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Filtered Dehomogenization Theory

For The Micro-structure sheves $\overset{\mu}{\underline{O}}Y$

By

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نظرية Dehomogenization المنقاه لحزمة البناء الدقيق $\underline{O}_y^{\mu}$

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تم في هذا البحث دراسة وتعميم نظرية Dehomogenization وذلك باستخدام حزم البناء الدقيق المنقاه $\underline{O}_y^{\mu}$ والمعرفة على الفراغ الهندسي(Y= sec^g (G(R)) وهذه النتائج لها معنى عند استخدام النسخة الكمية بالتقييد إلى الاجزاءات ذات الدرجة الصفرية .

Key Words: Micro-structure sheaves, Filtered sheaves, Quantum sheaves, Dehomogenization

ABSTRACT

In this work we study the theory of dehomogenization at the level of filtered sheaves using the micro-structure sheaves. For the sheaf of quantum section the results are still true.

0. INTRODUCTION

In projective algebraic geometry homogeneous coordinate rings appear together with a suitable dehomogenization. For example, if V (I) is a projective variety determined by a homogenous ideal I of the polynomial ring $k[x_0, \dots, x_n]$ and R is the graded coordinate ring $k[x_0, \dots, x_n]/I$ then $R/(1-\bar{x}_0)R$ is isomorphic to the coordinte ring of the open affine subvariety comlementary to the hyper plane at infinity in V (I). In a similar way every determinantal ring is a dehomogenization of a Schubert cycle (being the graded coordinate ring of a Schubert variety) and this dehomogenization principle is the basis for the study of determinantal rings. We now extend this to sheaf level. We study the theory of dehomogenization at the level of filtered sheaves using the micro-structure sheaf $\underline{O}^{\mu}_{Y}$ (see [1], [3], [131]) in this studying.

As in section 2 we prove that $\underline{O}^{\mu}_{Y}$ is a dehomogenization of its associated Rees sheaf $\tilde{\underline{O}}^{\mu}_{Y}$. For the sheaf of quantum sections (see [1], [131]) $(\underline{O}^{\mu}_{Y})^{q}=\mathcal{F}_0(\underline{O}^{\mu}_{Y})=\tilde{\underline{O}}^{\mu}_{Y}$ the result is still true.In section 3 we do the opposite, i. e. we show that by taking a suitable dehomogenization a graded sheaf may be made into Rees sheaf associated to Zariski filtered sheaf. Again, the opposite is true for the sheaf of quantum sections.

So in this way, one also gets that $\underline{O}^{\mu}_{Y}$ behaves as well as R with respect to this dehomogenization theory.

1. PRELIMINARIES

In fact we use the same preliminaries of [1] and let us recall them again.

All rings considered are associative rings with unit. Modules are left modules and ideals are two-sided ideals unless otherwise stated. A filtration FR on a ring R is given by an ascending chain ...$\subset F_{n-1}R \subset R \subset F_nR \subset$ ... of additive subgroups F_nR, $n\in\mathbb{Z}$, satisfying:

$1\in F_0R$, $F_nRF_mR \subset F_{n+m}R$ for n,

$m\in\mathbb{Z}$, $\bigcup_{n\in\mathbb{Z}} F_nR = R$.

A filtered R-module is an R-module M with filtration FM given by an ascending chain of additive subgroups ... $\subset F_{n-1}M \subset F_n M \subset$... satisfying: $F_nR F_mM \subset F_{n+m}M$ for n, $m\in\mathbb{Z}$, $\bigcup_{n\in\mathbb{Z}} F_nM = M$. We write R-filt for the (non-abelian) category of filtered modules with degree preserving morphisms.

To the filtration FR there corresponds the associated graded ring $G(R)=\oplus_{n\in\mathbb{Z}}F_nR/F_{n-1}R$ and similarly an associated graded module. $G(M) = \oplus_{n\in\mathbb{Z}}F_nM/F_{n-1}M\in G(R)\text{-gr}$ is associated to FM.

If $x\in F_nM-F_{n-1}M$ for some $n\in\mathbb{Z}$ then we say that x has filtrationdegree n and the principle symbol map σ is defined by putting $\sigma(x) = x$ modulo $F_{n-1}(M)$ when x has filtration degree n and $\sigma(x)=0$ when $x\in F_nM$ for all $n\in\mathbb{Z}$. A filtration FM is separated when $\bigcap_{n\in\mathbb{Z}}F_nM = 0$; for a separated filtration $\sigma(x) = 0$ if and only if $x = 0$ holds. There is another graded ring that may be associated to FR, namely $\oplus_{n\in\mathbb{Z}} F_nR$. We may identify this so called Rees ring of F R to the subring

$$\sum_{n\in\mathbb{Z}}F_n R X^n = \tilde{R} \text{ in } R[X, X^{-1}]$$

where X is a central variable over R; it is homogeneous of degree one. To an $M\in R\text{-filt}$ we may correspond a graded $\tilde{R}$-module $\tilde{M} = \sum_{n\in\mathbb{Z}}F_n M X^n \subset M[X, X^{-1}]$.

We denote by $\mathcal{F}_X$ the full subcategory in $\tilde{R}$-gr consisting of the graded $\tilde{R}$-modules that are also X-torsion free.

A filtration FM is said to be a good filtration if for every $n\in\mathbb{Z}$, $F_nM=\sum_{i=0}^{\nu} F_{n-d_i}R.m_i$ for certain $m_0, \dots, m_\nu\in M$ and some $d_0,\dots,d_\nu\in\mathbb{Z}$. A filtration FR is faithful when all good filtrations are automatically separated.

Amongst the filtered morphisms there are some that behave better with respect to the functor $G: R\text{-filt}\to G(R)\text{-gr}$ as far as exactness properties are being considered. A filtered morphism f: M $\to$ N is said to be strict when

$$f F_n M = f M \cap F_nN, \quad n\in\mathbb{Z}.$$

For a filtered inclusion i: M $\to$ N strictness of i just states that M is considered with the filtration induced by FN on

M. Note however that a good filtration FN need not induce a good filtration on a submodule M. Recall from [5], the following.

1.1. Lemma. With conventions and notation as above:

a. $\tilde{R}/X\tilde{R} \cong G(R)$, $\tilde{R}/(1-X)\tilde{R} \cong R$, $\tilde{R}_X \cong R[X, X^{-1}]$.

b. $\tilde{M}/X\tilde{M} \cong G(\tilde{M})$, $\tilde{M}/(1-X)\tilde{M} \cong M$, $\tilde{M}_X \cong M[X, X^{-1}]$

where $(-)_X$ denotes the (graded) localization at the multiplicative central set $\{1, X, X^2, \dots\}$.

c. The functor ~: R-filt $\longrightarrow$ $\tilde{R}$-gr defines an equivalence of categories between R-filt and $\mathcal{F}_X$.

