parameters in the function ${}_pF_{p-2}(a_1, ...a_p;b_1, ...b_{p-2};-\sigma g)$are totally determined from powers in the strong coupling expansion. For non-integer $a_i - a_j$, the hypergeometric function has the strong coupling expansion in the form [26]:

$${}_pF_{p-2}(a_1,...,a_p;b_1,...,b_{p-2};z) \propto \sum_{k=1}^{p} d_k(-z)^{-a_k}\left(1+O\left(\frac{1}{z}\right)\right) \quad (11)$$

From this expansion one concludes that the numerator parameters a_i can be obtained from the strong coupling asymptotic behavior of the perturbation series. So one can get the whole set of parameters in ${}_pF_{p-2}(a_1, ...a_p;b_1, ...b_{p-2};-\sigma g)$ from the available orders of the perturbation series (weak-coupling), large-order and strong-coupling information.

Sometimes one can find cases for which the differences between numerator parameters $(a_i - a_j)$ are integers and thus one might conclude that the strong-coupling asymptotic behavior of the given series can not be reproduced by any parametrization of the hypergeometric approximants. However, if the strong-coupling expansion of the approximant is alternating in sign, one can still extract the values of the parameters a_1, a_2......and a_p from the powers in the strong coupling expansion of the given series. We shall stress this point in Section 5 when studying the resummation of the ground state energy of the ϕ^4 theory in 0 + 1 dimensions.

3. The hypergeometric-Meijer resummation algorithm

For a divergent series that has a large-order $n!$ growth factor, the hypergeometric-Meijer resummation algorithm follows the following steps:

1. **Matching the given perturbation series with the series expansion of the hypergeometric approximant ${}_pF_{p-2}(a_1, ...a_p;b, ...b_{p-2};-\sigma g)$:**

(a) **Weak-coupling information as input**: In case we have only week coupling information, all the parameters in ${}_pF_{p-2}(a_1, ...a_p;b, ...b_{p-2};-\sigma g)$ are obtained by matching the expansion of ${}_pF_{p-2}$ with the available perturbative terms in the perturbation series in Eq. (3). Since there is an odd number $(2p-1)$ of unknown parameters, so one needs an odd number of terms from the given perturbation series. For the even number resummation, however, one can resum the once subtracted series $(Q(g)-\beta_\circ)/g$ instead of the original one [33]. As an example for the resummation of a series up to an odd order is the third order approximant ${}_2F_0(a_1, a_2;;-\sigma g)$ where the matching will lead to the result:

$$\begin{aligned} -a_1a_2\sigma &= \beta_1 \\ \tfrac{1}{2}a_1(1+a_1)a_2(1+a_2)\sigma^2 &= \beta_2 \\ -\tfrac{1}{6}a_1(1+a_1)(2+a_1)a_2(1+a_2)(2+a_2)\sigma^3 &= \beta_3 \end{aligned} \quad (12)$$

Solving these equations, the three parameters a_1, a_2 and σ are fully determined. Also, the fifth order hypergeometric approximant is ${}_3F_1(a_1, a_2, a_3; b_1;-\sigma g)$ while in using weak-coupling information up $O(g^M)$, one uses the approximant ${}_pF_{p-2}(a_1, ...a_p;b, ...b_{p-2};-\sigma g)$ which has $M=2p-1$ unknown parameters.

(b) **Weak-coupling and Large-order information as input**: If the available information about the given series include weak-coupling as well as large order information (but not strong-coupling), one can accelerate the convergence of the algorithm by using the large-order information. To illustrate this, consider for simplicity the approximant ${}_3F_1(a_1, a_2, a_3;b_1;-\sigma g)$ which needs three orders from the perturbation series represented by the coefficients β_1, β_2 and β_3 above. In this case, we have only four unknowns (a_1, a_2, a_3, b_1) but the needed fourth equation can be offered by the constraint:

$$\sum_{i=1}^{p} a_i - \sum_{i=1}^{p-2} b_i - 2 = b. \quad (13)$$

Accordingly, the large-order information lowers the previously fifth order parametrization of ${}_3F_1(a_1, a_2, a_3;b_1;-\sigma g)$ to a third order one. A note to be mentioned on using low-order approximants is that, except for rare cases, one usually can not extract good approximations from just first order of perturbation series as input and of course good approximations are always expected for second, third and higher orders. In general, in employing the large-order information, the number of unknowns is reduced by two and thus one can use the approximant ${}_{p+1}F_{p-1}(a_1, ...a_{p+1};b, ...b_{p-1};-\sigma g)$ to represent the up to $O(g^{(2p-2)})$ order of perturbation series.

(c) **Weak-coupling, strong-coupling and Large-order information as input**: In case we know the strong-coupling information, then the approximant ${}_pF_{p-2}(a_1, ...a_p;b, ...b_{p-2};-\sigma g)$ includes only $p-2$ unknowns since all numerator parameters $(a_1, ...a_p)$ are known from strong-coupling behavior while the parameter σ is known from the large order behavior. In this case, one needs only $(p-3)$ orders from perturbation series because of the constraint in Eq. (10). However, for the lowest order approximant ${}_2F_0(a_1, a_2;;-\sigma g)$ there will be redundancy as one can get all the parameters without using the condition in Eq. (10). In that case the lowest order approximant could be ${}_3F_1(a_1, a_2, a_3;b_1;-\sigma g)$ where in this case the given information can mach the number of unknown parameters. But for low order approximants like this one, it is not recommended to employ the condition in Eq. (10) as the large-order matching between the given series and the used approximant is correct up to $O(\frac{1}{n})$ which means that it will be more accurate for relatively high orders.

In case the approximant is exact, the constraint in Eq. (10) is satisfied automatically even for the lowest order approximant as we will see in the zero-dimensional partition function example to be studied in this work.

