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Bayes Estimation of the Parameter of Burr Type II Distribution in Case of Type II Censoring

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تقديرات بييز لمعلمة توزيع بر من النموذج الثاني في حالة التوقف من النوع الثاني

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بافتراض ان لدينا عينة عشوائية حجمها n من توزيع بر من النموذج الثاني ذي المعلمة θ فقد تم ايجاد تقديرات بييز لهذه المعلمة في حالة التوقف من النوع الثاني أي بمشاهدة أول r من أزمنة الفشل حيث $1 \leq r \leq n$ ، واستعمال الكثافة القبلية المرافقة وشبه الكثافة القبلية وذلك تحت اقتراني الخسارة من نوع مربع الخطأ والاقتران الخطي الأسي .

لقد تمت مقارنة التقديرات بحساب خطر بييز لها .

Key words : Burr type II distribution, Squared error loss, Linex loss function, Bayes estimator, risk function, Bayes risk, Type II Censoring.

ABSTRACT

Suppose that $X_1, X_2, ..., X_n$ is a random sample of failure times from a Burr type II (B2) distribution with parameter θ. Let $\underline{x} = (x_1, x_2, ..., x_r)$ with $x_1 \leq x_2 \leq x_3 \leq .... \leq x_r$ be the observed failure censored sample of the first r-failure times from the sample of size n, where $1 \leq r \leq n$.

We derive serveral Bayes estimators of θ assuming two priors for θ namely, quasi-prior and conjugate prior, and considering two types of loss functions.

The loss functions used are squared error and LINEX error loss functions. Comparison between the estimators was carried through computation of their Bayes risks.

1. INTRODUCTION

The Burr distribution was first introduced in the literature by Burr [1]. Inference on the parameters of Burr disributions has been the subject of investigation by many authors including Evans and Rajab [2], and Al-Marzoug and Ahmad [3].

Prediction intervals for future observations or statistics on future observations of Burr distributions are available in the literature. For example, Evans and Ragab [2] gave the prediction bounds for the k-th order statistic in a sample from Burr XII (B12)-distribution in the case of censored sample.

Nigm [4] gave the prediction interval for the k-th order statistic in a future sample from B12-distribution baed on a type II censored observed sample from the same distribution.

Sartawi and Abu-Salih [5] obtained the Bayesian prediction intervals for the order statistics in the two sample and one sample case when samples are from the one-parameter Burr type X distribution. Abu-Salih and Sartawi [6], obtained the Bayesian prediction bounds for the Burr type III model. Our interest is to find estimators of q of B2 distribution and study their properties. The probability density function (p.d.f) of B2 is given by

$$f(x|\theta) = \theta e^{-x} (1+e^{-x})^{-(\theta+1)y}, \; -\infty < x < \infty , \theta > 0 \qquad (1.1)$$

We shall derive the Bayes estimators of θ with respect to two loss functions using different priors of θ in case of type II censoring.

In Bayesian estimation the loss function and prior distribution play important roles. The symmetric loss function viz, squared error loss function has been used widely. Varian [7] introduced a class of asymetric functions defined by

$$L(\Delta) = b \left[e^{a\Delta} - a\Delta - 1 \right], \; a \neq 0, b > 0,$$

where $\Delta = \tilde{\theta} - \theta$,

denotes the scalar estimation error in using $\tilde{\theta}$ to estimate θ , *a* and b are constants. This class is known as a class of linear exponential (LINEX) loss functions, and it may be recommended to use such class when the use of symmetric loss function is not appropriate, Zellner [8].

2. Bayes Estimators of θ with respect to Squared error loss function

Let X be a random variable (r.v.) from Burr distribution of type II with p.d.f (1.1). Consider a random sample X_1, X_2, ... , X_n of n items whose life times have the distribution given in (1.1).

let $\underline{x} = (x_1, x_2, ...x_r)$ with $x_1 \le x_2 \le x_3 \le ..... \le x_r$ be the observed failure censored sample of the first r-failure times from the sample of size n where $1 \le r \le n$.

Let $Y_i = \text{In}(1+ \bar{e}^{x^i})$. It is seen that the p.d.f. of Y_i is

$$g(y|\theta) = \theta e^{-\theta y}, \; \theta > 0, y \ge 0 \qquad (2.1)$$

We shall derive the Bayes estimators of θ w.r.t. squared error loss function under several priors. In case of no information about θ , we sugget to use the non informative prior, given by :

$$\Pi_1(\theta) = \frac{1}{\theta^d}, \; \theta > 0, d \ge 0 \qquad (2.2)$$

Notice that this is a quasi-prior.