1.2. Lemma. With notation as before:

a. FM is good if and only if $\tilde{M}$ is finitely generated.

b. A filtered morphism f: M $\rightarrow$ N is strict exactly then when Coker $\tilde{f} \in \mathcal{F}_X$ i. e. when $\tilde{f}$ is a morphism in F_X. A strict sequence in R- filt transforms to an exact (graded) sequence in F_X.

c. FR is faithful if and only if $F_{-1}R \subset J(F_0R)$, , the Jacobson radical of F_0R, if and only if , the Jacobson graded radical of $\tilde{R}$

d. FR has the property that good filtration induce good filtrations on submodules if $\tilde{R}$.is left Noetherian.

We say that FR is a Zariskian filtration when $X \in J^g(\tilde{R})$ and $\tilde{R}$ is Noetherian (sinilar for the left or right notions). Unless other-wise stated we always consider a Zariskian filtration FR. Let S be a multiplicative set in R , $1 \in S$, $0 \notin S$, such that σ (S) is a multiplicative set in G (R), $1 \in \sigma(S)$, $9 \notin \sigma(S)$, satisfying the left Ore conditions (e. g. we may take

$$S = \{r \in R,\ \sigma(r) \in \sigma(S),\ r \neq 0\}).$$

For any $M \in R\text{-filt}$ we define the micro-localization of M at σ(S) by $\sigma(S)$ by $Q^{\mu}_{S}(M) = Q^{\mu}_{\tilde{S}}(\tilde{M})/(1 - X)Q^{\mu}_{\tilde{S}}(\tilde{M})$

where $Q^{\mu}_{\tilde{S}}(\tilde{M}) = \varprojlim_n Q^{g}_{\tilde{S}(n)}(\tilde{M}/X^n\tilde{M})l$ and

$\tilde{S} = \{\tilde{r} \in \tilde{R},\ \tilde{r} = sX^m$ for $s \in S$ with filtration degree $m\}$,

$\tilde{S}(n) = \{\tilde{r} \bmod X^n\tilde{R}$ for $\tilde{r} \in \tilde{S}\}$, cf. [5].

1.3. Theorem. WithThe following properties hold:

1. The $\tilde{R}$-module $Q^{\mu}_{\tilde{S}}(\tilde{M})$ is X-torsion free.
2. $FQ^{\mu}_{S}(M)$ is separated and complete.
3. $G(Q^{\mu}_{S}(M)) \cong Q^{\mu}_{S(s)}(G(M)) = \sigma(S)^{-1}G(M)$; $G(Q^{\mu}_{S}(R)) = \sigma(S)^{-1}G(R)$.
4. The right $\tilde{R}$ - module $Q^{\mu}_{\tilde{S}}(\tilde{R})$ is flat.
5. $Q^{\mu}_{S}(R)$ is the microlocalization of R at σ (S) in the sence of [5], i. e. it satisfies the universal property mentiond in Loc. cit.

We need some basic theory of sheaves and coherent sheaves but we refer to the classical paper [10] or to the books [6], [11]. In the sequel we assume that FR is a Zariskian filtration such that G (R) is a commutative domain, this situation is general enough in the sense that it allows application of the results to most of the important examples: enveloping algebras of Lie algebras, Weyl algebras, any rings of differential operators as well as the classical commutative Zariski rings that show up in singularity theory.

On the topological space Y=Proj G (R) with its Zariski topology having the Y (I) = {P∈Proj G (R), P⊃I}, I varies over the principle graded ideals of G (R), for a basis, we may define two structure-sheaves. First we may associative to Y (f), f∈G (R), the graded ring$Q^{g}_{G(R)}(G(R)) = G(R)[f^{-1}]$ and we obtain the fraded structuresheaf $\underline{O}_Y$ = Proj G(R). In a similar way we may associate to each graded G (R)-odule N the graded and the usual structure sheaf. $\underline{O}^{g}_{N}$ and $\underline{O}_N$ resp. over Y. Write $\mathcal{B}$ for the basis of the Zariski topology on Y consisting of the open sets Y (f), f∈G (R). When G (R) is not positively graded then consider Y=Spec^{g}G(R), the graded prime spectrum. We may define the Zariski topology by $Y(f)=\{P \in \mathrm{Spec}^g G(R),\ f \notin P\}$ and we may define $\underline{O}^{g}_{Y}$ and $\underline{O}_Y$ as before. Up to the final section the results we establish are insensitive to the definition of Y as Proj G (R) or as Spec^{g}G(R). Moreover, now as in [2], it is possible that Proj (G(R)) = Spec^g(G(R)). So unless otherwise specified Y is either ProjG (R) or Spec^{g}G(R); it is clear in the positively graded case Spec^{g}G(R) is a somewhat strange topological space so that in this case we will automativally assume Y = Proj G (R), [4].

Associating to an open set $Y(f) \in \mathcal{B}$ the micro-loalization $Q^{\mu}_{\sim}(\tilde{R})$, resp. $Q^{\mu}_{f}(R)$, we obtain sheaves $\tilde{\underline{O}}^{\mu}_{Y}$, resp. $^{f}\underline{O}^{\mu}_{Y}$, having as the completed stalks at $P\in Y$ the rings $Q^{\mu}_{P}(\tilde{R})$, resp. $Q^{\mu}_{P}(R)$ see [13]. Note that $\underline{O}^{\mu}_{Y}$ is a sheaf of Zariski rings.

Repacing R by any filtered R-module M in the above leads to the construction of an $\tilde{\underline{O}}^{\mu}_{Y}$-Module $\tilde{\underline{M}}^{\mu}_{Y}$ over the ringed space $\tilde{O}^{\mu}_{Y}$ and also an $\underline{O}^{\mu}_{Y}$-Module $\underline{M}^{\mu}_{Y}$ over the ringed space $\underline{O}^{\mu}_{Y}$, where we have written $\tilde{\underline{M}}^{\mu}_{Y}$ and $\underline{M}^{\mu}_{Y}$ for the sheaves defined by the presheaves constructed (note: in case M is such that G (M) is absolutely torsionfree then the presheaves of microlocalization are sheaves). The sheaves of quantum section for G (R), resp. G (M), are obtained by looking at the parts of filtration degree zero: $F_0\underline{O}^{\mu}_{Y}$, resp. $F_0\underline{M}^{\mu}_{Y}$. These may be viewed as parts of degree zero in the graded sense for the corresponding Rees sheaves $F_0\underline{O}^{\mu}_{Y} \cong (\underline{O}^{\mu}_{Y})_0$, $F_0\underline{M}^{\mu}_{Y} \cong (\underline{M}^{\mu}_{Y})_0$. We refere to [13] for some basic propereties and definition.