2. **Hypergeometric to Meijer-G approximants**: We use the representation of the hypergeometric function ${}_pF_{p-2}(a_1, \ldots\ldots a_p; b, \ldots\ldots\ldots b_{p-2}; -\sigma g)$ in terms of the Meijer-G function in Eq. (1) to get a convergent result out of the divergent series for ${}_pF_{p-2}$. For instance, the hypergeometric approximant ${}_3F_1$ is represented as:

$$ {}_3F_1\left(a_1, a_2, a_3; b_1; -\sigma z\right) = \frac{\Gamma(b_1)}{\prod_{k=1}^{3}\Gamma(a_k)} G_{3,2}^{1,3}\left(\begin{matrix} 1-a_1, \ldots, 1-a_3 \\ 0, 1-b_1 \end{matrix} \middle| -\sigma z\right). \tag{14} $$

3.1. Successive approximants of the algorithm

3.1.1. Weak-coupling parametrization

In case we have only weak-coupling information from the perturbation series as:

$$ Q(g) = \sum_{n=0}^{2p-2} \beta_n g^n + O(g^{2p-1}). \tag{15} $$

1. **Odd orders**: The successive odd orders approximants can be parametrized as follows:
   i) The third order approximant is taken as $\beta_0({}_2F_0(a_1, a_2;; -\sigma g))$where the parameters a_1, a_2, σ are obtained from the weak-coupling information as shown in Eq. (12).
   ii) The fifth order approximant is represented by $\beta_0({}_3F_1(a_1, a_2; b_1; -\sigma g))$and the five parameters are obtained all from the weak-coupling information.
   iii) The generalized odd $(2p-1)$ order is represented by the approximant $\beta_0({}_pF_{p-2}(a_1, \ldots\ldots a_p; b_1, \ldots\ldots\ldots b_{p-2}; -\sigma g))$.
2. **Even orders**: In case we have even terms from the perturbation series, we resum the subtracted series:

$$ Q_1(g) = \frac{Q(g)-\beta_0}{g}, \tag{16} $$

where in this case the fourth order approximant for $Q_1(g)$is represented by $\beta_1({}_2F_0(a_1, a_2;; -\sigma g))$while the $2p$ order is represented by the approximant $\beta_1({}_pF_{p-2}(a_1, \ldots\ldots a_p; b_1, \ldots\ldots\ldots b_{p-2}; -\sigma g))$.

3.1.2. Weak-coupling and large order parametrization

In this case the even order approximants are represented by:

1) The second order is represented by $\beta_0({}_2F_0(a_1, a_2;; -\sigma g))$ (σ is known from large-order asymptotic behavior.
2) The fourth order approximant is represented by $\beta_0({}_3F_1(a_1, a_2, a_3; b_1; -\sigma g))$
3) the $2p-2$ order approximant is represented by $\beta_0(pF_{p-2}(a_1, \ldots\ldots a_p; b_1, \ldots\ldots\ldots b_{p-2}; -\sigma g))$

For odd orders, however, we employ the constraint in Eq. (10) to obtain an even number of equations to solve for the $2p-2$ unknown parameters.

3.1.3. Strong-coupling, weak-coupling and large order parametrization

In this case, the numerator parameters as well as the large-order parameter σ are all known. Accordingly, the successive approximants are generated as follows:

1) The first order approximant is represented by $\beta_0({}_3F_1(a_1, a_2, a_3; b_1; -\sigma g))$ (b_1 is the only unknown parameter and we use only the first two orders ($\beta_0 + \beta_1 g$) from the perturbation series as input). Note that for low orders, we do not employ the constraint in Eq. (10) for the reasons mentioned above.
2) The second order approximant is represented by $\beta_0({}_4F_2(a_1, a_2, a_3, a_4; b_1, b_2; -\sigma g))$
3) The $p-2$ even order approximant is represented by $\beta_0({}_pF_{p-2}(a_1, \ldots\ldots a_p; b_1, \ldots\ldots\ldots b_{p-2}; -\sigma g))$
4) In case $p-2$ is an odd number but relatively high, the approximant is also $\beta_0({}_pF_{p-2}(a_1, \ldots\ldots a_p; b_1, \ldots\ldots\ldots b_{p-2}; -\sigma g))$ but in this case we use the constraint in Eq. (10).

4. Hypergeometric-Meijer resummation of the zero-dimensional partition function of the ϕ^4 scalar field theory

In this section, we give two examples for resummation of the partition function of ϕ^4 theory in zero dimension. In both cases, the partition function has a divergent series expansion with $n!$ growth factor. The first case is the single vacuum theory where the series is Borel summable and no complex ambiguity exists. The second example is the partition function of a double-vacua ϕ^4 theory where the series is non-Borel summable and thus one resorts to the resummation of the resurgent transseries.

4.1. Single-vacuum ϕ^4 scalar field theory

An example of a divergent series with zero radius of convergence that is always used to test the success of a resummation algorithm is the partition function of zero-dimensional ϕ^4 theory. Let us apply the algorithm here to resum the associated divergent perturbation series. We shall apply the algorithm three times for the same problem, one using weak-coupling information only, another by adding the large-order information and finally by adding strong-coupling information. The reason behind using that recipe is to test the validity of the new constraint set on the parameters in Eq. (10) using an exact resummation result. To do that, consider the partition function of that model given by:

$$ Z = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\phi \exp\left(-\frac{\phi^2}{2} - \frac{g}{4!}\phi^4\right), \tag{17} $$

with the associated weak-coupling perturbation series is of the form:

$$ Z(g) = 1 - \frac{g}{8} + \frac{35}{384}g^2 - \frac{385}{3072}g^3 + \frac{25025g^4}{98304} - \frac{1616615g^5}{2359296} + O(g^6). \tag{18} $$

In fact, the lowest order hypergeometric approximant is ${}_2F_0(a_1, a_2;; -\sigma g)$ with only three unknown parameters. To determine the parameters a_1, a_2 and σ we use Eq. (12) with the corresponding β_i coefficients:

$$ \begin{aligned} &-a_1 a_2 \sigma = -\tfrac{1}{8} \\ &\tfrac{1}{2} a_1(1+a_1)a_2(1+a_2)\sigma^2 = \tfrac{35}{384} \\ &-\frac{1}{6} a_1(1+a_1)(2+a_1)a_2(1+a_2)(2+a_2)\sigma^3 = -\frac{385}{3072}. \end{aligned} \tag{19} $$

The solution of these equations are given by: $a_1 = \frac{1}{4}$, $a_2 = \frac{3}{4}$ and $\sigma = \frac{2}{3}$. Accordingly, the hypergeometric-Meijer approximant of $Z(g)$is

$$ \begin{aligned} Z(g) &= {}_2F_0\left(\tfrac{1}{4}, \tfrac{3}{4};; -\tfrac{2}{3}g\right) \\ &= \frac{1}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} G_{2,1}^{1,2}\left(\begin{matrix} \frac{3}{4}, \frac{1}{4} \\ 0 \end{matrix} \middle| -\tfrac{2}{3}g\right) \end{aligned} \tag{20} $$

In using the identity (see Eq. (9) in Section 5.3.1 and Eq. (7) in Section 5.6 of Ref. [18])

$$ {}_2F_0\left(-n,\, n+1,\, x\right) = \frac{1}{\sqrt{\pi}}\sqrt{\frac{-1}{x}}\exp\left(-\frac{1}{2x}\right)K_{n+\frac{1}{2}}\left(-\frac{1}{2x}\right), \tag{21} $$

we get the exact result reported in Ref. [14] but it has been obtained there at the fifth order while we obtained it from knowing only the first three terms of the week coupling expansion. Note that the fifth order approximant ${}_3F_1(a_1, a_2, a_3;b_1;-\sigma g)$ is parametrized as:

$$ Z(g) = {}_3F_1\left(\frac{1}{4},\, \frac{3}{4},\, a;a;-\frac{2}{3}g\right) = {}_2F_0\left(\frac{1}{4},\, \frac{3}{4};;-\frac{2}{3}g\right). \tag{22} $$

Like wise the seventh order approximant reduces to the lowest order one which reflects the convergence of the algorithm.

