Determination of Optimal Safety Stock Policies

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ABSTRACT

Setting safety stock policies are especially important in the management of retail and wholesale inventories, as well as stores, spare parts, supply items, and in certain areas of production planning. From a practical perspective, determining the optimal safety stock policy and the optimal service level requires specifying the demand distribution. This paper develops optimal safety stock policies under several commonly used statistical demand distributions: normal, exponential, and poisson. In those situations where a manager has limited information on the shape of the demand distribution, Chebychev's Inequality Theorem is exploited to determine the optimal policies. The suggested computational approach enables the order quantity and the number of standard deviations that specifies the service level to be jointly determined by minimizing total relevant cost. A numerical example is also given to illustrate the computational process.

INTRODUCTION

Safety stock provides a "cushion" against the uncertainty of demand during lead-time. Setting safety stock policies remains an important issue in managing wholesale and retail inventories, as well as stores, spare parts, supply items, and in certain areas of production planning. In such applications, managers need to specify their demand distributions to improve the quality of their inventory policies. The purpose of this paper is to show how optimal safety stock policies under several commonly used statistical demand distributions can be determined. In those situations where a manager has limited information on the shape of the demand distribution, Chebychev's Inequality Theorem is exploited to determine the optimal policies.

Safety stock plays a significant role in production and inventory planning [e.g. 5, 6, 8, and 9]. Therefore, the determination of the optimal safety stock policy is an important issue for researchers and managers alike.

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Many researchers have studied safety stock and the different methods used to determine its policies [e.g. 1, 2, 3, and 4]. Aucamp [5] has presented a simplified model for determining the safety stock level. His model is limited to the Poisson distribution, and ignores the relationship between the order quantity and the service level.

In practice, other probability distributions, such as the Normal and the exponential, have useful properties for modeling lead-time demand. The normal distribution, for example, is appropriate when demand for the item is relatively large. The exponential distribution possesses convenient mathematical properties and is useful when demand for an item is at a relatively low rate per unit time. Using the appropriate probability distribution more accurately captures the behavior of demand and leads to a lower cost solution. In addition, the joint determination of the order size and service level provides the optimal solution, and therefore, represents an improvement over the approach suggested by Aucamp.

In the next section we present a marginal analysis approach for jointly determining the optimal order size (Q) and the optimal number of standard deviations that specifies the service level (Z). Next, we develop optimal policies for the normal, exponential, poisson, and unknown demand distributions. This is followed by a numerical example to illustrate the proposed approach. We conclude the paper by discussing the managerial implications of this research.

SERVICE LEVEL DEFINED

To determine the optimal safety stock policy, the service level must be clearly specified. Defining service level is most important when an organization does not know its stockout costs or feels very uneasy about estimating them. Under these conditions, it is common for management to set service levels from which recorder points can be ascertained. A service level, therefore, indicates the ability of a company to meet customer demands from available stock.

There are several ways to measure a service level. It can be measured by either order service level (OSL) or unit service level (USL). USL, which is sometimes known as "fill rate", counts the average number of units short expressed as the percentage of the order quantity. The OSL measures the percentage of cycles that will be out of stock or the probability of stockouts. If, for example, the desired service level is 80%, it is necessary to clarify whether this is 80% OSL or 80% USL because the safety stock level would be quite different in these two situations. In this paper we will use OSL.

SAFETY STOCK DETERMINATION USING MARGINAL ANALYSIS

Notation

Let:

Q = lot size

h = carrying cost per unit per year

D_L = demand during the lead time

$\overline{D}_L$ = average demand during the lead time

r = reorder point

σ_L = standard deviation of lead time demand

$\overline{D}_t$ = average demand per time period t

LT = average lead time

σ_{LT} = standard deviation of lead time

σ_{LTD} = standard deviation of the combined lead time and demand variations

Z = number of standard deviations that specifies the service level

σ_t = standard deviation per time period t

C_u = shortage cost per unit per cycle

D = annual demand

N_L = expected number of units short during the lead time

S = setup cost.

The safety stock, SS = $Z\sigma_L$, can be specified when shortage and carrying costs are known [5]. The intuition is that raising the reorder point by one unit of inventory will cost us hQ/D per cycle. On the other hand, if we do not increase the reorder point by one unit of inventory, it will cost us a shortage cost of C_u per unit per cycle with the probability of demand greater than the reorder point, i.e., $P(D_L > r)$. The probability of demand which is greater than the reorder point is commonly known as the stockout risk (SOR). Therefore, by marginal analysis, the above statement can be expressed in equation form in the backorder case (units are backordered because of shortage) for determining the optimal safety stock. In this case, the following relationship holds:

$$hQ/D = P(D_L>r)C_U$$

Accordingly, the condition of optimality can be expressed as:

$$SOR = P(D_L>r) = hQ/DC_u \qquad (1)$$

and the total relevant cost (TRC) including the shortage cost for the inventory problem is given by

$$TRC = Qh/2 + hZ\sigma_L + DS/Q + C_uN_LD/Q \qquad (2)$$

The first two terms in (2) are carrying costs and the last two terms are the setup and shortage costs, respectively. Equations (1) and (2) are the guiding equations for the determination of optimal service levels and lot sizes under several demand distributions assuming constant lead time. Because of the interrelationship between Q and Z, an iterative process will be used to determine the optimal safety stock policy.

