λ-point anomaly in view of the seven-loop hypergeometric resummation for the critical exponent ν of the $O(2)$ ϕ^4 model

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In this work, we use a specific parametrization of the hypergeometric approximants [the one by Mera *et al.* in Phys. Rev. Let. **115**, 143001 (2015)] to approximate the seven-loop critical exponent ν for the $O(2)$-symmetric ϕ^4 model. Our prediction gives the result $\nu = 0.6711(7)$ which is compatible with the value $\nu = 0.6709(1)$ from the famous experiment carried on the space shuttle *Columbia*. On the other hand, our result is also compatible with recent precise theoretical predictions that are excluding the experimental result. These theoretical results include nonperturbative renormalization group calculations [$\nu = 0.6716(6)$], the most precise result from Monte Carlo simulations [$\nu = 0.67169(7)$] as well as the recent conformal bootstrap calculations [$\nu = 0.67175(10)$]. Although our result is compatible with experiment, the plot of the renormalization group result versus the number of loops suggests that higher orders are expected to add significantly to the accuracy and precision of the ν exponent in a way that may favor the theoretical predictions.

Rychkov wrote a nice commentary [1] on the recent conformal bootstrap (CB) prediction for the critical exponent ν of the $O(2)$ model [2] which can describe the ^{4}He superfluid phase transition. The CB work in Ref. [2] confirmed the most precise result of Monte Carlo (MC) simulations in Ref. [3,4] but excluded the experimental result in Ref. [5]. These facts have been summarized by Rychkov where he outlined the current status of the predictions of both theoretical and experimental results. The CB and MC results, in conjunction with the recent result from nonperturbative renormalization group (NPRG) calculations [6], have asserted what can be called the λ-point dispute between theory and experiment which lasts for a decade. Resummation of the perturbation series for the associated renormalization group (RG) function, on the other hand, so far is not precise enough to favor either experiment or the mentioned nonperturbative theoretical calculations.

The λ-shape behavior describing the change in specific heat vs temperature for the helium superfluid transition is characterized by the critical exponent α. In fact, the specific heat critical exponent α is related to the exponent ν by the hyperscaling relation $\alpha = 2 - 3\nu$. Using this relation, the

critical exponent ν for the ^{4}He superfluid transition can be extracted from the α value in Ref. [5] as $\nu = 0.6709(1)$. As mentioned above, this result is in contradiction with both the recent Monte Carlo (MC) simulation result of $\nu = 0.67169(7)$ [3] and bootstrap calculations of $\nu = 0.67175(10)$ [1,2]. Note that in Refs. [6,7], it has been stated that the recent NPRG result [$\nu = 0.6716(6)$] also excludes the experimental result. In other words, while MC, CB and NPRG predictions are compatible with each other, they exclude the result of the famous experiment by Lipa *et al.* in Ref. [5] that leads to the above result for ν (all these results and others are listed in Fig. 1).

The best-known result from resummation of renormalization group functions at fixed dimension ($d = 3$) is $\nu = 0.6703(15)$ [9]. It is clear that this result is not precise enough and has not been improved since its appearance in 1998 as explained by Rychkov in Ref. [1]. Also, the resummation of the ε expansion of RG functions is known to have slower convergence than the resummation of RG functions at fixed dimensions [10]. This may explain the significant difference (see Fig. 1) between theoretical (MC, CB, NPRG) as well as experimental results and the recent six-loop (ε-expansion) resummation result [$\nu=0.6690(10)$] [8]. In fact, although the fixed-dimension and the ε-expansion RG results have uncertainties of the same order of magnitude, one can realize that the experimental as well as MC and CB results exclude the prediction of the six-loop ε expansion in Ref. [8]. In this work, we show that (unlike the six-loop expansion) the seven-loop ε expansion for the exponent ν of the $O(2)$ model gives a precise result that enables it to play a role in the λ-point dispute. To do
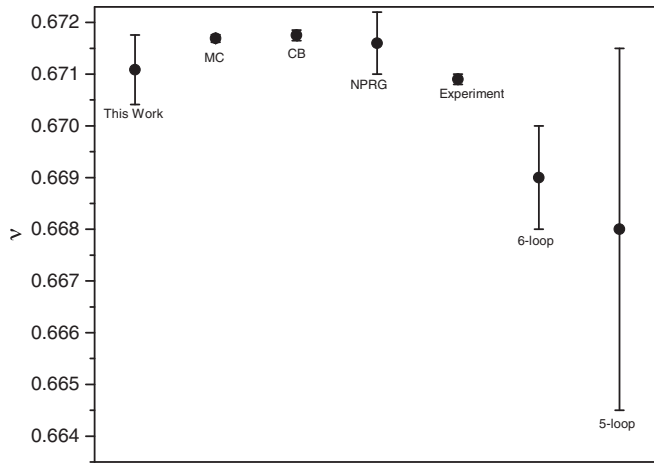
FIG. 1. In this graph, we show the predictions from seven-loop resummation (this work), the recent MC prediction from Ref. [3], the conformal bootstrap result [2], the NPRG result [6], the experimental result from Ref. [5], six-loop resummation from Ref. [8] and the five-loop result from Ref. [9].