Using (2.1), we find the likelihood function of θ given $\underline{y}$ as

$$L(\theta|t,y_r) = \frac{n!}{(n-r)!} \theta^r e^{-\theta t} e^{-(n-r)\theta y_r}, \; \theta > 0 \qquad (2.3)$$

where $T = \sum_{i=1}^{r} Y_i$

The posterior distribution of Θ in case of $\Pi_1(\theta)$ prior and censored sample is :

$$\Pi_1^*(\theta|t,y_r) = \frac{u^{r-d+1} \theta^{r-d} e^{-\theta u}}{\Gamma(r-d+1)}, \; \theta > 0, d < r+1 \qquad (2.4)$$

where $U = T + (n-r) Y_r$

The Bayes estimator $\tilde{\theta}$ of θ under squared error loss function is the posterior mean of Θ , given by

$$\tilde{\theta}_{sd} = \frac{r-d+1}{U}, \; d < r+1 \qquad (2.5)$$

Now let the prior of Θ be the natural conjugate, i.e. the gamma density

$$\Pi_2(\theta) = \frac{\beta^{\alpha}\theta^{\alpha-1}e^{-\theta\beta}}{\Gamma(\alpha)}, \alpha, \beta > 0, \theta > 0 \quad (2.6)$$

The posterior distribution of Θ under prior $\Pi_2(\theta)$ and censored sample is :

$$\Pi_2^*(\theta|t, y_r) = \frac{(u+\beta)^{r+\alpha}\theta^{r+a-1}e^{-\theta(u+\beta)}}{\Gamma(r+\alpha)}, \theta > 0 \quad (2.7)$$

The Bayes estimator of q under $\Pi_2(\theta)$ w.r.t squared error loss function is the posterior-mean of Θ, given by

$$\theta_{sG} = \frac{r+\alpha}{U+\beta} \quad (2.8)$$

We shall discuss the properties of these estimators in section 4.

3. Bayes Estimators of θ w.r. t. LINEX loss function

Using both the non informative and the conjugate priors of Θ, we shall consider estimation of θ when the loss function is the LINEX loss function

$$L(\Delta) = b\left[e^{a\Delta} - a\Delta - 1\right], a \neq 0, b > 0.$$

where $\Delta = \hat{\theta} - \theta$ denotes the error if we estimate θ by $\tilde{\theta}$.

Zellner [8] gave the unique bayes estimator of θ using LINEX loss function as :

$$\hat{\theta}_L = \frac{-1}{a} \ln\left[E(e^{-a\Theta}|Y)\right] \quad (3.1)$$

where expectation is taken w.r.t. the posterior p.d.f. of Θ, provided that the expectation is finite. From (2.4) and (3.1), the bayes estimator of θ w.r.t LINEX loss function and noninformative prior is

$$\tilde{\theta}_{Ld} = \frac{r-d+1}{a} \ln\left(\frac{a+U}{U}\right), \text{ whenever } a > 0 \quad (3.2)$$

When $a < 0$, $E\left[e^{-a\Theta}|\underline{Y}\right]$ does not exist, when $u > -a$, and in such a case we use the following estimator of θ.

$$\theta^*_{Ld} = \begin{cases} \dfrac{r-d+1}{a} \ln\left(\dfrac{a+U}{a}\right), & U > -a \\ \\ \dfrac{r-d+1}{U}, & U \leq -a \end{cases} \quad (3.3)$$

Notice that θ^*_{LD} is not a Bayes estimator, when $U \leq -a$. In this case we chose $\theta^*_{Ld} = \tilde{\theta}_{sd}$ as given in 2.5 for simplicity and because $\tilde{\theta}_{sd}$ is admissible w.r.t. squared error loss when $d = 3$ and $r > 2$ (Lemma 2 below).

Again, from (2.4) and (3.1), we get the Bayes estimator of θ w.r.t. LINEX loss function and conjugate prior to be

$$\tilde{\theta}_{LG} = \frac{(n+\alpha)}{a} \ln \frac{n\bar{Y}+\beta+\alpha}{n\bar{Y}+\beta}, a > 0 \quad (3.4)$$

When $a < 0$, $E\left[e^{-a\Theta}|\underline{Y}\right]$ is not finite except when $u+\beta+a > 0$. Since $u \geq 0$, it is sufficient to assume that $\beta + a \geq 0$. Therefore, whenever $a < 0$, we retrict our consideration to the conjugate prior with $\beta > -a$, and get $\tilde{\theta}_{LG}$ as given in (3.4).