In [5], [8] and [6], [12], the use of graded techniques in $\tilde{R}$-gr allowed to obtain the desired results for filtered objects. Here we plan to use the same strategy when studying coherent sheaves of modules and their relations with structure sheaves of modules of sections, but we have to modify the technique somewhat when dealing with micro-localization and micro-structuresheaves because the elements of sections and stalks are not fractions. By a theorem from [5] we know that $Q^{\mu}_{f}(R)=(S_f^{-1}R)^{\wedge}$ where $S_f=\{r\in R,\ \sigma(r)\in\{1,f,\dots\}\}$ and ^ stands for completion of the ring of fractions with respect to the localized filtration. Therefore it makes sense to use $\underleftarrow{\mathrm{Lim}}$ and the "slicing-up of $\tilde{R}$" " in terms of $\tilde{R}/X^n\tilde{R}$ for every $n\in\mathbb{N}$ in order to reuce all problems to the graded case and consideration of homogeous fractions $Q^{g}_{S_f}(\tilde{R}/X^n\tilde{R})$ for every $n\in\mathbb{N}$.

When looking at sheaves of filtered and modules over some sheaf of filtered rings it may happen that the restriction morphisms in the sheaf change the filtration degree of some elements even if these morphisms are filtered morphisms, in other words it may not be possible to define a "principle symbole" globally on the sheaf of filtered modules even though the sheaf of associated graded modules is perfectly well-defined. This technicality is avoided if we define a filtered sheaf $\underline{M}$ of modules over a filtered sheaf of rings $\underline{R}$ by assuming that we have subsheaves of additive groups $\mathcal{F}_n\underline{R}$, $\mathcal{F}_n\underline{M}$ for $n\in\mathbb{Z}$ satisfying the expected conditions. Of course $\underline{O}^{\mu}_{Y}$ is such a filtered sheaf of rings because the restriction morphisms on principle Zariski open sets are strict filtered morphisms and under assumptions they are even strict filtered monomorphisms. A filtered sheaf morphism between filtered sheaves, $f\colon \underline{M} \to \underline{N}$ say, is then one such that for all $n\in\mathbb{Z}$, $f(\mathcal{F}_n\underline{M}) \subset \mathcal{F}_n\underline{N}$ and we say that f is a strict sheaf morphism if for all stalks the induced stalk-mophism is strict. For coherent and filtered sheaves of modules we now introduce the equivalent of a good filtration and fillows. If $\underline{M}$ is a coherent filtered sheaf of (filtered) $\underline{R}$-modules then we say that $\underline{M}$ is coherently filtered if for every open U we have an exact sequence of sheaf morphisms

$$(\underline{R}|_U)^m \xrightarrow{f_u} (\underline{R}|_U)^n \xrightarrow{g_v} \underline{M}|_U \longrightarrow 0$$

for some $n, m\in\mathbb{N}$, where f_u and g_v are now supposed to be a strict sheaf morphism.

When $\underline{M}$ is a filtered sheaf of $\underline{O}^{\mu}_{Y}$-modules then we may define a graded sheaf $\tilde{\underline{O}}^{\mu}_{Y}$-modules by putting $\tilde{\underline{M}} = \oplus_{n\in\mathbb{Z}}\mathcal{F}_n\underline{M}$ where for all $\tilde{\underline{M}}(U) = \oplus_{n\in\mathbb{Z}}(\mathcal{F}_n\underline{M})(U) = \oplus_{n\in\mathbb{Z}}F_n(\underline{M}(U))$ and $\tilde{\underline{M}}_y = \oplus_{n\in\mathbb{Z}}F_n\underline{M}_y$ for all Zariski open sets $U\in Y$ and every $y\in Y$.

We say that $\tilde{\underline{M}}$ is the (graded) Rees sheaf of $\underline{M}$ and the sheaf determined by $\oplus_{n\in\mathbb{Z}}[\mathcal{F}_n\underline{M}(U)/\mathcal{F}_{n-1}\underline{M}(U)]=G(\underline{M})$ is called the associated graded sheaf of $\underline{O}^{g}_{G(R)}$-modules.. It is not a too difficult excercise to verify that the structure sheaf $\underline{M}_Y$ constructed before is in fact a coherently filtered sheaf of complete $\underline{O}^{\mu}_{Y}$-modules.

For a general filtered sheaf $\underline{O}^{\mu}_{Y}$-modules $\underline{M}$ we still have: $\tilde{\underline{M}}/\underline{X}\tilde{\underline{M}} \cong G(\underline{M})$ and $\tilde{\underline{M}}/(1-\underline{X})\tilde{\underline{M}} \cong \underline{M}$ as sheaves where $\underline{X}$ is the "constant" global section. Again it makes sense to say that $\underline{M}$ is $\underline{X}$-torsion-free and as in [1]. The coherence of $\tilde{\underline{M}}$ implies the coherence of G and the coherence of $\underline{M}$. Under some conditions, cf. P. Shapira's book [11], the coherence of $G(\underline{M})$ alone will imply the coherence of $\underline{M}$, but we do not need this here.

We need too some basic theory of dehomogenizations but we refer to [7], and [9].

2. GRADED SHEAVES FROM FILTERED SHEAVES AND DEHOMOGENIZATION

There are two associated functors from the category of coherently filtered sheavves $\underline{O}_Y^{\mu}$-modules to the category of graded sheaves $G(\underline{O}_Y^{\mu})$-modules and the category of graded sheaves $\tilde{\underline{O}}_Y^{\mu}$-modules respectively. That is, if $\underline{M}_Y \in \underline{O}_Y^{\mu}\text{-filt}$ is coherently filtered that

$\tilde{\underline{M}}_Y \in \tilde{\underline{O}}_Y^{\mu}\text{-Gr}$ and $G(\underline{M}_Y) \in G(\underline{O}_Y^{\mu})\text{-Gr}$.

