A Note on Norlund Summability of Fourier Series

by

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ABSTRACT

In this paper we consider point-wise summability in the Norlund sense of Fourier series by assuming

$H(n) \, . \, F(n) = o\,(P_n)$ as $n \to \infty$, where $(H(x) = {}_1\int^x g(t)\,dt$, $g(t)$ is continuous, non-increasing, $H(x)$ is slowly varying, and $F(x)$ is positive non-decreasing. THE CASE $F(x) = 1$, and $g(x) = \frac{1}{x}$ IS THE RESULT IN [1].

Introduction

1. let $\sum_{k=1}^{\infty} U_k$ be a given series, and $\{S_n\}$ denote the sequence of its partial sums. Let

$\{p_n\}$ be a sequence of real numbers with $p_o > o$, $p_n \geq 0$ for $n = 1,2, \ldots\ldots$, and

$$p_n = \sum_{k=o}^{n} p_k .$$

Define $t_n = \frac{1}{P_n} \sum_{k=o}^{n} p_{n-k} S_k$. If $\lim_{n \to \infty} t_n = S$

we say that $\sum_{k=1}^{\infty} u_k$ is summable to S in the Norlund sense or $S(N,p_n)$.

The regularity conditions for the (N,p_n) method are:

1. $\frac{p_n}{p_n} = o(1)$ as $n \to \infty$, and
2. $\sum_{k=o}^{n} |P_k| = O(|P_n|)$ as $n \to \infty$.

Key words : Fourier series, Norlund summability.

2. Assume that f(t) is a periodic function with period 2π, integrable in the sense of Lebesgue over the interval $(-\pi, \pi)$.

Let the Fourier series of f(t) be:

$$\frac{1}{2} a_o + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt. \text{ Let}$$

$$\varnothing(t) = f(x+t) + f(x-t) - 2 f(x),$$

$$\Phi(t) = {}_0\int^t |\varnothing(u)| du, \text{ and } T = \left[\frac{1}{t}\right] \text{ be}$$

the integral part of $\frac{1}{t}$.

MAIN RESULT: We have the following theorem:

Theorem: If (N,p_n) is a regular Norlund method, defined by real, non-negative, monotonic, non-increasing sequence of coefficients $\{p_n\}$, such that $p_n \to \infty$, and $H(n) . F(n) = O(P_n)$ as $n \to \infty$,

where $H(x) = {}_1\int^x g(t)\, dt$, g(t) is continuous, non-increasing, H(X) is slowly varying, and is positive non-decreasing.

and if

$$\Phi(t) = {}_0\int^t | \varnothing(u) | du = o \left(\frac{g(\frac{1}{t}) . F(\frac{1}{t})}{P_\tau} \right) \text{ as } t \to +0,$$

then the Fourier series of f(t) at t= x, is summable (N,p_n) to f(x).

THE CASE F(x) = 1, AND $g(x) = \frac{1}{x}$ IS THE RESULT IN [1].

PROOF: We have

$$S_n(x) = \sum_{k=1}^{n} A_k(x) ;$$

$$S_n(x) - f(x) = \frac{1}{2\pi} {}_0\int^{\pi} \varnothing(t) \frac{\sin (n + \frac{1}{2}) t}{\sin \frac{1}{2} t} dt;$$

hence

$$t_n(x)-f(x) = \frac{1}{P_n} \sum_{k=o}^{n} p_n \; S_{n-k}(x) - f(x),$$

$$= \frac{1}{P_n} \sum_{k=o}^{n} p_k \frac{1}{2\pi} \int_o^{\pi} \varnothing(t) \frac{\sin\left(n-k+\frac{1}{2}\right)}{\sin \frac{1}{2} t} dt$$

$$= \int_0^{\pi} \varnothing(t) \frac{1}{2\pi P_n} \sum_{k=o}^{n} p_k \frac{\sin\left(n-k+\frac{1}{2}\right) t}{\sin \frac{1}{2} t} dt$$

$$= \int_o^{\pi} \varnothing(t) \; K_n(t) \; dt, \text{ say.}$$

In order to prove the theorem we show

$$\int_0^{\pi} \varnothing(t) K_n(t) \; dt = o(1) \qquad \text{as } n \to \infty .$$