4. Properties of the Estimators

In the following we shall disuss some properties of the estimators given in sections 2 and 3 .

Lemma 1 :

When $d = 3$, the estimator $\tilde{\theta}_{sd}$ given in (2.5) becomes $\delta^*_1 = \frac{r-2}{U}$, $r > 2$, and has minimum mean squared error among all estimators of the form $c. \frac{1}{U}$, c constant.

Proof is straightforward.

Results :

1 - Any estimator $\frac{c}{U}$, $c \neq r-2$ is inadmissible w.r.t squared error loss.

2 - Bayes estimators of θ w.r.t squared error loss and quasi priors with $d \neq 3$ are inadmissible.

Lemma 2 :

$\delta_1^* = \dfrac{r-2}{U}$, r > 2, is admissible w.r.t squared error loss.

The proof of this lemma appears in Blyth and Roberts [9].

Lemma 3 :

(i) $\tilde{\theta}_{sG} = \dfrac{r-\alpha}{U+\beta}$ is admissible w.r.t squared error loss for all $\alpha, \beta > 0$.

(ii) $\tilde{\theta}_{LG} = \dfrac{r-\alpha}{a}\, ln \left(\dfrac{a+\beta+U}{\beta+U} \right)$ is admissible w.r.t LINEX error loss for all α , β > when a > 0, but when a < 0 we have to take β > - a .

Proof by lemma 8.7.1 (c.f Zacks [10] p.343).

We remark that the question of admissibility of $\tilde{\theta}_{Ld} = \dfrac{r-d+1}{a}\, ln \left(\dfrac{a+U}{U} \right)$ cannot be established by lemma 8.7.1 (c.f Zacks [10] p.343). since

$\Pi_1(\theta) = \dfrac{1}{\theta^d}$ is a quasi-prior.

5. Bayes risks of the estimators

In the following we shall give the formulas for Bayes risks of the estimators.

Let $r_s(.)$ and $r_L(.)$ denote the Bayes risks under conjugate prior w.r.t. squared error loss and LINEX loss functions, respectively, then :

$$r_s(\tilde{\theta}_{sd}) = \frac{\alpha(\alpha+1)}{(r-1)\beta^2}\,\frac{r+d^2-6d+7}{(r-2)} \tag{5.1}$$

$$r_L(\tilde{\theta}_{sd}) = b\left[\frac{\Gamma(r+\alpha)}{\Gamma(\alpha)}\beta^a \int_0 \frac{u^{r-1}}{(u+\beta+\alpha)r+\alpha} e^{-\theta\left(\frac{r-2}{u}\right)}du + \frac{\alpha a}{\beta(r-1)} - 1 \right] \tag{5.2}$$

Putting d = 3 in (5.1), and (5.2) we get :

$$r_s(\delta_1^*) = \frac{\alpha(\alpha+1)}{(r-1)\beta^2} \tag{5.3}$$

$$r_L(\delta_1^*) = b\left\{ \frac{\Gamma(r+\alpha)}{\Gamma(r)\,\Gamma(\alpha)}\beta^\alpha \int_0^\infty \frac{u^{r-1}}{(u+\beta+a)^{r+\alpha}} e^{a\left(\frac{r-2}{u}\right)} du + \frac{\alpha a}{\beta(r-1)} - 1 \right\} \tag{5.4}$$

$$r_s(\tilde{\theta}_{Ld}) = \frac{\left(\frac{r-d+1}{a}\right)}{\Gamma(r)}\,\frac{\beta^\alpha}{\Gamma(\alpha)}\Gamma(r+\alpha)\int_0^\infty \frac{u^{r-1}}{(u+\beta)^{r+\alpha}} \ln^2\left(\frac{a+u}{u}\right)du - \frac{2(r-d+1)}{a\,\Gamma(r)}\,\frac{\Gamma(r+\alpha+1)}{\Gamma(\alpha)}\beta^\alpha \int_0^\infty \frac{u^{r-1}}{(u+\beta)^{r+\alpha+1}}\ln\left(\frac{a+u}{u}\right)du + \frac{\alpha(\alpha+1)}{\beta^2}$$