As in [8] denote by $\underline{X}$ the global central regular section of $\tilde{\underline{O}}_Y^{\mu}$ determined by $X\tilde{Q}_{\tilde{f}}^{\mu}(R)$ for each $Y(f) \in \beta$. . In fact that $\underline{X}$ is the image of the unit section $\underline{1} \in \mathcal{F}_1 \underline{O}_Y^{\mu}$ in $(\tilde{\underline{O}}_Y^{\mu})_1$. $\underline{M}_Y \in \tilde{\underline{O}}_Y^{\mu}\text{-Gr}$ is said to be $\underline{X}$- torsionfrelocally.

2.1. Lemma. For every $\underline{M}_Y \in \underline{O}_Y^{\mu}\text{-filt}$ that is coherently filtered. It follows that $\tilde{\underline{M}}_Y$ is $\underline{X}$-torsionfree.

Proof. From tha above definition and [12], [13].

These $\tilde{\underline{M}}_Y$, with $\underline{M}_Y \in \underline{O}_Y^{\mu}\text{-filt}$ such that $\underline{M}_Y$ is coherently filtered, form a full subcategory of $\tilde{\underline{O}}_Y^{\mu}\text{-Gr}$ denoted by $\mathcal{F}_{s\underline{X}}$.

Denote by $\underline{I}_Y = \underline{X}\,\tilde{\underline{O}}_Y^{\mu}$ the sheaf of graded ideals in $\tilde{\underline{O}}_Y^{\mu}$, i. e., for every $Y(f) \in \beta$: $\underline{I}_Y(Y(f)) = X\,\tilde{Q}_{\tilde{f}}^{\mu}(\tilde{R})$ which is a graded ideal in $\tilde{\underline{O}}_Y^{\mu}(Y(f)) = \tilde{Q}_{\tilde{f}}^{\mu}(\tilde{R})$.

2.2. Theorm. With notation as above:

a. $\tilde{\underline{O}}_Y^{\mu}/\underline{X}\tilde{\underline{O}}_Y^{\mu} \cong G(\underline{O}_Y^{\mu})$, $\tilde{\underline{M}}_Y^{\mu}/\underline{X}\tilde{\underline{M}}_Y^{\mu} \cong G(\underline{M}_Y^{\mu})$ as graded sheaves.

b. $\tilde{\underline{O}}_Y^{\mu}/(1-\underline{X})\tilde{\underline{O}}_Y^{\mu} \cong \underline{O}_Y^{\mu}$, $\tilde{\underline{M}}_Y^{\mu}/(1-\underline{X})\tilde{\underline{M}}_Y^{\mu} \cong \underline{M}_Y^{\mu}$ as filtedered sheaves. Moreover,

$\tilde{\underline{M}}_Y^{\mu}/(1-\underline{X})\tilde{\underline{M}}_Y^{\mu} \cong \varinjlim_{n} (\tilde{\underline{M}}_Y^{\mu})_n$ where the map in the inductive system of sheaves of groups are given by the multiplication of $\underline{X}$ and there are isomorphisms of sheaves of additive groups $\mathcal{F}_n \tilde{\underline{M}}_Y^{\mu} \cong ((\tilde{\underline{M}}_Y^{\mu})_n + (1-\underline{X})\tilde{\underline{M}}_Y^{\mu} \cong \underline{M}_Y^{\mu})/(\underline{1}-\underline{X})\tilde{\underline{M}}_Y^{\mu}$, $n \in \mathbb{Z}$.

Proof. a. For each $Y(f) \in \beta$ we can see:

$$(\tilde{\underline{O}}_Y^{\mu}/\underline{X}\tilde{\underline{O}}_Y^{\mu})(Y(f)) = \tilde{Q}_{\tilde{f}}^{\mu}(\tilde{R})/X\tilde{Q}_{\tilde{f}}^{\mu}(\tilde{R}) = (Q_f^{\mu}(R))^{\sim}/X(Q_f^{\mu}(R))^{\sim} = G(Q_f^{\mu}(R)) = Q_f^{\mu}(G(R)),$$

where the final equality follows from theorem 3.8 in [13]. On the other hand $G(\underline{O}_Y^{\mu})Y(f) = Q_f^{\mu}(G(R))$. . Hence $\tilde{\underline{O}}_Y^{\mu}/\underline{X}\tilde{\underline{O}}_Y^{\mu} \cong G(\underline{O}_Y^{\mu})$, for each $Y(f) \in \beta$, so $\tilde{\underline{O}}_Y^{\mu}/\underline{X}\tilde{\underline{O}}_Y^{\mu} \cong G(\underline{O}_Y^{\mu})$. . Similarly we can arrive at $\tilde{\underline{M}}_Y^{\mu}/\underline{X}\tilde{\underline{M}}_Y^{\mu} \cong G(\underline{M}_Y^{\mu})$. . It is observed that they are sheaf isomorphisms because there are isomorphisms on the stalks.

b. As in a., we can prove that $\tilde{\underline{O}}_Y^{\mu}/(1-\underline{X})\tilde{\underline{O}}_Y^{\mu} \cong \underline{O}_Y^{\mu}$ and $\tilde{\underline{M}}_Y^{\mu}/(1-\underline{X})\tilde{\underline{M}}_Y^{\mu} \cong \underline{M}_Y^{\mu}$.

But for the sheaf $\varinjlim_{n} (\tilde{\underline{M}}_Y^{\mu})_n$ we see, for $Y(f) \in \beta$:

$$(\varinjlim_{n} (\tilde{\underline{M}}_Y^{\mu})_n)\, Y(f) = \varinjlim ((\tilde{\underline{M}}_Y^{\mu})_n(Y(f))) = Q_f^{\mu}(M) = \underline{M}_Y^{\mu}(Y(f))$$

So $\varinjlim_{n} (\tilde{\underline{M}}_Y^{\mu})_n \cong \underline{M}_Y^{\mu}$ as filtered sheaves over $\underline{O}_Y^{\mu}$. .