Now

$$\int_\sigma^{\pi} \varnothing(t) \; K_n(t) \; dt = dt \left(\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{1} + \int_1^{\pi} \right) \varnothing(t) K_n(t) \; dt,$$

$$= I_1 + I_2 + I_3, \text{ say.}$$

First by [2] we have:

$$I_1 = \int_0^{\frac{I}{n}} \varnothing(t) \; K_n \; (t) \; dt$$

$$= O\left(\int_0^{\frac{I}{n}} | \varnothing \; (t) | \; | K_n(t) | \; dt \right)$$

$$= O(n\int_{0}^{\frac{1}{n}} |\varnothing(t)|\, dt)$$, and hence by our hyprthesis we have:

$$I_1 = O\left(n. \frac{g(n).F(n)}{P_n}\right)$$

$$= O\left(n. \frac{g(n).F(n)}{H(n).F(n)}\right)$$

$$= O\left(\frac{n.g(n)}{H(n)}\right).$$

Now since $H(x)$ is slowly varying with a monotone, non-increasing derivative $H'(x) = g(x)$ it can be shown easily (see [3]) that

$$\frac{x.H'(x)}{H(x)} = \frac{x.g(x)}{H(x)} = o(1) \text{ as } x \rightarrow \infty.$$

Hence $I_1 = o(1)$ as $n \rightarrow \infty$.

Second it follows by the Riemann Lebesgue theorem, and the regularity of the metod (N,p_n) that,

$$I_3 = \int_{1}^{\pi} \varnothing(t)k_n(t)\, dt = o(1) \quad \text{as } n \rightarrow \infty .$$

Third by Tamarkin and Hie's Lemma [4] we have:

$$I_2 = \int_{\frac{1}{n}}^{1} \varnothing(t)\, K_n(t)\, dt = O\left(\frac{1}{P_n} \int_{\frac{1}{n}}^{1} |\varnothing(t)|\, \frac{P_T}{t}\, dt\right).$$

Now

$$\frac{1}{P_n} \int_{\frac{1}{n}}^{1} |\varnothing(t)|\, \frac{P_T}{t}\, dt = \left(\int_{\frac{1}{n}}^{\frac{1}{n-1}} + \int_{\frac{1}{n-1}}^{\frac{1}{n-2}} + \dots + \int_{\frac{1}{2}}^{1}\right) |\varnothing(t)|\, \frac{P_T}{t}\, dt .$$

Hence integrating by parts and simplifying we obtain :

$$\frac{1}{P_n} \int_{\frac{I}{n}}^{I} |\varnothing(t)| \frac{P_T}{t} dt - o(1) = \frac{1}{P_n} \check{\Phi}(t) \frac{P_T}{t} \Bigg]_{\frac{1}{n}}^{1} + \frac{1}{P_n} \int_{\frac{I}{n}}^{I} \check{\Phi}(t) \frac{P_T}{t^2} dt$$

Now

$$\frac{1}{P_n} \tilde{\Phi}(t) \frac{P_T}{t} \Bigg]_{\frac{1}{n}}^{1} = O\left(\frac{1}{P_n}\right) + o\left(\frac{1}{P_n} \cdot \frac{g(n).\ f(n)}{P_n} . n.P_n\right)$$

$$= O\left(\frac{1}{P_n}\right) + o\left(\frac{n.g(n)}{H(n)}\right) = o\ (1) \text{ as } n \to \infty$$

$$\frac{1}{P_n} \int_{\frac{I}{n}}^{I} \tilde{\Phi}(t) \frac{P_T}{t^2} dt = \frac{1}{P_n} \int_{I}^{n} \tilde{\Phi}\left(\frac{1}{u}\right) P_u\ du$$

$$= o\left(\frac{1}{P_n} .F(n) \int_{I}^{n} g(u)\ du\right)$$

$$= o\left(\frac{F(n).H(n)}{F(n).H(n)}\right) = o\ (1) \quad \text{as } n \to \infty .$$

This completes the proof of the theorem.

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مجمـــــوع نورلند لسـلاسل فورير

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كليـــة العلـوم ـ جامعة الإمارات العربية المتحدة

ملخـــــــــــــص

في هذا البحث ننظر إلى المجموع النفطي لسلاسل فورير بطريقة نورلند ونحصل على نتيجة أعم من التي حصل عليها في (١) حيث نفرض أن

$$H(n).F(n) = 0(P_n) \; n \to \infty, \qquad H(x) = \int^{x} g(t)dt,$$

$g(t)$ هي دالة مستمرة ، غير متزايدة ، $H(x)$

دالة متغيرة ببطىء ، $F(x)$ موجبة وغير متناقضة .

الحالة $F(x) = 1$ و $g(x) = \frac{1}{x}$ هي النتيجة في (١)