$$\text{,when } a > 0 \tag{5.5}$$

$$\begin{aligned} r_s(\theta_{Ld}^*) = {} & \frac{\beta^\alpha}{\Gamma(\alpha)\Gamma(r)} \int_0^{-a} \frac{u^{r-3}\,\Gamma(r+\alpha)}{(u+\beta)^{r+\alpha}}du - \frac{2\beta^\alpha}{\Gamma(r)\Gamma(\alpha)}\int_0^{-a}\frac{u^{r-2}\,\Gamma(r+\alpha+1)}{(u+\beta)^{r+\alpha+1}}du \\ & + \frac{\beta^\alpha}{\Gamma(\alpha)\Gamma(r)}\Gamma(r+\alpha+2)\int_0^{-a}\frac{u^{r-1}}{(u+\beta)^{r+\alpha+2}}du \\ & + \frac{\left(\frac{r-d+1}{a}\right)}{\Gamma(r)\Gamma(\alpha)}\beta^\alpha\,\Gamma(r+\alpha)\int_{-a}^{\infty}\frac{u^{r-1}}{(u+\beta)^{r+\alpha}}\ln^2 du \\ & - \frac{2(r-d+1)}{a\Gamma(r)}\,\frac{\Gamma(r+\alpha+1)}{\Gamma(\alpha)}\beta^\alpha\int_{-a}^{\infty}\frac{u^{r-1}}{(u+\beta)^{r+\alpha+1}}\ln\left(\frac{a+u}{u}\right)du + \frac{\alpha(\alpha+1)}{\beta^2} \end{aligned}$$

$$\text{, when } a < 0 \tag{5.6}$$

$$r_L(\tilde{\theta}_{Ld}) = b\left\{ \frac{\beta^a}{\Gamma(\alpha)\,\Gamma(r)}\Gamma(r+\alpha)\int_0^\infty \frac{u^{d-2}(a+u)^{r-d+1}}{(a+u+\beta)^{r+\alpha}}du + \frac{a\alpha}{\beta} - \frac{(r-d+1)}{\Gamma(r)\Gamma(\alpha)}\Gamma(r+\alpha)\beta^\alpha \int_0^\infty ln\left(\frac{a+u}{u}\right)\frac{u^{(r-1)}}{(u+\beta)^{r+\alpha}}du - 1 \right\}$$

$$\text{, when } a < 0 \tag{5.7}$$

$$r_L(\theta^{*}_{Ld}) = b\left\{\left[\left[\frac{\beta^{\alpha}\,\Gamma(r+\alpha)}{\Gamma(\alpha)\Gamma(r)}\int_0^{-a}\frac{e^{\frac{a}{u}}\;u^{r-1}}{(a+\beta+u)^{r+\alpha}}\,du + \int_{-a}^{\infty}\frac{u^{d-2}\;(a+u)^{r-d+1}}{(a+u+\beta)^{r+\alpha}}\,du\right] - \frac{\beta^{\alpha}\,\Gamma(r+\alpha)}{\Gamma(r)\Gamma(\alpha)}\left[\int_0^{-a}\frac{u^{r-2}}{(u+\beta)^{r+\alpha}}\,d - (r-d+1)\int_0^{\infty}\ln\left(\frac{a+u}{u}\right)\frac{r-1}{(u+\beta)^{r+\alpha}}\,du\right] + \frac{2\alpha}{\beta} - 2\right\},$$

,when $a < 0$ (5.8)

$$r_s(\tilde{\theta}_{sG}) = \frac{\left(\frac{r+\alpha}{r}\right)^2 \beta^{\alpha}}{\Gamma(r)\Gamma(\alpha)}\,\Gamma(r+\alpha)\int_0^{\infty}\frac{u^{r-1}}{(u+\beta)^{r+\alpha+2}}\,du - 2\left(\frac{r+\alpha}{r}\right)\frac{\beta^{\alpha}}{\Gamma(\alpha)}\,\frac{\Gamma(r+\alpha)}{\Gamma(r)}\int_0^{\infty}\frac{u^{r-1}}{u+\beta}\,\frac{1}{(u+\beta)^{r+\alpha}}\,du + \frac{\alpha(\alpha+1)}{\beta^2} \qquad (5.9)$$

$$r_L(\tilde{\theta}_{sG}) = b\left\{\frac{\beta^{\alpha}}{\Gamma(\alpha)\Gamma(r)}\,\Gamma(r+\alpha)\int_0^{\infty}\frac{e^{a\left(\frac{r+\alpha}{u+\beta}\right)}\;u^{r-1}}{(u+\beta+a)^{r+\alpha}}\,du + \frac{a\alpha}{\beta} - \frac{a}{\Gamma(r)}\,\frac{\beta^{\alpha}\,\Gamma(r+\alpha)}{\Gamma(\alpha)}\int_0^{\infty}\frac{u^{r-1}}{(u+\beta)^{r+\alpha+1}}\,du - 1\right\} \qquad (5.10)$$