Normal Distribution

To determine the optimal safety stock, the optimal Z value must be found. Instead of deriving it mathematically, we can use the condition of optimality in equation (1) presented above. An iterative process is used to determine the values of Q and Z. The steps of the iterative process are:

1. Initially, set $Q = (2DS/h)^{1/2}$, i.e. the well known EOQ

2. Calculate $P(D_L>r) = hQ/C_uD$. The value of Z corresponding to $P(D_L>r)$ can be found from the normal distribution table. The total relevant cost is

$$TRC = Qh/2 + hZ\sigma_L + DS/Q + C_u\sigma_L g(Z)D/Q$$

where $\sigma_L g(Z) = N_L$, and g(Z) is the expected number of units short under normal distribution with $\sigma_L = 1$. Letting the partial derivative of TRC with respect to Q equal to 0, we obtain:

$$Q = (2D(S+C_u\sigma_L g(Z))/h)^{1/2} \qquad (3)$$

3. Substitute Z into equation (3), and determine the revised Q.

4. Substitute the revised Q into step 2 to find the revised Z.

5. Repeat steps 3 and 4 until Z and Q values are stabilized.

Once the optimal Z and Q are found, the optimal safety stock is defined by:

$SS^* = Z^*\sigma_L$

Exponential Distribution

When actual demand data can be properly described by the exponential distribution, it can be shown that (The derivations are given in the appendix):

$$SOR = P(D_L > r) = e^{-(1+Z)} \quad (4)$$

$$\sigma_L = \overline{D}_L$$

$$N_L = \sigma_L e^{-(1+Z)}$$

Using the condition of optimality in equation (1), the optimal service level under the exponential distribution must satisfy

$$e^{-(1+Z)} = hQ/C_u D \quad (5)$$

It is clear that shortage can occur only during the lead time period when the fixed order quantity inventory control system is used. Therefore, the ordering cost and shortage cost can be combined. The optimal order quantity including the shortage cost is given by:

$$Q = \{2D(S + C_u\sigma_L e^{-(1+Z)})/h\}^{1/2} \quad (6)$$

The Z value of equation (5) can be numerically evaluated using the initial value $Q = (2DS/h)^{1/2}$. Using the iterative process, as shown in the normal distribution case, we can find the optimal values of Z and Q. As a result, the optimal safety stock is given by

$$SS^* = Z^*\sigma_L = Z^*\overline{D}_L \quad (7)$$

The total relevant cost can still be determined by equation (2).

Poisson Distribution

For slow moving items, the poisson distribution would be most appropriate because of its special mathematical properties. The standard deviation for this distribution is equal to % mean. Aucamp [5] has shown that if the demand can be estimated by the poisson distribution, the optimal safety stock can be established by finding the optimal Z value. The formulas for Z and TRC are, respectively,

$$Z = \{2 \ln [C_u\sigma_L/(4\pi ICST)^{1/2}]\}^{1/2} \quad (8)$$

$$TRC = Qh/2 + DS/Q + Z\sigma_L h + C_U\sigma_L aD/Q$$

If we let dTRC/dQ = 0, we obtain

$$Q = [2D(S + C_u\alpha\,\sigma_L)/h]^{1/2} \quad (9)$$

where $a = P(D_L > r)$ and, T = lead time.

Variable Lead Time

If lead time is uncertain, the standard deviation must include both the variation of demand and the variation of lead time, i.e., σ_{LTD}. If the demand distribution is normal, then as shown in [4]:

$$\sigma_{LTD} = [LT\sigma_t^2 + (\overline{D}_t\,\sigma_{LT})^2]^{1/2}$$

Where the first term is the demand variation given mean lead time and the second term is the lead time variation given mean demand. Similarly, if the distribution is exponential, then

$$\sigma_{LTD} = [LT\sigma_t^2 + (\overline{D}_t{}^2 LT)^2]^{1/2}, \text{ since } \sigma_{LT} = LT$$

and, if the distribution is poisson,

$$\sigma_{LTD} = [LT(\sigma_t^2 + \overline{D}_t]^{1/2}, \text{ since } \sigma_{LT} = (LT)^{1/2}.$$

It is obvious that the safety stock would be much higher under the variable lead time situation ($Z\sigma_{LTD} > Z\sigma_L$). The total relevant cost can be calculated using equation (2) by replacing σ_L with σ_{LTD}.