We have seen that $\tilde{\underline{M}}_Y^{\mu}/(1-\underline{X})\tilde{\underline{M}}_Y^{\mu}$ is a filtered sheaf such that for every $n \in \mathbb{Z}$ we get

$$((\tilde{\underline{M}}_Y^{\mu})_n + (\underline{1}-\underline{X})\tilde{\underline{M}}_Y^{\mu})/(\underline{1}-\underline{X})\tilde{\underline{M}}_Y^{\mu}$$

is a sheaf of groups defined on Y by:

$$\left[((\tilde{\underline{M}}_Y^{\mu})_n + (\underline{1}-\underline{X})\tilde{\underline{M}}_Y^{\mu})/(\underline{1}-\underline{X})\tilde{\underline{M}}_Y^{\mu}\right](Y(f))$$
$$= \left((Q_{\tilde{f}}^{\mu}(\tilde{M}))_n + (1-X)Q_{\tilde{f}}^{\mu}(\tilde{M})\right)\Big/(1-X)Q_{\tilde{f}}^{\mu}(\tilde{M})$$
$$= \left[\left((Q_f^{\mu}(M))^{\sim}\right)_n + (1-X)(Q_f^{\mu}(M))^{\sim}\right]\Big/(1-X)(Q_f^{\mu}(M))^{\sim}$$
$$\cong F_n(Q_f^{\mu}(R)) = (\mathcal{F}_n\,\underline{M}_Y^{\mu})(Y(f)).$$

This is for each $Y(f) \in \beta$, , so

$$\mathcal{F}_n\,\underline{M}_Y^{\mu} \cong ((\tilde{\underline{M}}_Y^{\mu})_n + (\underline{1}-\underline{X})\tilde{\underline{M}}_Y^{\mu})/(\underline{1}-\underline{X})\tilde{\underline{M}}_Y^{\mu}, \qquad n \in \mathbb{Z}$$

2.3. Theorem. ForWith hypothesis as above:

a. The functor $(\quad)^{\sim}$ defined above determines an equivalence of categories coherently filtered $\underline{O}_Y^{\mu}$ -torsionfree $\tilde{\underline{O}}_Y^{\mu}$ -modules in $\mathcal{F}_{s\underline{X}}$. In particular, every coherent $\underline{X}$- torsionfree $\tilde{\underline{O}}_Y^{\mu}$-module $\underline{M}_Y$ is of the form for some $\underline{M}_Y^{\mu} \in \underline{O}_Y^{\mu}\text{-filt}$ which is coming from a $M \in R\text{-filt}$ with good filtration FM.

b. The localization of $\tilde{\underline{O}}_Y^{\mu}$ at the multiplicative closed set of global sections $\{\underline{1}, \underline{X}, \underline{X}^2, \ldots\}$ equals to $\underline{O}_Y^{\mu}[\underline{X}, \underline{X}^{-1}]$ denoted by $(\tilde{\underline{O}}_Y^{\mu})_{\underline{X}}$. Also $(\tilde{\underline{M}}_Y^{\mu})_{\underline{X}} = \underline{M}_Y^{\mu}[\underline{X}, \underline{X}^{-1}]$.

Proof. a. It follows from Lemma 2.1. above and Theorem 2.6 in [1].

b. To prove these statements we need some ideas about localization of sheaf of modules, but for this we refer to [14]. Now the statements are local hence for each Y (f) we see that

$$\underline{O}_Y^{\mu}[\underline{X}, \underline{X}^{-1}](Y(f)) = \underline{O}_Y^{\mu}(Y(f))[\underline{X}, \underline{X}^{-1}] = Q_f^{\mu}(R)[\underline{X}, \underline{X}^{-1}] =$$
$$\left((Q_f^{\mu}(R))^{\sim}\right)_X = \left[\tilde{\underline{O}}_Y^{\mu}(Y(f))\right]_X = (\tilde{\underline{O}}_Y^{\mu})_{\underline{X}}(Y(f)).$$

Hence $(\tilde{\underline{O}}_Y^{\mu})_{\underline{X}} \cong \underline{O}_Y^{\mu}[\underline{X}, \underline{X}^{-1}]$ for each $Y(f) \in \beta$. So

$$(\tilde{\underline{O}}_Y^{\mu})_{\underline{X}} \cong \underline{O}_Y^{\mu}[\underline{X}, \underline{X}^{-1}].$$

Similarly, we may parove that $(\tilde{\underline{M}}_Y^{\mu})_{\underline{X}} \cong \underline{M}_Y^{\mu}[\underline{X}, \underline{X}^{-1}]$.

2.4. Remark. The foregoing Theorems 2.2 and 2.3. are, in fact, the basis for many results on filtered sheaf theory (in particular, on Zarishian filtered sheaf theory). From both it is clear that $\underline{O}^{\mu}_{Y}$ is a dehomogenization of its associated graded Rees sheaf $\tilde{\underline{O}}^{\mu}_{Y}$.

Given the micro-structure sheaf $\underline{O}^{\mu}_{Y}$ and $\underline{M}^{\mu}_{Y}\in\underline{O}^{\mu}_{Y}$-filt that is coherently filtered. Hece by theorem 2.2. we see that $\underline{O}^{\mu}_{Y} \cong \tilde{\underline{O}}^{\mu}_{Y}/(\underline{1}-\underline{X})\tilde{\underline{O}}^{\mu}_{Y}$, $\underline{M}^{\mu}_{Y} \cong \tilde{\underline{M}}^{\mu}_{Y}/(\underline{1}-\underline{X})\tilde{\underline{M}}^{\mu}_{Y}$

are filtered sheaves defined on Y with respect to the filtrations

$\mathcal{F}_n(\tilde{\underline{O}}^{\mu}_{Y}/(\underline{1} - \underline{X})\tilde{\underline{O}}^{\mu}_{Y}) = ((\tilde{\underline{O}}^{\mu}_{Y})_n + (1 - \underline{X})\tilde{\underline{O}}^{\mu}_{Y})/(\underline{1} - \underline{X})\tilde{\underline{O}}^{\mu}_{Y}$,

$\mathcal{F}_n(\tilde{\underline{M}}^{\mu}_{Y}/(\underline{1}-\underline{X})\tilde{\underline{M}}^{\mu}_{Y})=((\tilde{\underline{M}}^{\mu}_{Y})_n + (1-\underline{X})\tilde{\underline{M}}^{\mu}_{Y})/(\underline{1}-\underline{X})\tilde{\underline{M}}^{\mu}_{Y}$

respectively. There are sheaf isomorphisms

$\mathcal{F}_n(\tilde{\underline{O}}^{\mu}_{Y}/(\underline{1} - \underline{X})\tilde{\underline{O}}^{\mu}_{Y}) = \mathcal{F}_n\underline{O}^{\mu}_{Y}$, $\mathcal{F}_n(\tilde{\underline{M}}^{\mu}_{Y}/(\underline{1} - \underline{X})\tilde{\underline{M}}^{\mu}_{Y})=\mathcal{F}_n\underline{M}^{\mu}_{Y}$.