that, we use a modified parametrization of the hypergeometric resummation algorithm [11].

The hypergeometric approximants suggested by Mera *et al.* in Ref. [11] use the hypergeometric function $_2F_1(a_1, a_2; b_1; -\sigma x)$ to approximate a divergent series with an $n!$ growth factor (zero radius of convergence). Although this approximant can give accurate results [12–17], it has been realized (by the same authors) that it suffers from a genuine problem as the expansion of $_2F_1(a_1, a_2; b_1; -\sigma x)$ has a finite radius of convergence [12,14,18]. To overcome this issue, the tip of the branch cut is forced to lie at the origin [14,18]. In Ref. [19], we have shown that the numerator parameters (a_1, a_2) represent the strong-coupling asymptotic behavior and thus knowing them can accelerate the convergence. On the other hand, employing parameters from the large-order asymptotic behavior of the given perturbation series is well known to accelerate the convergence of resummation algorithms too [10,20]. For the problem under consideration in this work, the large-order parameters are more important than the strong-coupling parameters as the first are well known in quantum field problems while the second are not. However, the approximant $_2F_1(a_1, a_2; b_1; -\sigma x)$ cannot accommodate the large-order parameters because its expansion does not have the same form of large-order behavior that the given perturbation series has. In fact, as stated by Mera *et al.*, the parameter σ ought to take large values to account for the missing $n!$ factorial growth factor [12,18]. Apart from this, one needs to find a way to change the parametrization of the approximant to make it able to give explicitly the $n!$ growth factor of the given series.

In Ref. [21], we suggested the parametrization of the form $_2F_1(a_1, a_2; b_1; -b_1\sigma x)$. If the parameter b_1 takes large values, then we have the following limit:

$$\lim_{b_1 \to \infty} (_2F_1(a_1, a_2; b_1; -b_1\sigma x)) = {}_2F_0(a_1, a_2; -\sigma x). \quad (1)$$

In fact the expansion of $_2F_0(a_1, a_2; -\sigma x)$ has an $n!$ growth factor [22] and thus the parameter σ can be taken from the large-order behavior of the given series.

For more details of how to apply the new parametrization, consider a quantity Q whose first four perturbative terms are known:

$$Q(x) \approx 1 + \sum_{1}^{3} d_i x^i. \quad (2)$$

This quantity can be approximated by the hypergeometric approximant $_2F_1(a_1, a_2; b_1; -b_1\sigma x)$ such that

$$a_1 a_2 \sigma = d_1,$$

$$\frac{a_1(1 + a_1)a_2(1 + a_2)}{2(1 + b_1)} b_1 \sigma^2 = d_2,$$

$$\frac{a_1(1 + a_1)(2 + a_1)a_2(1 + a_2)(2 + a_2)(b_1)^2 \sigma^3}{6(1 + b_1)(2 + b_1)} = d_3. \quad (3)$$

The three unknown parameters (a_1, a_2, b_1) are then obtained by solving the above set of equations.

In case one knows $M + 1$ orders, one can approximate the series for $Q(x)$ by the generalized hypergeometric function:

$$Q(x) \approx {}_{k+1}F_k(a_1, a_2, \ldots\ldots a_{k+1}; b, b_2, \ldots b_k; -b\sigma x), \quad (4)$$

with $M = 2k + 1$. This strategy has been followed in Ref. [21] and led to accurate results for different examples.