Lost Sales Case

The above safety stock policies are assumed to be of the backorder shortage case. In the lost sales case, the stockout cost, C_u, includes foregone revenue. Adding one unit to the order point incurs hQ/D in carrying costs. If we do not add the unit, the penalty of stockout is C_u and one extra unit of inventory will be held through the next cycle (because a full supply of Q is on hand to start the next cycle and in the backorder case the beginning inventory is one unit less). Therefore,

$$hQ/D = P(D_L > r)(C_u + hQ/D)$$

$$SOR = P(D_L > r) = hQ/(hQ + C_u D). \qquad (10)$$

is the condition of optimality. The optimal service level for the various distributions must be equal to $1 - hQ/(hQ + C_u D)$. This indicates that a higher service level will be realized because of increased safety stock size in comparison to the backorder case.

Unknown Distribution

There are cases in which we do not know or have sufficient data to construct the specific demand distribution. Our knowledge may be confined to only the average demand and standard deviation. Since there are many probability distributions with same mean and standard deviation, it is very difficult to determine the optimal safety stock without the knowledge of the specific distribution. In order to provide some protection against possible stockouts under this situation, we may be forced to use the well-known Chebychev inequality theory. Of course, the safety stock policy can be improved when more information becomes available. Cheychev's inequality theory [10] states that

"Given probability distribution with the mean and standard deviation, the probability of obtaining a value within k standard deviations of the mean is at least $1-1/k^2$."

This approach is a very conservative approach. The total relevant cost, in this case, is given by:

$$TRC = Qh/2 + DS/Q + Z\sigma_L h + C_u \sigma_L D/QZ^2$$

where z = k, and, $\sigma_L/Z^2 = N_L$

Solving the TRC equation, we obtain the following optimal values for Z and Q:

$$Z = (2C_u D/hQ)^{1/3} \qquad (11)$$

$$Q = [2D(S + C_u \sigma_L/Z^2)]^{1/2} \qquad (12)$$

We set the initial value Q = (2DS/h), then the iterative process described earlier could be used to find the optimal Z and Q. Thus,

$$SS^* = Z^* \sigma_L.$$

Illustrative Example

Suppose we have h = \$10, D = 1250 units, C_u = \$18.8, S = \$500. Lead time = 1 week. The demand distribution during the lead time period is given as follows:

Demand (units)	Probability
0	0.22
10	0.22
20	0.14
30	0.11
40	0.11
50	0.08
60	0.06
70	0.06

The actual mean = 25, standard deviation = 22. To determine whether a particular theoretical distribution applies to the actual data, the chi-square (C^2) goodness-of-fit test can be used. The C^2 test indicates that the exponential distribution fits the actual data best in this case. The theoretical exponential distribution has the property that mean = standard deviation = 25. Assuming the backorder case, the condition of optimality is

SOR = $P(D_L > r) = hQ/C_uD$ = 10(353)/ 18.8(1250) = 0.15

where the initial value $Q = (2SD/h)^{1/2} = 353$.

By equation (5), thus, $e^{-(1+Z)} = hQ/C_uD = 0.15$

and by evaluating Z numerically, we obtain Z = 0.9 and the revised

$$Q = \{ \frac{2(1250)(500 + (18.8)(25)(0.15)}{10} \}^{1/2} = 377$$

By iteration, the revised SOR = 0.16 = $e^{-(1+Z)}$. The values of Z and Q are stabilized at Z = 0.85, Q = 379. The optimal safety stock

$SS^* = Z^*\sigma_L = 0.85(25) = 21$. By equation (2) we obtain:

TRC = 379($10)/2 + 1250($500)/379 + 0.85 (25) ($10)
+ $18.8(25(0.516) = $3997.

The SS = 21 with the SOR = 0.16 satisfies the order service level of 84%. If the demand distribution was wrongly assumed to be normal or poisson, the results, by equations (1) through (3) and (9), would have been as follows:

	Normal	Poisson
Mean	25	25
Std. dev.	22	5
Z	1.01	1.02
Q	364	358
SS	22	5
TRC	$4004	$4084

For example, if the poisson distribution was selected to estimate the demand distribution, we would have assumed that the standard deviation = the square root of the mean. Therefore, we would have found that SS = 5 and the order point = 30. It is clear that the safety stock level, in this case, would be too low for the demand distribution which was actually exponential. The probability of shortage (SOR) for SS = 5 is approximately 30% (SS = Z(25)=5, Z=0.2, therefore, $e^{-(1+0.2)} = 0.30$) which is much higher than the condition of optimality at 16%. As a result, the TRC = $4084. Similarly, the TRC would have been $4004 if the normal distribution was used.