Hence at n = 0 we get that:

$((\tilde{\underline{O}}^{\mu}_{Y})_o + (\underline{1} - \underline{X})\tilde{\underline{O}}^{\mu}_{Y})/(\underline{1}-\underline{X})\tilde{\underline{O}}^{\mu}_{Y}\cong\mathcal{F}_n\underline{O}^{\mu}_{Y}\cong(\tilde{\underline{O}}^{\mu}_{Y})_o$,

$((\tilde{\underline{M}}^{\mu}_{Y})_o + (\underline{1} - \underline{X})\ \underline{M}^{\mu}_{Y})/(\underline{1} - \underline{X})\tilde{\underline{M}}^{\mu}_{Y} \cong \mathcal{F}_n\ \underline{M}^{\mu}_{Y} \cong (\tilde{\underline{M}}^{\mu}_{Y})_o$.

This implies that the quantum of $\underline{O}^{\mu}_{Y}$ is a dehomogenization of its shifted associated quantum Rees sheaf.

3. FILTERED SHEAVES FROM GRADED SHEAVES AND DEHOMOGENIZATION

In this section we do the opposite, i. e. we show that by taking a suitable dehomogenization many graded sheaves of rings may be made into Rees sheaves of rings associated to Zariski filtered sheaves of rings. So in this way one also gets information for a given graded sheaf from its associated filtered sheaf.

Let $\underline{H}_Y = \oplus_{n\in\mathbb{Z}}\underline{H}_{Y,n}$ be any $\mathbb{Z}$-graded sheaf of rings defined on a topological space Y. Associate to Y a base of open subsets β. Let $\underline{X}$ be a homogeneous global section of degree 1 in $\underline{H}_Y$. This $\underline{X}$ determines for each U∈β a global section $\underline{X}|_U = X_{\underline{H}_Y(U)}$ whereas

$\underline{H}_Y(U) = \oplus_{n\in\mathbb{Z}}\ \underline{H}_{Y,n}(U) = \oplus_{n\in\mathbb{Z}}(\underline{H}_Y(U))_n$.

For every homogeneous section h_n of $\underline{H}_{Y,n}$ over U and t>0 we have $h_n = X^t h_n + (1-X^t)h_n$.

The sheaf $\underline{D}_Y = \underline{H}_Y/(\underline{1}-\underline{X})\underline{H}_Y$, Where $(\underline{1}-\underline{X})\underline{H}_Y$ is a sheaf of ideals in $\underline{H}_Y$ defined on Y, may be made into a filtered sheaf of rings by endowing it with the filtration:

$\mathcal{F}_n\underline{D}_Y=(\underline{H}_{Y,n}+(\underline{1}-\underline{X})\underline{H}_Y)/(\underline{1}-\underline{X})\underline{H}_Y$, $n\in\mathbb{Z}$.

We can see that the filtration $\mathcal{F}\ \underline{D}_Y$ on $\underline{D}_Y$ defined above is such that $\bigcup_{n\in\mathbb{Z}} \mathcal{F}_n\ \underline{D}_Y$ and $\underline{1}\in\mathcal{F}_o\underline{D}_Y$.

3.1. Lemma. Witth notations as above, if $\underline{X}$ is a regular homogeneous global section in $\underline{H}_Y$ then

a. $(\underline{1}-\underline{X})\underline{H}_Y \cap \underline{H}_{Y,n} = 0$: the zero sheaf for all $n\in\mathbb{Z}$.

b. If $\underline{X}$ is also a normal global section ($\underline{X}\ \underline{H}_Y = \underline{H}_Y\ \underline{X}$) then $\underline{X}$ is a central global section if and only if and only if $(\underline{1} - \underline{X})\ \underline{H}_Y$ is a sheaf of ideals in $\underline{H}_Y$.

Proof. a. Clear.

b. To prove that $(\underline{1} - \underline{X})\ \underline{H}_Y$ is a sheaf of ideals in $\underline{H}_Y$, let U∈β, $(\underline{1} - \underline{X})\underline{H}_Y(U) = (1-X)(\underline{H}_Y(U))$ which is an ideal in $\underline{H}_Y(U)$. Similarly$(\underline{H}_Y(\underline{1} - \underline{X}))(U) = \underline{H}_Y(U)(1 - X) = (1-X)\underline{H}_Y(U) = ((\underline{1}-\underline{X})\underline{H}_Y)(U)$, since $X_{\underline{H}_Y(U)}$ is regular homogeneous normal central in $\underline{H}_Y(U)$. But to prove that $\underline{X}$ is regular homogeneous normal central in $\underline{H}_Y(U)$.. But to prove that $\underline{X}$ is central it will be sufficient to be central over U, U∈β. So let us consider the sheaf morphism $\underline{H}_Y \xrightarrow{\phi} \underline{H}_Y/(\underline{1}-\underline{X})\underline{H}_Y$, hence for U∈β we have $\underline{X}|_U = X_{\underline{H}_Y(U)}$, $\phi(U)$: $\underline{H}_Y(U) \longrightarrow \underline{H}_Y(U)/(1 - X)\underline{H}_Y(U)$.

Now $\underline{H}_Y(U)$ is a graded ring and $X_{\underline{H}_Y(U)}$ is normal. Let us assume that X is not central this implies that there is a $s\in\underline{H}_Y(U)$, , $X\ s\neq s\ X$ i.e. $(s-Xs)\neq(s-sX)$ and have the same image under $\phi(U)$. This is contradiction, therefore $\underline{X}$ is central for each U∈β. Hence $\underline{X}$ is central.

Let $\underline{H}_Y=\oplus_{n\in\mathbb{Z}}\underline{H}_{Y,n}$ be a graded sheaf on Y. Define $\underline{J}^g(\underline{H}_Y)$, graded sheaf of ideals in $\underline{H}_Y$, such that for U∈β: $\underline{J}^g(\underline{H}_Y(U))$, the graded Jacobson radical sheaf of $\underline{H}_Y$. Here we have to associate a sheaf $\underline{J}^g(\underline{H}_Y) \cap \underline{H}_{Y,o}$ which be defined by for each U∈β:

$$(\underline{J}^g(\underline{H}_Y) \cap \underline{H}_{Y,o})(U) = \underline{J}^g(\underline{H}_Y(U)) \cap (\underline{H}_Y(U))_o$$

3.2. Theorem. Let $\underline{H}_Y$ be a graded sheaf of graded rings and $\underline{X}$ a regular central homogeneous blobal section of degree 1 in $\underline{H}_Y$, with notations as above then

a. $\tilde{\underline{D}}_Y \cong \underline{H}_Y$ as graded sheaf on Y.

b. $G(\underline{D}_Y) \cong \underline{H}_Y/\underline{X}\ \underline{H}_Y$ as graded sheaf.