In this work, however, instead of solving for the parameter b, we shall use it as an input and set it as large as possible and then solve for the other parameters. This setting is very suitable to approach the limit in Eq. (1).

Before we use the algorithm to tackle the problem of the critical exponent ν of the model under consideration, let us first apply it to an example for which the exact value is known. To do that, consider the seven-loop expansion of the reciprocal of the critical exponent ν for the Ising case of the $O(N)$-vector model ($N = 1$) [23]:

$$\nu^{-1} = 2.0000 - 0.33333\varepsilon - 0.11728\varepsilon^2 + 0.12453\varepsilon^3$$
$$- 0.30685\varepsilon^4 + 0.95124\varepsilon^5 - 3.5726\varepsilon^6 + 15.287\varepsilon^7, \quad (5)$$

with $\sigma = \frac{1}{3}$ [10]. The exact value for $\varepsilon = 2$ (two dimensions) is $\nu = 1$. Taking $b = 170$ (the maximum value one can use due to Γ functions defining the coefficients of the hypergeometric series), one can approximate this series by

$$\nu^{-1}(\varepsilon) \approx 2.0000 - 0.33333\varepsilon_4 F_3\left(a_1, a_2, a_3, a_4; 170, b_2, b_3; 170\left(-\frac{1}{3}\right)\varepsilon\right). \tag{6}$$

We find the following values for the parameters:

$$a_1 = 0.0411154, \qquad a_2 = 13.3176,$$
$$a_3 = -0.470683, \qquad a_4 = -2.29587$$
$$b_1 = -2.28074, \qquad b_2 = 0.245782.$$

For $\varepsilon = 2$, these parameters lead to the result $\nu(2) = 0.977203$. To see how our idea improved the prediction of the algorithm, we obtained ν^{-1} by using the original hypergeometric approximant (with the parametrization presented in Ref. [11]) which has the parametrization

$$\nu^{-1}(\varepsilon) \approx 2.0000 - 0.33333\varepsilon_3 F_2(a_1, a_2, a_3; b_1, b_2; d\varepsilon). \tag{7}$$

This parametrization gives the result $\nu(2) \approx 0.964952$. Note that, the more sophisticated Borel algorithm with conformal mapping resummation for the six-loop expansion of the same series in Ref. [8] gives the result $\nu(2) \approx 0.952(14)$. Although the algorithm we follow here in this work might be the simplest one, it gives an accurate result as shown above. One can even refine the result by taking larger values for the parameter b but in this case to overcome the machine limit one can use the limiting case

$$\nu^{-1}(\varepsilon) \approx \lim_{b \to \infty}\left(2.0000 - 0.33333\varepsilon_4 F_3\left(a_1, a_2, a_3, a_4; b, b_2, b_3; b\left(-\frac{1}{3}\right)\varepsilon\right)\right)$$
$$= \left(2.0000 - 0.33333\varepsilon_4 F_2\left(a_1, a_2, a_3, a_4; b_2, b_3; \left(-\frac{1}{3}\right)\varepsilon\right)\right).$$

In such a case one has to resort to the representation of the hypergeometric function $_4F_2(a_1, a_2, a_3, a_4; b_2, b_3; (-\frac{1}{3})\varepsilon)$ in terms of the Meijer-G function [24]. Using this and the value of $b = 10^5$, we get the value $\nu(2) = 0.98499$. This example shows clearly that the simple algorithm we use in this work can give challenging results and thus is trusted to tackle the λ-point anomaly for the ^{4}He superfluid phase transition as our main problem.