One can even accelerate the convergence to the exact result by using the large-order information. The large-order behavior for the series $Z(g)$ can also be obtained as $n!n^{-1}\left(\frac{-2}{3}g\right)^n\left(1+O\left(\frac{1}{n}\right)\right)$ for $n\to\infty$. Accordingly, we have $\sigma=\frac{2}{3}$. For the parameters a_1 and a_2 , we use one equation from matching the weak-coupling expansion with expansion of ${}_2F_0(a_1, a_2;;-\sigma g)$ to get:

$$ \frac{-2}{3}a_1a_2 = -\frac{1}{8}, \tag{23} $$

while the other equation from matching the large-order behavior in Eq. (10):

$$ a_1 + a_2 - 2 = -1. \tag{24} $$

Solving these equations one gets: $a_1=\frac{3}{4}$ and $a_2=\frac{1}{4}$. So in using the large-order data, the exact result has been obtained from first order in perturbation series. This result assures the validity of the new constraint obtained in this work (Eq. (10)). Also, higher order approximants reduces to the lowest order approximant ${}_2F_0(a_1, a_2;;-\sigma g)$.

One can also make the convergence even faster in case we know also the strong-coupling information. The strong coupling expansion of the integral in Eq. (17) can be obtained as:

$$ Z(g) = \frac{\sqrt[4]{\frac{24}{16}}\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}g^{-\frac{1}{4}} - \sqrt[4]{\frac{3}{2}}\frac{\Gamma\left(\frac{3}{4}\right)}{\sqrt{\frac{\pi}{3}}}g^{-\frac{3}{4}} + O\left(g^{-\frac{5}{4}}\right). \tag{25} $$

When a_1-a_2 is not an integer, the asymptotic behavior of ${}_2F_0(a_1, a_2;;-\sigma g)$ for large g values takes the form:

$$ {}_2F_0(a_1, a_2;;\sigma g) \sim c_1g^{-a_1} + c_2g^{-a_2}. \tag{26} $$

Accordingly, we get $a_1=\frac{3}{4}$ and $a_2=\frac{1}{4}$. In other words we know σ from large-order data and $a_1$and a_2 from strong-coupling data. Thus the exact partition function has been obtained from the knowledge of the large-order and strong-coupling information only (no week-coupling data needed). The higher orders approximants like ${}_4F_2(a_1, a_2, a_3, a_4;b_1, b_2;-\sigma g)$ do involve weak-coupling, strong-coupling as well as large order information but they again reduce to the lowest order approximant or equivalently:

$$ {}_2F_0\left(\tfrac{1}{4}, \tfrac{3}{4};;-\tfrac{2}{3}g\right) = {}_3F_1\left(\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{5}{4};\tfrac{5}{4};-\tfrac{2}{3}g\right) = {}_4F_2\left(\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{5}{4}, \tfrac{7}{4};\tfrac{5}{4}, \tfrac{7}{4};-\tfrac{2}{3}g\right) \tag{27} $$

4.2. Double-vacua ϕ^4 theory

In some cases, the perturbation series is not Borel-summable and the Borel-summation of the series leads to complex ambiguities [20]. An example of such kind of perturbation series is the one associated with the integral representing the zero-dimensional partition function of the degenerate-vacua ϕ^4 theory [14,19,21]:

$$ Z = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\phi\exp\left(-\frac{\phi^2}{2}(1-\sqrt{g}\phi^2)^2\right), \tag{28} $$

where it has an expansion of the from:

$$ Z(g) \sim 1 + 6g + 210g^2 + 13860g^3 + O(g^4). \tag{29} $$

It is clear that this series is not Borel-summable and the Borel sum will result in a complex ambiguity. The reason behind that is the existence of singular points on the contour used in the Borel transform which results in the existence of Stokes phenomena. A similar situation can exist for the Meijer-G resummation since the Meijer-G function is defined through a Mellin-Barnes integrals where Stokes phenomena can exist too [21]. In such case a resurgent transseries can be obtained that can account for non-perturbative contributions for small coupling values associated with the expansion around the non-perturbative saddle point [20,19,21]. The transseries for the zero-dimensional partition function of the degenerate-vacua ϕ^4 theory has been reported in Ref. [14] as:

$$ \begin{aligned} Z(g) &= \pm i\sqrt{2}\exp\left(\frac{-1}{32g}\right)(1-6g+210g^2-13860g^3+O(g^4)) + 2 \\ &(1+6g+210g^2+13860g^3+O(g^4)), \end{aligned} \tag{30} $$

where the $+$ sign for $Im(g)>0$ and $-$ sign for $Im(g)<0$. This transseries has in fact incorporated the contributions from the Gaussian saddle point and the instanton saddle point [22]. The two separate series in the transseries above can be resummed using the hypergeometric-Meijer resummation followed in this work and the exact result is obtained at the third order where we have:

$$ \begin{aligned} Z(g) = &\frac{2}{\prod_{k=1}^{2}\Gamma(a_k)}G_{2,1}^{1,2}\left(\begin{matrix}1-a_1, 1-a_2\\ 0\end{matrix}\middle| -32g\right) \\ &+ \frac{\pm i\sqrt{2}e^{-\frac{1}{32}/g}}{\prod_{k=1}^{2}\Gamma(a_k)}G_{2,1}^{1,2}\left(\begin{matrix}1-a_1, 1-a_2\\ 0\end{matrix}\middle| 32g\right), \end{aligned} \tag{31} $$

with $a_1=\frac{1}{4}$ and $a_2=\frac{3}{4}$. Note that this result is real and exact. In Ref. [31], the exact result is listed as (for $Re(g)>0$):

$$ Z(g) = \frac{e^{-\frac{1}{64}/g}D_{-\frac{1}{2}}\left(-\frac{1}{4\sqrt{g}}\right)}{\sqrt{2}\sqrt[4]{g}}, \tag{32} $$

where $D_\nu(z)$ is the parabolic cylinder function which is equivalent to the result we obtained.