Proof. a. Since

$\tilde{\underline{D}}_Y = \oplus_{n\in\mathbb{Z}}(\tilde{\underline{D}}_Y)_n = \oplus_{n\in\mathbb{N}} \mathcal{F}_n\ \underline{D}_Y = \oplus_{n\in\mathbb{N}} \frac{\underline{H}_{Y,n}+(\underline{1}-\underline{X})\underline{H}_Y}{(\underline{1}-\underline{X})\underline{H}}$

and $\underline{H}_Y = \oplus_{n\in\mathbb{Z}} \underline{H}_{Y,n}$. For each n one may define a sheaf isomorphism of sheaves of groups:

$\phi_n: (\underline{H}_{Y,n}+(\underline{1}-\underline{X})\underline{H}_Y)/(\underline{1}-\underline{X})\underline{H}_Y \longrightarrow \underline{H}_{Y,n}$ as follows: let $U\in\beta$ be a basic open subset of Y, we have the group homomrphism :

$\phi_n(U)': \underline{H}_{Y,n}(U) + (1-X)\underline{H}_Y(U) \longrightarrow \underline{H}_{Y,n}(U)$,

defined by $h_n + (1-X)\underline{H}_Y(U) \longmapsto h_n$,

which is surjective with kernel

$\ker \phi_n(U)' = \{h_n + (1-X)\underline{H}_Y(U) : h_n = 0\} = (1-X)\underline{H}_Y(U)$.

So $[(\underline{H}_{Y,n} + (\underline{1}-\underline{X})\underline{H}_Y)/(\underline{1}-\underline{X})\underline{H}_Y](U) \rightarrow \underline{H}_{Y,n}(U)$

is a group isomorphism. Hence ϕ_n is an isomorphism for each $U\in\beta$, so ϕ_n is a sheaf isomorphism. Combine all ϕ_n to obtain the required sheaf isomorphism. This proves $\tilde{\underline{D}}_Y \cong \underline{H}_Y$ as graded sheaves defined on Y.

b. By definition

$$G(\underline{D}_Y) = \oplus_{n\in\mathbb{N}} \frac{(\underline{H}_{Y,n} + (\underline{1}-\underline{X})\underline{H}_Y)/(\underline{1}-\underline{X})\underline{H}_Y}{(\underline{H}_{Y,n-1}+(\underline{1}-\underline{X})\underline{H}_Y)/(\underline{1}-\underline{X})\underline{H}_Y}$$

$$= \oplus \left[(\underline{H}_{Y,n} + (\underline{1}-\underline{X})\underline{H}_Y)/(\underline{H}_{Y,n-1} + (\underline{1}-\underline{X})\underline{H}_Y)\right]$$

and

$$\underline{H}_Y/\underline{X}\ \underline{H}_Y \cong \oplus_{n\in\mathbb{N}} (\underline{H}_Y/\underline{X}\ \underline{H}_Y)_n = \oplus_{n\in\mathbb{N}} [(\underline{H}_{Y,n} + \underline{X}\ \underline{H}_Y)/X\ \underline{H}_Y]$$

For each n, one may define a sheaf isomorphism of sheaves of additive groups;

$$\phi_n: \frac{\underline{H}_{Y,n} + (\underline{1}-\underline{X})\underline{H}_Y}{\underline{H}_{Y,n-1}+(\underline{1}-\underline{X})\underline{H}_Y} \longrightarrow \frac{\underline{H}_{Y,n} + \underline{X}\ \underline{H}_Y}{\underline{X}\ \underline{H}_Y}$$

as follows, for each $U\in\beta$ define

$$\phi_n(U): (\underline{H}_{Y,n}(U) + (1-X)\underline{H}_Y(U))/(\underline{H}_{Y,n-1}(U) + (1-X)\underline{H}_Y(U) \longrightarrow$$

$$\longrightarrow (\underline{H}_{Y,n}(U) + X\ \underline{H}_Y(U))/X\underline{H}_Y(U)$$

by

$$h_n + \underline{H}_{Y,n-1}(U) + (1-X)\underline{H}_Y(U) \longmapsto h_n + X\underline{H}_Y(U)$$

From; since for every $h_{n-1}\in\underline{H}_{Y,n-1}(U)$, we have $h_{n-1}-(1-X)h_{n-1}=Xh_{n-1}$ it follows that $\phi_n(U)$ is well defined. Clearly, $\phi_n(U)$ is a surjective map. Moreover, if if $h\in X\ \underline{H}_Y(U)$ then $h_n = X\ h_{n-1}$, $h_{n-1}\in\underline{H}_{Y,n-1}(U)$ hence

$h_n = X\ h_{n-1} = h_{n-1} - (1-X)\ h_{n-1}\in\underline{H}_{Y,n-1}(U) + (1-X)\underline{H}_Y(U)$

then, $\phi_n(U)$ is injective and this shows that $\phi_n(U)$ is a group isomorphism. So ϕ_n is a group isomorphism for each $U\in\beta$ then ϕ_n is a sheaf isomorphism. We shall obtain the required sheaf isomorphism if we combine all ϕ_n.

3.3. Proposition. Consider the same hypothesis in theorem 3.2. then $\underline{X}\in\underline{J}^g(\underline{H}_Y)$ if and only if $\mathcal{F}_{-1}\underline{D}_Y \subset \underline{J}(\mathcal{F}_0\underline{D}_Y)$ where $\underline{J}(\mathcal{F}_0\underline{D}_Y)$ denotes the Jacobson radical sheaf of ideals in $\mathcal{F}_0\ \underline{D}_Y$.

Proof. The statement is local so, let be a basic open subset of Y, we have to show that

$\underline{X}|_U\in\underline{J}^g(\underline{H}_Y)(U)$ if and only if $\mathcal{F}_{-1}\underline{D}_Y(U) \subset (\underline{J}(\mathcal{F}_0\underline{D}_Y))(U)$.