Now consider the seven-loop critical exponent for the $O(2)$ case given by [23]

$$\nu^{-1} = 2.0000 - 0.40000\varepsilon - 0.14000\varepsilon^2 + 0.12244\varepsilon^3$$
$$- 0.30473\varepsilon^4 + 0.87924\varepsilon^5 - 3.1030\varepsilon^6 + 12.419\varepsilon^7, \tag{8}$$

which again can be approximated by ($\sigma = 3/10$)

$$\nu^{-1}(\varepsilon) \approx 2.0000 - 0.40000\varepsilon_4 F_3$$
$$\times \left(a_1, a_2, a_3, a_4; b, b_2, b_3; b\left(-\frac{3}{10}\right)\varepsilon\right). \tag{9}$$

again taking $b = 10^5$ and $\varepsilon = 1$, we get the result $\nu = 0.67094$. Note that the five-loop resummation for the same series gives $\nu = 0.6680(35)$ [9] while the six-loop resummation gives $\nu = 0.6690(10)$ [8] and the experimental result is $0.6709(1)$ [5]. Apart from the uncertainty in the calculation which we did not discuss yet, one can realize that our result shows a significant improvement of the accuracy of the critical exponent ν obtained previously from the resummation of the ε expansion of RG functions. This can be clarified by comparing with other nonperturbative predictions like the Monte Carlo result which gives $\nu = 0.67169(7)$ [3], the NPRG result that gives the prediction $\nu = 0.6716(6)$ [6] and the conformal bootstrap prediction result $\nu = 0.6719(11)$ [25]. However, as we mentioned above, a more precise bootstrap result of $\nu = 0.67175(10)$ recently appeared in Refs. [1,2].

To know what is the situation of the precision of our resummation result among all other predictions, one needs to offer an estimate for the size of the error in the calculated result. In fact, the sources of errors differ from one theoretical method to another. For the algorithm we use, the error is due to the unknown higher terms in the perturbation series as well as the arbitrary parameters in the resummation algorithm. Also for MC calculations, errors are due to Monte Carlo statistical errors and systematic errors associated with the correction to scaling [1]. Other methods like NPRG and CB also have their own sources of errors. So it is very natural to have different precisions from different methods. An improvement of some calculation can be decided by comparison with previous calculations within the same method. For instance, looking at Fig. 1, one can realize that our seven-loop prediction shows a significant improvement for both accuracy and precession when compared to the five-loop and six-loop predictions.

The algorithm we follow seems to have no arbitrary parameters which one can optimize and thus find the uncertainty in the result. However, a deep understanding of the simple algorithm we follow can find some implicit

$$\nu^{-1}(\varepsilon) \approx 2.0000\,_5F_4(a_1, a_2, a_3, a_4, a_5; b, b_2, b_3, b_4; b(-\sigma)\varepsilon),$$

$$\nu^{-1}(\varepsilon) \approx 2.0000 - 0.40000\varepsilon\,_4F_3(a_1, a_2, a_3, a_4; b, b_2, b_3; b(-\sigma)\varepsilon),$$

$$\nu^{-1}(\varepsilon) \approx 2.0000 - 0.40000\varepsilon - 0.14000\varepsilon^2 + 0.12244\varepsilon^3\,_3F_2(a_1, a_2, a_3; b, b_2; b(-\sigma)\varepsilon),$$

$$\nu^{-1}(\varepsilon) \approx 2.0000 - 0.40000\varepsilon - 0.14000\varepsilon^2 + 0.12244\varepsilon^3 - 0.30473\varepsilon^4 + 0.87924\varepsilon^5\,_2F_1(a_1, a_2; b; b(-\sigma)\varepsilon). \tag{10}$$

All these approximants are legal and use the same content of information and have the $n!$ growth factor at the limit $b \to \infty$. Of course they give different approximations and the question is which one we should select. To answer this question, one also notices that the six-loop resummation can also be approximated by different hypergeometric functions. A natural choice is then to select a pair of approximants for six and seven loops that has the fastest convergence or equivalently we select the pair that minimizes the difference (Δ) defined as

$$\Delta = |\nu_k^7 - \nu_{k'}^6|, \tag{11}$$

where k defines the hypergeometric approximant ($_kF_{k-1}$) used while superscripts denote the number of loops involved. We found that the approximant

$$(\nu^{-1})_4^7 = 2.0000$$
$$- 0.40000\varepsilon\,_4F_3(a_1, a_2, a_3, a_4; b, b_2, b_3; b(-\sigma)\varepsilon),$$

arbitrariness in its parametrization. As an example, one can find different hypergeometric functions that can approximate a given order of the perturbation series. For instance, the seven-loop expansion in Eq. (8) can be approximated by

for the seven-loop resummation and the approximant

$$(\nu^{-1})_4^6 = 2.0000 - 0.40000\varepsilon$$
$$- 0.14000\varepsilon^2\,_3F_2(a_1, a_2, a_3; b, b_2; b(-\sigma)\varepsilon),$$

for the six-loop resummation give the smallest difference of $\Delta = 0.0004$. This could be used as an uncertainty and our seven-loop resummation result can be taken as $\nu = 0.67094(4)$. This method of error calculation has been used in different references (see for instance Sec. 16.6.1 in Ref. [10]).