5. Resummation of the vacuum energy perturbation series of the ϕ^4_{0+1} Scalar field theory

As another testing example, we apply the algorithm to resum the ground-state energy of the anharmonic oscillator where it is equivalent to the scalar ϕ^4 theory in $0+1$ space-time dimensions. We shall resum the same series using two different parametrizations. The first parametrization is using weak-coupling, large-order and strong-coupling data. In the second parametrization, we use weak-coupling and large-order data while the strong-coupling parameters are extracted from the approximant. Up to the best of our knowledge, a closed form strong-coupling asymptotic behavior has not been obtained yet even for simple quantum field theories like the ϕ^4-scalar field theory in space-time dimensions higher than $0+1$. Accordingly, the second parametrization is very important in obtaining the asymptotic strong-coupling behavior in quantum field theories where other resummation algorithms can give different results for the same problem [37–40].

5.1. Weak-coupling, large-order and strong-coupling parametrization of resummation approximants for ϕ^4_{0+1} vacuum energy

The Hamiltonian density for this example is given by:

$$ H = \frac{\pi^2}{2} + \frac{m}{2}\phi^2 + \frac{g}{4}\phi^4. \tag{33} $$

In 0 + 1 space-time dimensions and for $m = 1$, the perturbation series of the ground state energy has the form [27]:

$$E_0 = \frac{1}{2} + \frac{3}{4}g - \frac{21}{8}g^2 + \frac{333}{16}g^3 - \frac{30885}{128}g^4 + \frac{916731}{256}g^5 + O(g^6). \quad (34)$$

The large order behavior is given also by $-(-3)^n\sqrt{\frac{6}{\pi^3}}\Gamma\left(n+\frac{1}{2}\right)$ [27]. It is clear here that the parameter σ is then given by $\sigma = 3$. A scaling operation can lead to the strong coupling expansion [6] from which one can extract a_i as:

$$a_1 = -\frac{1}{3},\ a_2 = \frac{1}{3},\ a_3 = 1,\ a_4 = \frac{5}{3},\ ...... \quad (35)$$

For the approximants ${}_2F_0$ and ${}_3F_1$, the difference between any two numerator parameters $(a_i - a_j)$ is not an integer and thus the numerator parameters lead to the well-known strong coupling asymptotic behavior [26]. For all higher orders approximants (${}_4F_2$, ${}_5F_3$ ....), however, $a_i - a_j$ has the possibility to take integer values and thus lead to logarithmic factors in the strong-coupling asymptotic behavior[26] which does not mach with the known strong-coupling expansion of the given series. In fact, the logarithmic factors in the strong coupling asymptotic behavior are multiplied by g^{-a_4} and g^{-a_5} for ${}_5F_3$ (for instance) which means that such terms will be led by the power behaviors g^{-a_4} and g^{-a_5} while the logarithmic factors have minor effect at large g. This means that although for some approximants, $a_i - a_j$ have the possibility to be integers, one can still consider the numerator parameters matching the exact ones known from strong-coupling expansion of the given series. To test these expectations, we parametrized the approximant ${}_5F_3$ in two ways, one by setting a_1, a_2, a_3, a_4 and a_5 to equal the known exact ones while for the other parametrization we take a_1, a_2, a_3 from known exact ones while predicting a_4 and a_5 by considering more terms from the weak-coupling data. We found only marginal differences between the predictions of the two parametrization (Fig. 1). For more clarification of how it is important to incorporate the strong-coupling parameters, we generated the data in Table 1. From that table, it is clear that the fourth order approximant ${}_6F_4$ for which all numerator parameters are set according to the known strong-coupling behavior gives more accurate results than the fourth order approximant ${}_5F_3$ for which one of the numerator parameters is taken as an unknown.

A concrete advocate of the irrelevance of existing singular coefficients in the strong coupling expansion of the hypergeometric approximants can be introduced by more deep analysis of its properties. For

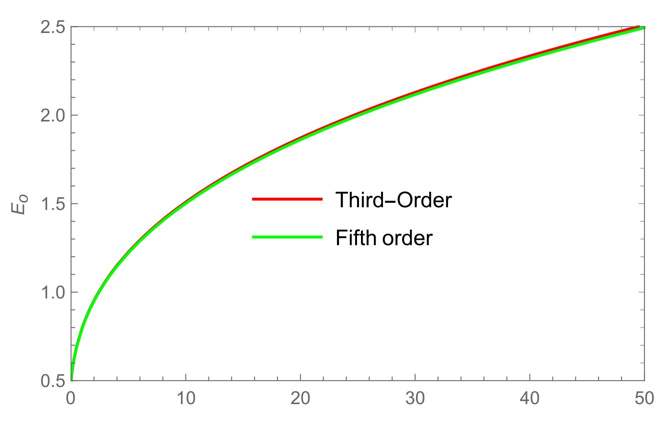

Fig. 1. Comparison between resummation of the ground state energy for the ϕ^4_{0+1} theory using two different parametrization of the ${}_5F_3$ approximant. In the third order parametrization (red color) we set all values of numerator parameters to be $-\frac{1}{3}, \frac{1}{3}, 1, \frac{5}{3}$ and $\frac{7}{3}$ from the known strong coupling expansion of the theory. For the other parametrization (fifth order, green color), we take $a_1 = -\frac{1}{3}$, $a_2 = \frac{1}{3}$, $a_3 = 1$ while a_4 and a_5 are obtained by considering two more orders from perturbation series.

Table 1
The predictions of the fourth order approximants of the hypergeometric-Meijer resummation for the ground state Energy in Eq. (34) compared to the exact results from Ref. [30]. For the fourth order approximant ${}_6F_4$, all numerator parameters are set from the strong-coupling behavior. In the other fourth order approximant (${}_5F_3$), we set the first four numerator parameters from the strong coupling behavior while the last numerator parameter as well as the three denominator parameters are determined using weak-coupling information.

g	$({}_5F_3)_{4th}$	$({}_6F_4)_{4th}$	Exact
0.5	0.696872	0.696614	0.696176
1	0.806154	0.805178	0.803771
50	2.61212	2.54504	2.49971
1000	7.18095	6.85807	6.69422
20000	19.6354	18.6059	18.1372

the given series, the strong coupling behavior of the approximant $\frac{{}_5F_3}{2}$ (for instance) is given by:

$$\frac{{}_5F_3}{2} \propto g^{-a_1}\left(c_1 + c_2 g^{a_1-a_2} + c_3 g^{a_1-a_3} + c_4 g^{a_1-a_4} + c_5 g^{a_1-a_5}\right) \quad (36)$$