Now let $\underline{X}|_U\in\underline{J}^g(\underline{H}_Y)(U) = \underline{J}^g(\underline{H}_Y(U))$, $\underline{H}_Y(U)$ graded ring it follows that

$X\,\underline{H}_{Y,-1}(U) \subset \underline{J}^g(\underline{H}_Y(U)) \cap \underline{H}_{Y,0}(U)$ i.e. $1-Xh_{-1}$ is invertible in $\underline{H}_{Y,0}(U)$ for every $h_{-1}\in\underline{H}_{Y,-1}(U)$, but, then it fillows from $h_{-1} = Xh_{-1} + (1-X)h_{-1}$, $h_{-1}\in\underline{H}_{Y,-1}(U)$, that $(\underline{H}_{Y,-1}(U)+(1-X)\underline{H}_Y(U))/(1-X)\underline{H}_Y(U) \subset J[(\underline{H}_{Y,0}(U)+(1-X)\underline{H}_Y(U))/(1-X)\underline{H}_Y(U)]$. So $\mathcal{F}_{-1}\underline{D}_Y(U) \subset (\underline{J}(\mathcal{F}_0\underline{D}_Y))(U)$ for each $U\in\beta$. Hence $\mathcal{F}_{-1}\underline{D}_Y \subset \underline{J}(\mathcal{F}_0\underline{D}_Y)$ as sheaves of groups. Conversely, since for each $U\in\beta$ we have $\underline{H}_{Y,n}(U) \cap (\underline{1}-\underline{X})\underline{H}_Y(U) = \underline{0}$, for every n it follows that $\underline{X}|_u\in\underline{J}^g(\underline{H}_Y)(U) = \underline{J}^g(\underline{H}_Y(U))$. Hence $\underline{X}\in\underline{J}^g(\underline{H}_Y)$.

3.4. Corollary. With notation and conventions as above the localization of $\underline{H}_Y = \oplus_{n\in\mathbb{Z}}\underline{H}_{Y,n}$ at the Ore sheaf set $\{\underline{1},\underline{X},\underline{X}^2,\dots\}$ exists, it is denoted by $(\underline{H}_Y)_{\underline{X}}$, which is a graded sheaf of rings defined on Y such that there is a commutative diagram of sheaf morphism

$$\begin{array}{ccc} \underline{H}_Y & \longrightarrow & (\underline{H}_Y)_{\underline{X}} \\ \downarrow & & \downarrow \\ \underline{H}_Y \mid (\underline{1}-\underline{X})\underline{H}_Y & \longrightarrow & \underline{D}_Y \end{array}$$

Proof. One can define $(\underline{H}_Y)_{\underline{X}}$ as follows: for each $U\in\beta$, $(\underline{H}_Y)_{\underline{X}}(U)=(\underline{H}_Y(U))_{\underline{X}}$, where $X=X_{\underline{H}_Y(U)}$, the localization of the graded ring $\underline{H}_Y(U)$ at the Ore set $\{\underline{1},\underline{X},\underline{X}^2,\dots\}$. since

$(\underline{H}_Y(U))_{\underline{X}}$ is a graded ring hence $(\underline{H}_Y)_{\underline{X}}$ is a graded sheaf on Y such that

$(\underline{H}_Y)_{\underline{X}} = \oplus_{n\in N}((\underline{H}_Y)_{\underline{X}})_n ; ((\underline{H}_Y)_{\underline{X}})_n(U) = ((\underline{H}_Y(U))_{\underline{X}})_n, n\in\mathbb{Z}.$

For each $U\in\beta$ one may define a homomorphism of rings: $(\underline{H}_Y)_{\underline{X}}(U) \rightarrow \underline{D}_Y(U)$; $X^j h \longmapsto h + (1-X)\underline{H}_Y(U)$

where $h\in\underline{H}_{Y,n-j}(U)$, $j\in\mathbb{Z}$

and this define a sheaf morphism $(\underline{H}_Y)_{\underline{X}} \rightarrow \underline{H}_Y/(\underline{1}-\underline{X})\underline{H}_Y$.

3.5. Example. Let $\underline{H}_Y$ be any $\mathbb{Z}$- graded sheaf of graded rings defined on Y. Then $\underline{H}_Y[T]$ is a graded sheaf; for each $U\in\beta$: $\underline{H}_Y[\underline{T}](U) = \underline{H}_Y(U)[T]$ and $\underline{H}_Y[\underline{T}]/(\underline{1}-\underline{X})\underline{H}_Y[\underline{T}]$ is a filtered sheaf (i. e., sheaf of filtered rings), has an associated graded sheaf $\underline{H}_Y[\underline{T}]/\underline{T}\ \underline{H}_Y[\underline{T}] \cong \underline{H}_Y$ as graded sheaves and a Rees graded sheaf isomophic to $\underline{H}_Y[\underline{T}]$ as graded sheaves.

3.6. Remark. Todo the opposite of the result concerning the quantum case, let $\underline{H}_Y$ be any coherent graded sheaf defined on Y. Hence

$$\left((\underline{H}_Y/(\underline{1}-\underline{X})\underline{H}_Y)^{\sim}\right)_0 \cong (\underline{H}_Y)_0 \qquad (*)$$

where $\underline{H}_Y | (\underline{1}-\underline{X})\underline{H}_Y = \underline{D}_Y$ may be made as above into a filtered sheaf by

$\mathcal{F}_n\underline{D}_Y = ((\underline{H}_Y)_n + (\underline{1}-\underline{X})\underline{H}_Y)/(\underline{1}-\underline{X})\underline{H}_Y$.

But from (*) we can conclude the required result. So the theory dehomogenization can be applied to the quantum level.

A filtered sheaf $\underline{O}_Y$ defined on Y, that has a base β as above, is said to be a Zariski filtered sheaf on Y if its associated Rees sheaf $\tilde{\underline{O}}_Y$ is a Noetherian graded sheaf on Y and $\mathcal{F}_{-1}\underline{O}_Y \subset \underline{J}(\mathcal{F}_0\underline{O}_Y)$ as sheaves on Y.

Now we are ready to prove the final result of this note that $\underline{D}_Y$ mentioned above may be made into a Zariski filtered sheaf deffined on Y.

3.7. Corollary. With notation and considerations as before, if $\underline{X}\in\underline{J}^g(\underline{H}_Y)$ and $\underline{H}_Y$ is a Noetherian graded sheaf on Y then $\tilde{\underline{D}}_Y$ will be Zariski filtered on Y.

Proof. From Theorem 3.2. we have seen that $\tilde{\underline{D}}_Y \cong \underline{H}_Y$, hence $\underline{D}_Y$ is Noetherian graded on Y. From Proposition 3.3. we have seen that $\underline{X}\in\underline{J}^g(\underline{H}_Y)$ is equivalent to $\mathcal{F}_{-1}\underline{D}_Y \subset \underline{J}(\mathcal{F}_0\underline{D}_Y)$ as sheaves on Y, hence the result follows.

3.8. Final remark. This is an important dehomogenization theory. For more cases and results we may continue to find applications. We shall do this in the near future.

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