We have another source of arbitrariness which can be taken from the form of the large-order behavior of the given perturbation series as
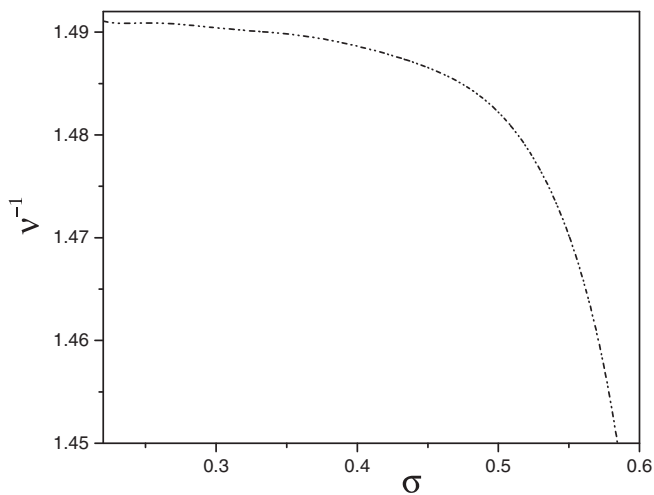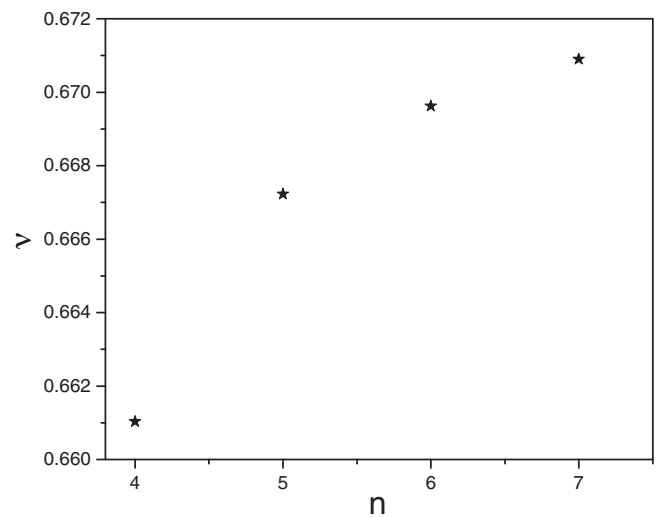

FIG. 2. The hypergeometric approximant in Eq. (9) for the seven-loop resummation of ν^{-1} vs σ which defines the width $w = x - y$ of the most stable region of the curve. The curve falls down for $\sigma < 0.2$ (not shown in the figure).

FIG. 3. In this figure, we plot our hypergeometric resummation results for the three-dimensional critical exponent ν of the $O(2)$ model vs the number of loops (n) used. One can realize that the distribution of the data is not saturated enough to claim that the seven-loop result is the best RG result that one can achieve and thus shows the need for more orders.

TABLE I. Our resummation for the seven-loop critical exponent ν for the $O(2)$ scalar ϕ^4 model in three dimensions. We list also in the table the famous experimental result from Ref. [5], the most precise result from Monte Carlo simulations [3], the NPRG prediction from Ref. [6] as well as conformal bootstrap results [1,2]. To show the significant improvement that the seven-loop resummation adds to the previous resummation results of the same series, we list the Borel algorithm with conformal mapping (BCM) results for five loops [9] and six loops from Ref. [8].