Here c_1, c_2 and c_3 are finite butc$_4$ and c_5 are singular. Let us write them explicitly:

$$c_4 = \frac{(3)^{-a_4}\Gamma(b_3)\Gamma(b_2)\Gamma(b_1)\Gamma(a_2-a_4)\Gamma(a_3-a_4)\Gamma(a_5-a_4)\Gamma(a_1-a_4)}{2\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_5)\Gamma(b_1-a_4)\Gamma(b_2-a_4)\Gamma(b_3-a_4)}$$
$$c_5 = \frac{(3)^{-a_5}\Gamma(b_3)\Gamma(b_2)\Gamma(b_1)\Gamma(a_2-a_5)\Gamma(a_3-a_5)\Gamma(a_4-a_5)\Gamma(a_1-a_5)}{2\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)\Gamma(b_1-a_5)\Gamma(b_2-a_5)\Gamma(b_3-a_5)} \quad (37)$$

Clearly these coefficients are singular but also their factors of the singular terms having opposite signs. One thus can hope to regularize the fourth and the fifth terms in the strong-coupling expansion. To do that let us substitute the given parameters and add a fictitious variable ϵ in the singular terms and then take the limit as $\epsilon \to 0$. In this case we have:

$$c_4 g^{-5/3} + c_5 g^{-7/3} = \frac{(0.0210944)\Gamma(\epsilon-2)}{g^{5/3}} - \frac{(0.0564589)\Gamma(\epsilon-2)}{g^{7/3}}$$
$$= \frac{1}{\epsilon}\left(\frac{1.\ 0547\times10^{-2}}{g^{\frac{5}{3}}} - \frac{2.\ 8229\times10^{-2}}{g^{\frac{7}{3}}}\right) + \left(\frac{9.\ 7328\times10^{-3}}{g^{\frac{5}{3}}} - \frac{2.\ 6050\times10^{-2}}{g^{\frac{7}{3}}}\right) + O(\epsilon) \quad (38)$$

Now taking the limit as $(g, \epsilon) \to (\infty, 0)$, we get

$$c_4 g^{5/3} + c_5 g^{7/3} = \left(\frac{9.7328\times10^{-3}}{g^{\frac{5}{3}}} - \frac{2.6050\times10^{-2}}{g^{\frac{7}{3}}}\right) \quad (39)$$

In this case the strong-coupling approximation for the ground state energy then takes the form:

$$E_0(g) \approx 0.66649 g^{\frac{1}{3}} + \frac{0.13506}{g^{1/3}} - \frac{0.072736}{g} + \frac{0.00973}{g^{\frac{5}{3}}} \quad (40)$$

One can test the validity of this result by taking $g = 50$ to get $E_0(50) \approx 2.\ 4906$ while the full ${}_5F_3$ approximant gives $E_0 = 2.4936141$ compared to exact result as $E_0 = 2.49971$.

In incorporating the numerator parameters from the known strong-coupling behavior and in going from low-order approximants to higher orders, the convergence is improved in a systematic way. The predictions from successive approximants are shown in Table 2 which clearly reflects the improvement of the results as we increase the order. In the following, we will present the details about ${}_8F_6$ approximant only (sixth order in using weak-coupling, large-order and strong coupling data). The b_i-parameters in the ${}_8F_6$ function can then be obtained from matching the coefficients of the series expansion of ${}_8F_6$ term by term with the coefficients in the perturbation series in Eq. (34). We obtained the following values for the parameters b_i:

Table 2
The predictions of the first, second,⋯and sixth order of approximants of the hypergeometric-Meijer resummation for the ground state Energy in Eq. (34) compared to the exact results from Ref. [30].

g	${}_3F_1$	${}_4F_2$	${}_5F_3$	${}_6F_4$	${}_7F_5$	${}_8F_6$	Exact
0.5	0.728373	0.692206	0.695864	0.696614	0.696026	0.696120	0.696176
1	0.864509	0.794639	0.803068	0.805178	0.803157	0.803716	0.803771
50	3.04726	2.37129	2.49361	2.54504	2.46220	2.50062	2.49971
1000	8.33438	6.28700	6.67868	6.85807	6.54297	6.70275	6.69422
20000	22.66	17.0027	18.0962	18.6059	17.6947	18.1657	18.1372

$$b_1 = 0.448491,\ \ b_2 = 1.02679 - 2.64427i,\ \ b_3 = b_2^*,$$
$$b_4 = 0.585824 - 0.748355I,\ \ b_5 = b_4^*,\ \ b_6 = 12.6379. \tag{41}$$

Accordingly, the sixth order resummation gives;

$$E_0 = \frac{1}{2}\,{}_8F_6(a_1, \ldots a_8; b_1 \ldots b_6; -3g) = \frac{\prod_{k=1}^{6}\Gamma(b_k)}{2\prod_{k=1}^{8}\Gamma(a_k)} G_{8,7}^{1,8}\left(\begin{matrix}1-a_1,\ldots,1-a_8\\ 0,\ 1-b_1,\ldots,1-b_6\end{matrix}\middle| -3g\right). \tag{42}$$

5.2. Predicting the asymptotic strong-coupling behavior using hypergeometric-Meijer algorithm

In resummation techniques, the parameter representing the asymptotic strong-coupling behavior is taken arbitrary and is determined through optimization technique. For instance, in the Borel with conformal mapping algorithm in Ref. [33], results are optimized to give the best convergence. However, it has been shown in the literature [37–40] that different optimizations can lead to different strong-coupling behaviors for the same theory. In the following we extract the asymptotic strong-coupling behavior of the ground state energy of the ϕ^4_{0+1} theory and compare it with known exact results. In fact, in our algorithm no optimization is needed to determine the numerator parameters and thus the asymptotic strong-coupling behavior is unique for the same theory provided that we incorporate the same information as input.

For the approximant ${}_4F_2(a_1, \ldots a_4; b_1, b_2; \sigma g)$ for instance, the above discussions telling us that the strong coupling behavior is given by ${}_4F_2(a_1, \ldots a_4; b_1, b_2; \sigma g) \propto g^s$ where $s = Max(-a_1, -a_2, -a_3, -a_4)$. We parametrized the approximant ${}_4F_2(a_1, \ldots a_4; b_1, b_2; \sigma g)$ for the ground state energy using weak-coupling and large-order data and found that as $g \to \infty$ we have $E_0 \propto g^s$ with $S = 0.325731$. This is a fifth order prediction for s while the exact value is $s = 1/3 \approx 0.333333$ as shown above. Of course higher order approximants shall give better prediction for s. Accordingly, one can claim that our algorithm can be used to predict accurate asymptotic strong-coupling behavior of a divergent series from the knowledge of weak-coupling and large-order data.