Unknown Distributions

If the manager has only limited information and has estimated that the mean of the demand during the lead time = 25 and the standard deviation = 22, then by equations (11) and (12) we get:

$$Z = \frac{\{2(18.8)(1250)\}^{1/2}}{(1)(335)} = 2.37$$

$Q = [2(1250)[500 + 18.8/(2.37)_2]/10]^{1/2} = 355;$

the optimal safety stock policy is SS = 2.37(22) = 52, and,

$SL \geq 1 - \frac{1}{(2.37)^2} = 82\%$. This means that if we set safety stock level at 52 units, the service level is at least 82%. This is the best a firm can hope for to provide some protection against stockout situations without sufficient information. Of course, the safety stock level can be improved when more information becomes available.

SUMMARY

Under conditions of demand uncertainty, safety stock must be established to provide protection against possible stockouts during the lead time period. In this paper, we have emphasized the following in order to determine the lot size and the optimal safety stock policy:

1. Service level must be clearly defined, OSL or USL;
2. In order to properly determine the optimal safety stock policy, an appropriate theoretical demand distribution should be chosen using the chi-square test;
3. Because of the interrelationship between safety stock level and lot size, Z and Q must be jointly determined;
4. In case of insufficient data, the well-known Chebychev inequality theory can be used to set up the safety stock level until more information becomes available.

CONCLUSIONS AND MANAGERIAL IMPLICATIONS

Generally, it is believed that production and inventory managers are not interested in optimal solutions. Reasons often given include:

(1) they are used to rules of thumb that have provided satisfactory solutions in the past; and,

(2) they are not aware of the availability of better solution methods.

In this paper, we have shown that settling for a satisfactory solution is not enough. These findings are consistent with findings of many other researchers on decision-making [e.g. 14, and 15]. These researchers have found that cognitive limitations, cost, and limits on time force individual and group decision-makers to choose simplistic/heuristic models which provide approximate solutions to problems facing them. A number of additional researchers have provided consistent evidence of direct and indirect antecedent effects of environmental characteristics upon information utilization within organizations (e.g. 16, and 17). The environmental variables reported in these studies include: environmental uncertainty, complexity and threat.

Production and inventory management decisions are highly complex and subject to uncertainty. They are influenced by the aforementioned cognitive and environmental variables. Managers may gain a time advantage in seeking approximate solutions rather than optimal solutions to their inventory control problems. Nevertheless, these approaches do not minimize the overall inventory cost and do not take the consequences of environmental uncertainty into consideration. Over the long run, the firm might be vulnerable to losing its competitive position in the market place. For example, consider the intense competition that exists among the major mega-retailers, such as Wal-Mart, K-Mart, and so on. In these and similar situations, managers need to improve the quality of their inventory policies. Deciding to use optimal inventory policies such as those presented here do not take much additional manager's time and can dramatically improve the firm's ability to gain a competitive advantage.

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Matthew J. Liberatore, Ph.D. is the John F. Connelly Chair in Management and Professor of Decision and Information Technologies in the College of Commerce and Finance at Villanova University. Dr. Liberatore received a BA in mathematics, and MS and Ph.D. degrees in operations research, all from the University of Pennsylvania. He previously taught at Temple University and held management positions at RCA and FMC Corporation. At Villanova, he previously served as Chair of the Department of Management and as Associate Dean. Dr. Liberatore has published extensively in the fields of management science, information systems, project management, and research and engineering management. His current research focuses on project planning and scheduling and the use of decision support systems for technology-based organizations and health care decision-making applications. He currently serves on the editorial boards of the *American Journal of mathematical and Management Sciences and IEEE Transactions on Engineering Management*, and previously served on the board of *Entrepreneurship Theory* and Practice and as area editor for Production and operations management for *Interfaces*. He is a member of DSI, INFORMS, and the Project Management Institute.

Appendix

Assume demand for an item in lead time "L" has distribution $P(D_L)$, where D_L is demand during "L". The exponential distribution is defined as

$$P\,(D_L)\,dD_L = \lambda \exp\,(-\,\lambda\, D_L) * dD_L$$

where

$$\lambda = \frac{1}{\overline{D}_L}, \qquad \text{and} \quad \sigma_L = \overline{D}_L$$

$\overline{D}_L$ being the mean demand during "L".

Then the following two formulas can be derived:

$$P(D_L>r) = \qquad \lambda e^{-\lambda D_L}\, dD_L = e^{-\lambda r}$$

substituting

$$r = \overline{D}_L + Z_L\sigma_L$$

we get

$$P(D_L>r) = e^{-\frac{1}{\overline{D}_L}(\overline{D}_L+Z_L\sigma_L)} = e^{-(1+Z)}$$

where Z is the number of standard deviations and,

$$N_L = \sigma_L e^{-\lambda r} = \sigma_L e^{-(1+Z)}$$

where

$$\sigma_L = \overline{D}_L$$