Method	ϵ^7: This work	MC: [3]	Experiment: [5]	CB: [1,2]	NPRG: [6]	ϵ^6; BCM: [8]	ϵ^5; BCM: [9]
ν	0.6711(7)	0.67169(7)	0.6709(1)	0.67175(10)	0.6716(6)	0.6690(10)	0.6680(35)

$$Q(x) \approx \sum_0^n d_i x^i,$$

$$d_n = \alpha n! (-\sigma)^n n^b \left(1 + O\left(\frac{1}{n}\right)\right), \qquad n \to \infty. \quad (12)$$

In fact, the parameter α for the given model is known and has the value $\alpha = 5.892 \times 10^{-4}$ [26] but none of our equations have been constrained to account for it. The explicit form of α [Eqs. (4.19,4.10) in Ref. [26]] depends on $\frac{1}{N+8} = \frac{\sigma}{3}$ which means that one can account for its variation by varying the parameter σ and then finding the variance of ν_4^7 defined as [8]

$$\text{Var}_\sigma(\nu_k^7(\sigma)) = \min_{x \le \sigma \le y} \left(\max_{x \le \sigma \le y} (\nu_k^7(\sigma) - \nu_t^7(\sigma')) \right).$$

The width $w = y - x$ is chosen according to the stability region in the curve of the seven-loop exponent $\nu_k^7(\sigma)$ (see Fig. 2).

Based on the shape of the curve in Fig. 2, we choose $w = 0.2$ where we vary σ around its exact value of 0.3 from 0.2 to 0.4. We use an adapted form of the method detailed in Ref. [8] where our error can be taken as the minimum of

$$\Delta(\sigma, k, k') = |\nu_k^7(\sigma) - \nu_{k'}^6(\sigma)| + \text{Var}_\sigma(\nu_k^7(\sigma)). \quad (13)$$

This form of error is closer in shape to the one used in Ref. [8] but adapted in view of the shape in Fig. 2 as well as the behavior of the six-loop result vs σ (not shown in figures). The error value obtained is $\Delta_{\min} \approx 7 \times 10^{-4}$ while the predicted value of ν is 0.6711(7). As shown in Fig. 1, this result shows a significant improvement to the resummation results of the same series in the literature. In fact, one can easily realize that by moving from six to seven loops, the RG result is more precise and accurate as it has been shifted toward the MC and BC results.

Our prediction is compatible with both experiment as well as MC, BC and NPRG results. However, in view of Fig. 3, we see that the seven-loop result is not sufficiently leveling off the curve for ν vs the number of loops n. In fact, the tangent of the curve becomes smaller as a function of n but not small enough to claim a stable shape. This shape of the curve is thus suggesting a possibility for smaller error as

well as a higher value in the exponent ν to come from future higher orders. In other words, the future eight-loop result might exclude the experimental result the same way NPRG, MC and CB results do. For a summary of comparison between our results and other methods, see Table I.

To conclude, we used a simple parametrization of the hypergeometric approximant that enables it to accommodate the large-order parameters for the sake of convergence acceleration. The modified hypergeometric algorithm was tested first for the two-dimensional Ising case where the exact critical exponent is well known. The prediction of the modified hypergeometric algorithm for the seven-loop ε expansion is very close to the exact result and better than the prediction of the unmodified algorithm as well as the six-loop resummation result from the Borel algorithm.

Motivated by the success of the modified hypergeometric algorithm in the two-dimensional case, we tackled the controversial three-dimensional case for ν of the $O(2)$ model. In fact, for $\varepsilon = 1$, one expects even better convergence than the two-dimensional case. We calculated the exponent ν for the $O(2)$-symmetric case and got the value $\nu = 0.6711(7)$ which is compatible with the experimental result $[\nu = 0.6709(1)]$ as well as the theoretical calculations from the NPRG method $[\nu = 0.6716(6)]$, the more precise CB result ($\nu = 0.67175(10)$) and the MC result $[\nu = 0.67169(7)]$. Note that NPRG, CB and MC results exclude the experimental one.

The plot of the exponent ν vs the number of loops suggests that the seven-loop result in this work might not be the most precise or accurate prediction that one can obtain from a resummation of RG perturbations. The shape of the curve reinforces the expectation of a more accurate as well as precise result to come from the future eight-loop order of the perturbation series. In other words, there is still room for the RG result to agree with both MC and CB predictions but exclude the experimental result.

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