6. Vacuum energy of the $\mathcal{PT}$-symmetric $i\phi^3_{0+1}$ theory

Another example for a divergent series with zero radius of convergence is the ground state energy of the $\mathcal{PT}$-symmetric $i\phi^3$ theory with Hamiltonian density operator in the form:

$$H = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2(x) + \frac{i\sqrt{g}}{6}\phi^3(x). \tag{43}$$

In 0 + 1 space-time dimensions, the ground state energy of this theory has the perturbation series [28]

$$E_0 = \frac{1}{2} + \frac{11g}{288} - \frac{930}{288^2}g^2 + \frac{158836}{288^2}g^3 + \frac{38501610}{288^4}g^4 + O(g^5). \tag{44}$$

Also, the strong coupling parameters are given in Ref. [2] as:

$$a_1 = 1,\ a_2 = -\frac{1}{5},\ a_3 = \frac{3}{5},\ a_4 = \frac{7}{5},\ a_5 = \frac{11}{5},\ a_6 = \frac{15}{5},\ \ldots. \tag{45}$$

Table 3
The sequence of approximants from first (${}_3F_1$) to fourth (${}_6F_4$) order hypergeometric-Meijer resummation for the ground state energy corresponding to the Hamiltonian in Eq. (43) compared to the 150^{th} order of ODMs resummation methods in Ref. [2] and also to exact results.

g	${}_3F_1$	${}_4F_2$	${}_5F_3$	${}_6F_4$	ODMs	Exact
0.5	0.51749	0.516869	0.516891	0.516892	0.516892	⋯
1	0.53258	0.530669	0.530775	0.530785	0.530782	0.530782
288/49	0.628506	0.610237	0.612557	0.613031	0.612738	0.612738

In using these parameters and matching the expansion of ${}_6F_4$ with expansion in Eq. (44), we get the numerators parameters b_i as:

$$b_1 = 0.43189086698613627` - 1.2561659803549978`i,\ b_2 = b_1^*,$$
$$b_3 = 4.605221446564435,\ b_4 = 0.3721464374405092. \tag{46}$$

Accordingly, the fourth order hypergeometric-Meijer resummation for the vacuum energy is:

$$E_0 = \frac{1}{2}\,{}_6F_4\left(a_1, \ldots a_6; b_1 \ldots b_4; -\sigma g\right) = \frac{\prod_{k=1}^{4}\Gamma(b_k)}{2\prod_{k=1}^{6}\Gamma(a_k)} G_{6,5}^{1,6}\left(\begin{matrix}1-a_1,\ldots,1-a_6\\ 0,\ 1-b_1,\ldots,1-b_4\end{matrix}\middle| -\sigma g\right), \tag{47}$$

where $\sigma = \frac{5}{24}$ [2]. The vacuum energy of the $\mathcal{PT}$-symmetric $i\phi^3_{0+1}$ theory has been resummed using different techniques in Ref. [2]. Our calculations from successive approximants are shown in table-3 where it is compared to the 150^{th} order of order-dependent mappings method (ODMs) in Ref. [2] and also compared to exact results. Again the hypergeometric-Meijer algorithm in this work gives very accurate results using a relatively low order of the perturbation sires as an input.

7. Critical exponents of the $O(4)$-symmetric quantum field model

The Lagrangian density of the $O(N)$-vector quantum field model is given by:

$$\mathcal{L} = \frac{1}{2}(\partial\Phi)^2 + \frac{m^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4, \tag{48}$$

with $\Phi = (\phi_1, \phi_2, \phi_3, \ldots\ldots\ldots\phi_N)$ is an N-component field having $O(N)$ symmetry where $\Phi^4 = (\phi_1^2 + \phi_2^2 + \phi_3^2 + \ldots\ldots\ldots\phi_N^2)^2$. For $N = 4$, it can describe the the phase transition in QCD with two light flavors at finite temperature [32]. Recently, the six-loops order for the renormalization group functions β, γ_{ϕ^2} and γ_{m^2} has been obtained in Ref. [33]. In Minimal-subtraction technique and in three dimensions, the six-loops order for the β-function in three dimensions is given by:

$$\beta(g) \approx -g + 4g^2 - 8.667g^3 + 55.66g^4 - 533.0g^5 + 6318g^6 - 86768g^7. \tag{49}$$

The large-order asymptotic behavior of this series is characterized by the parameters $\sigma = 1$ and $b = 5$ [4]. The strong-coupling asymptotic behavior is not yet known (up to the best of our knowledge). The suitable weak-coupling and large order parametrized hypergeometric-

Meijer approximant for β is then:

$$\beta(g) \approx -g_4F_2\left(a_1, \ldots a_4; b_1, b_2; -g\right) = -g\frac{\prod_{k=1}^{2}\Gamma(b_k)}{\prod_{k=1}^{4}\Gamma(a_k)}G_{4,3}^{1,4}\left(\begin{matrix}1-a_1,\ldots,1-a_4\\ 0,\ 1-b_1,\ 1-b_2\end{matrix}\middle| -g\right), \quad (50)$$

where $a_1 = 15.4564$, $a_2 = -2.02503$, $a_3 = -0.598824$, $a_4 = -0.136248$ and $b_1 = 0.315181$, $b_2 = -2.02558$. zeros of $\beta(g)$ defines fixed points where our resummation result gives $\beta(g_c) = 0$ at $g = g_c = 0.358733$. Here g_c is the critical value of the coupling where it has been predicted (but at five-loops) in Ref. [25] to be $g_c = 0.34375$. The critical exponent ω is defined as $\beta'(g_c)$ which gives $\omega = 0.781617$ compared to Borel with conformal mapping result as 0.794(9) from Ref. [33] and Monte Carlo simulations result that gives the value 0.765 [34] while the recent conformal bootstrap calculations gives the result $\omega = 0.817(30)$ [41,33].

The six-loops series for the anomalous mass dimension γ_{m^2} has been obtained in the same reference (Ref. [33]) where:

$$\gamma_{m^2}(g) \approx -2g + 1.6667g^2 - 9.500g^3 + 64.39g^4 - 571.9g^5 + 5983g^6, \quad (51)$$

and the corresponding large-order parameters are $\sigma = 1$ and $b = 5$. The hypergeometric-Meijer resummation gives the exponent ν as:

$$\nu^{-1} = 2 + \gamma_{m^2}(g_c) = 1 + {}_4F_2((a_1, \ldots a_4; b_1 \ldots b_2; -g_c) = 1 + \frac{\prod_{k=1}^{2}\Gamma(b_k)}{\prod_{k=1}^{4}\Gamma(a_k)}G_{4,3}^{1,4}\left(\begin{matrix}1-a_1,\ldots,1-a_4\\ 0,\ 1-b_1,\ 1-b_2\end{matrix}\middle| -g_c\right), \quad (52)$$

where $a_1 = -2.45317$, $a_2 = -0.893616$, $a_3 = 9.62099$, $a_4 = 0.058413$, $b_1 = -0.256423$, $b_2 = -2.40227$. That parametrization of the approximant yields the result $\nu = 0.744181$. The Monte Carlo simulations result from Ref. [34] gives 0.750(2) and the recent Borel with conformal mapping result is 0.7397(35) [33] while conformal bootstrap gives the result $\nu = 0.751(3)$ in Ref. [41].