$$r_s(\tilde{\theta}_{LG}) = \frac{\left(\frac{r+\alpha}{a}\right)^2}{\Gamma(r)}\,\frac{\beta^{\alpha}}{\Gamma(\alpha)}\,\Gamma(r+\alpha)\int_0^{\infty}\ln^2\left(\frac{\alpha+\beta+u}{\beta+u}\right)\frac{u^{r-1}}{(u+\beta)^{r+\alpha}}\,du - \frac{\left(\frac{r+\alpha}{\alpha}\right)}{\Gamma(r)}\,\frac{\Gamma(r+\alpha+1)}{\Gamma(\alpha)}\,\beta^{\alpha}\int_0^{\infty}\frac{u^{r-1}}{(u+\beta)^{r+\alpha+1}}\,\ln\left(\frac{a+\beta+u}{\beta+u}\right)du + \frac{\alpha(\alpha+1)}{\beta^2} \qquad (5.11)$$

$$r_L(\tilde{\theta}_{LG}) = b\left\{\frac{\beta^{\alpha}\,\Gamma(r+\alpha)}{\Gamma(r)\Gamma(\alpha)}\int_0^{\infty}\frac{u^{r-1}}{(\beta+u)^{r+\alpha}}\,du - \frac{\Gamma(r+\alpha)\beta^{\alpha}\,\Gamma(r+\alpha)}{\Gamma(\alpha)\Gamma(r)}\int_0^{\infty}\ln\left(\frac{a+\beta+u}{\beta+u}\right)\frac{u^{r-1}}{(u+\beta)^{r+\alpha}}\,du + \frac{a\alpha}{\beta} - 1\right\} \qquad (5.12)$$

We remark that formulas (5.1), and (5.12), are valid whenever $a > 0$, but if $a < 0$ then they are valid provided $\beta > -a$.

6. Comparison between the estimators

Since many of the formulas for Bayes risks of the estimators are not given in closed form, we used Mathematica, Welfram [11] to evaluate the integrals involved. Assuming $d = 3$, $\alpha = 2$, $\beta = 3$, $a = 2$ and without loss of generality $b = 1$, we calculated the Bayes risks of the estimators for different sample sizes and values of r. The results are given in Table 1.

Table 1

Bayes risk of the estimators when d = 3, α = 2, β = and a = 2

n	r	$r_s(\tilde{\theta}_{Ld})$	$r_L(\tilde{\theta}_{Ld})$	$r_s(\hat{\theta}_{sd})$	$r_L(\hat{\theta}_{sd})$	$r_s(\tilde{\theta}_{sG})$	$r_L(\tilde{\theta}_{sG})$	$r_s(\tilde{\theta}_{LG})$	$r_L(\tilde{\theta}_{LG})$
20	15	0.0476	0.0800	0.0468	0.0836	0.0370	0.0744	0.0400	0.0679
30	20	0.0351	0.6217	0.0346	0,0638	0.0290	0.0583	0.0309	0.0541
35	25	0.0278	0.0504	0.0274	0.0515	0.0238	0.0479	0.0252	0.0449
40	30	0.0230	0.0420	0.0227	0.0432	0.0202	0.0407	0.0213	0.0384

By studying Table 1, we notice the following :

1. $r_s(\tilde{\theta}_{sG}) < r_s(\tilde{\theta}_{LG}) < r_s(\tilde{\theta}_{sd}) < r_s(\tilde{\theta}_{Ld})$
2. $r_L(\tilde{\theta}_{LG}) < r_L(\tilde{\theta}_{sG}) < r_L(\tilde{\theta}_{Ld}) < r_L(\tilde{\theta}_{sd})$
3. $r_s(\tilde{\theta}_{sG}) < r_L(\tilde{\theta}_{sG})$

   $r_s(\tilde{\theta}_{LG}) < r_L(\tilde{\theta}_{LG})$

   $r_s(\tilde{\theta}_{Ld}) < r_L(\tilde{\theta}_{Ld})$

   $r_s(\tilde{\theta}_{sd}) < r_L(\tilde{\theta}_{sd})$

(1) and (2) are natural ordering of Bayes risks where it is expected that the Bayes risks in the case of informative prior is smaller than that of the non-informative one .

(3) shows that for the same estimator the Bayes risk under squared error loss is always smaller than that in the case of LINEX error loss.

These result suggest that except when the symmetric quadratic loss is completely inappropriate, it is preferred to the LINEX loss function. Also natural inappropriate conjugate priors lead to smaller Bayes risks than non-informative ones. Table 1 shows that θ_{sG} is the preferred estimator.

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