The six-loops order for the field anomalous dimension γ_{ϕ^2} is [33]

$$\gamma_{\phi^2}(g) \approx g^2(0.16667 - 0.16667g + 0.9028g^2 - 6.5636g^3 + 55.93g^4), \quad (53)$$

with $\sigma = 1$ and $b = 4$ [4]. The resummation result is

$$\gamma_{\phi^2}(g) = 0.16667g_3^2F_1\left(a_1, \ldots a_1; b_1; -g\right) = 0.16667g^2\frac{\Gamma(b_1)}{\prod_{k=1}^{3}\Gamma(a_k)}G_{3,2}^{1,3}\left(\begin{matrix}1-a_1,\ldots,1-a_3\\ 0,\ 1-b_1\end{matrix}\middle| -g\right), \quad (54)$$

where $a_1 = 7.62 + 7.85714i$, $a_2 = 0.10777$, $a_3 = a_1^*$, $b_1 = 12.9107$. Our resummation result gives $\eta = 2\gamma_\phi(g_c) = 0.036695$ compared to Monte Carlo result 0.0360(3) [34] and recent Borel with conformal mapping result 0.0360(4)[33] while recent conformal bootstrap calculations for η is 0.0378(32) [42]. The critical exponents predictions of this work are summarized in table-4 and compared to recent resummation results as well as simulations results. The five-loop resummation results are also listed in the table (second) to give an idea about precision of the algorithm. We will not go far for such type of calculations as a full discussion of the critical exponents of the $O(N)$ model will appear in another work.

8. Resummation of the seven-loops β-function of the four dimensional ϕ^4 scalar field theory

In $\overline{MS}$-Scheme, the seven-loop for the perturbation series of the β-function for the ϕ^4_{3+1} scalar field theory has been recently obtained in Ref. [35] as:

Table 4
In this table we list the six-loop (6L) hypergeometric-Meijer resummation for the critical exponents ν, η and ω for $O(4)$-symmetric model. To give an idea about the precision of our calculations, we list (second) the five-loop resummation results (5L). Also, the results are compared to recent conformal bootstrap calculations (third), Borel with conformal mapping resummation (fourth) from Ref. [33] and also recent Monte Carlo simulations methods (last) from Ref. [34].

N	ν	η	ω	Reference
4	0.74418	0.0366948	0.78162	This work-6L
	0.74858	0.0365246	0.789063	~ 5-L
	0.751(3)	0.0378(32)	0.817(30)	[33,41,42]
	0.7397(35)	0.0366(4)	0.794(9)	[33]
	0.750(2)	0.0360(3)	0.765	[34]

$$\beta \approx 3.000g^2 - 5.667g^3 + 32.55g^4 - 271.6g^5 + 2849g^6 - 34776g^7 + 474651 g^8. \quad (55)$$

The ϕ^4_{3+1} theory is well known to have no fixed points and the series above has been recently resummed using the Borel-hypergeometric resummation algorithm [36]. The results of the Borel-hypergeometric resummation assured the non-existence of fixed points for the theory but on the other hand the convergence of the calculations was not perfect. We resummed the same series using our algorithm where we get:

$$\beta = 3g_4^2F_2\left(a_1, \ldots a_4; b_1 \ldots b_2; -g\right) = 3g^2\frac{\prod_{k=1}^{2}\Gamma(b_k)}{\prod_{k=1}^{4}\Gamma(a_k)}G_{4,3}^{1,4}\left(\begin{matrix}1-a_1,\ldots,1-a_4\\ 0,\ 1-b_1,\ 1-b_2\end{matrix}\middle| -g\right), \quad (56)$$

where $a_1 = 10.5029$, $a_2 = -0.552104 + 2.06665i$, $a_3 = a_2^*$, $a_4 = 0.234073$, $b_1 = -0.711745 - 2.33358i$, $b_2 = b_1^*$. To monitor the convergence of the calculations and thus compare with those presented in Fig.1 of Ref. [36], we generated the five and six loops resummation results too and plot all the results in Fig. 2. In the figure, the calculations prove also the non-existence of any fixed points for the theory but our calculations show a clear improvement of the convergence when compared to the Borel-hypergeometric results in Fig.1 in Ref. [36].

9. Resummation of the seven-loop ε-expansion for the fractal dimension of the critical curves for the self avoiding polymer

Recently, in Ref. [43], the authors obtained the six-loop ε expansion for the fractal dimension d_f for the case $N = 0$ of the $O(N)$-symmetric ϕ^4 model. They introduced what they called self-consistent resummation

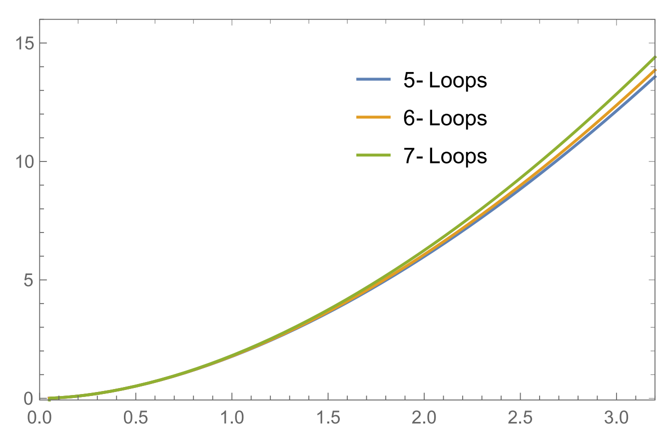

Fig. 2. The hypergeometric-Meijer resummation of the five, six and seven loops of the β-function of the four-dimensional ϕ^4 theory.

procedure and used it to resum the associated divergent series. However, the seven-loop g-expansion has been recently obtained [35] from which one can extract the seven-loop order of the ε-expansion for any N. We shall stress her only the $N = 0$ case which is in the same class of universality with the self-avoiding polymer. In fact, the resummation for the seven-loop critical exponents for different N values will appear in another work[44]. Using the seven-loop expansions in Ref. [35], we obtain the flowing perturbation series up to ε^7 for the fractal dimension d_f:

$$d_f(\varepsilon) = 2.0000 - 0.25000\varepsilon - 0.085938\varepsilon^2 + 0.11443\varepsilon^3 - 0.28751\varepsilon^4 + 0.95613\varepsilon^5 - 3.8558\varepsilon^6 + 17.784\varepsilon^7. \tag{57}$$

Note that the first six terms in this result are compatible with the six-loop result in Ref. [43]. The large order parameters are $\sigma = \frac{3}{8}$ and $b = 4$ [4]. We used this series to parametrize the hypergeometric approximant $2\ {}_5F_3(a_1, ...a_5; b_1, b_2, b_3; -\sigma\varepsilon)$ which in turn leads to the result:

$$d_f \approx \frac{2\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)\Gamma(a_5)} G_{5,4}^{1,5}\left(\begin{matrix} 1-a_1, 1-a_2, 1-a_3, 1-a_4, 1-a_5 \\ 0, 1-b_1. ,1-b_2, 1-b_3 \end{matrix} \middle| -\frac{3}{8}\varepsilon\right) \tag{58}$$

The parameters in this approximant take the values $a_1 = -3.68209$, $a_2 = 10.5985 + 10.5125i$, $a_3 = a_2^*$, $a_4 = -1.18266$, $a_5 = -0.0115771$, $b_1 = 0.688263$, $b_2 = -3.67874$, $b_3 = 13.3112$. This approximant yields the result $d_f = 1.3307$ for the two dimensional ($\varepsilon = 2$) self-avoiding polymer. Note that the conformal field theory (also exact) result is $\frac{4}{3} = 1.3333$ [43,45–48] while the recent self-consistent resummation result is 1.354(5). It is very clear that our seven-loop resummation result is very close to the exact result. In Table 5, we listed the five, six and seven loops resummation results to monitor the convergence of the results as the order increases. For comparison, we listed also the SC as well as exact results.

10. Summary and conclusions

We introduced what we can call it the hypergeometric-Meijer algorithm for a resummation of a divergent series with zero radius of convergence. The suggested algorithm is capable of accommodating the large-order and strong coupling information and thus is able to accelerate the convergence to the exact results. In Ref. [14], Héctor Mera *et.al* followed a Borel-hypergeometric technique that led to a Meijer-G approximant algorithm which has been shown to produce precise results from weak coupling information as input. The algorithm we introduced however avoids Borel or Padé techniques used in Ref. [14] and instead starting from the parametrization of a hypergeometric function that has the same $n!$ growth factor characterizing the divergent series and then use the equivalent integral representation of Meijer G function as an approximant to the given perturbation series. In fact, using weak coupling information in both the hypergeometric-Meijer G approximant in our work and that in Ref. [14] leads to different parametrizations. This can be seen from the exact partition function of zero-dimensional ϕ^4 theory which has been obtained by a third order parametrization of hypergeometric-Meijer G approximant in our work while in Ref. [14] the same result has been obtained at the fifth order.

Incorporation of the large-order information has been shown to accelerate the convergence and in adding the strong coupling data into the resummation technique, the convergence is even faster a fact that is traditionally known in resummation techniques [4]. In our work, however, we obtained a new constraint on the parameters of the hypergeometric approximant which relates them to one of the parameters in the large-order asymptotic behavior of the given perturbation sires. The validity of this constraint has been tested in our work by obtaining the exact result of the zero-dimensional partition function of the ϕ^4 theory at the first order parametrization using weak-coupling and large-order data while in adding strong coupling data, the exact result is completely parametrized from large-order and strong-coupling data. In both of these different parametrizations that lead to the exact result, the constraint ($\sum_{i=1}^{p} a_i - \sum_{i=1}^{p-2} b_i - 2 = b$) on the parameters has been applied.

The algorithm is also applied to resum the ground state energies of the ϕ^4 as well as the $\mathscr{PT}$-symmetric $i\phi^3$ field theories in 0 + 1 space-time dimensions (quantum mechanics). It shows accurate predictions although few number of perturbative terms are employed.

It is well known that till now a closed form of the strong-coupling asymptotic behavior for quantum field theories in dimensions greater than one has not been obtained yet. In literature one can find that some predictions for the asymptotic behavior can be extracted from resummation techniques. These resummation algorithms include arbitrary parameters that have to be optimized to give the best convergence. Such optimization can lead to a prediction of the strong-coupling behavior. However, different optimizations can lead to different results for the same theory. In our algorithm, on the other hand, all the parameters are uniquely determined from the input information and thus the predicted asymptotic strong-coupling behavior is unique for the same theory with the same information as input. We tested our algorithm regarding this fact and obtained an accurate prediction for the asymptotic behavior of the ground state energy of the $(\phi^4)_{0+1}$ theory using weak-coupling and large order data as input.

Since the type of divergent series stressed in this work shares the same properties of the divergent series representing the renormalization group functions in quantum field theory [4], we applied it to resum the recent six-loops orders for the β, γ_{m^2} and γ_{ϕ^2} renormalization group functions of the $O(4)$-symmetric model in three dimensions. Precise results of the corresponding critical coupling as well as critical exponents have been extracted from resummation of the renormalization group functions.

The ϕ^4 scalar field theory is well known to have no fixed points in four dimensions. The hypergeometric-Borel resummation algorithm has been applied recently to resum the recent seven-loops perturbative order of the β-function. The result of that algorithm asserts the non-existence of fixed points but the convergence of calculations is questionable. We resummed the same series using our algorithm where our calculations shows also no fixed points but on the other hand convergence of the calculations has been greatly improved.

The seven-loop perturbation series (ε-expansion) for the fractal dimension d_f of the self-avoiding polymer has been listed in this work. Resumming that series using our algorithm introduced in this work gives a very accurate result for the two dimensional case ($\varepsilon = 2$). Note that, in two dimensions, the ε-series is well known to have a slower convergence than the three dimensional case. Accordngly, its resummation offers a challenging test to our algorithm. The accurate result we obtained ($d_f = 1.3307$) reflects an extraordinary success to our rsummation method specially when we know that the exact value is $d_f = 4/3 \approx 1.3333$. Our resummation result might be the most accurate resummation prediction for the same series in literature.

Since the Meijer-G function is represented by a Mellin-Barnes type of integrals, there is a possibility for the existence of Stokes phenomena [19]. So one can have hypergeometric-Meijer non-summability like cases of non-Borel summability. For such cases one resorts to the resummation of resurgent transseries which kills the complex ambiguity [20]. We applied our algorithm to resum the transeeries of the partition

Table 5
The five, six and seven-loop hypergeometric-Meijer resummation for ε-expansion of fractal dimension of the critical curves for the self avoiding polymer. The results are compared to self-consistent (SC) resummation result from Ref. [43] and the exact result [48] is also listed.

Method	5-loop	6-loop	7-loop	SC	Exact
d_f	1.3571	1.3464	1.3307	1.354(5)	1.3333

function of degenerate vacua ϕ^4 theory where we obtained exact result at the third order parametrization of hypergeometric-Meijer approximant. After incorporating the large order data, the same result has been obtained using first order parametrization.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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