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A NOTE ON PRE-J-GROUP RINGS

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ABSTRACT

Given an associative ring R satisfying $a^m b = Ba^m$ for any pair a and b of its elements, is being a positive integer, and a group G satisfying $x^{m-1} = y^{m-1}$, we give a sufficient and necessary condition on the group ring GR so that its elements satisfy the same condition assigned to R. A similar result can be obtained if the group G is replaced by a semigroup.

In this note we obtain a necessary and sufficient condition for a group ring RG to be a pre J-ring. We call group rings which are pre-J-rings as pre-J-group rings. The author in [1] calls an associative and commutative ring R to be a pre-J-ring if there exists a positive integer m such that $x^m y = xy^m$ for every x and $y \in R$.

Theorem 1. Let G be a commutative group in which $x^{m-1} = y^{m-1}$ for every x and y in G and R a pre-J-ring with $x^m y = xy^m$ for every x,y in R, where m is an integer > 0 Then the group ring RG is a pre-J-ring if and only if for every α and β with formal representations $\alpha = \sum_{i=1}^{s,t} \alpha_i g_i$ and $\beta = \sum_{j=1}^{s,t} \beta_i h_j$ in RG we have

$$\alpha^m \beta - \left(\sum_{i=1}^{s} \alpha_i^m g_i^m \right) \beta = \alpha \beta^m - \alpha \left(\sum_{j=1}^{t} \beta_j^m h_j^m \right).$$

Proof. Given R is a pre-J-ring in which $xy^m = x^m y$ for every x and y in R and m a positive integer and G is commutative group in which $x^{m-1} = y^{m-1}$ for every x,y in G. Given RG is a pre-J-group ring to prove

$$\left(\sum_{i=1}^{s} \alpha_i g_i \right)^m \sum_{j=1}^{t} \beta_j h_j - \sum_{i=1}^{s} \alpha_i^m g_i^m \left(\sum_{j=1}^{t} \beta_j h_j \right)$$

$$= \sum_{i=1}^{s} \alpha_i g_i \left(\sum_{j=1}^{t} \beta_j h_j \right)^m - \sum_{i=1}^{s} \alpha_i g_i \left(\sum_{j=1}^{t} \beta_j^m h_j^m \right)$$

holds in RG for every α, β in RG where $\alpha = \sum_{i=1}^{s} \alpha_i g_i$ and $\beta = \sum_{j=1}^{t} \beta_j h_j$ with $\beta_j, \alpha_i \in R$

and g_i, $h_j \in G$ for $j = 1, 2, \ldots, t$ and $i = 1, 2, \ldots, s$. Since $\alpha^m\beta = \alpha\beta^m$ we have

$$\left(\sum_{i=1}^{s} \alpha_i g_i\right)^m \sum_{j=1}^{t} \beta_j h_j = \sum_{i=1}^{s} \alpha_i g_i \left(\sum_{j=1}^{t} \beta_j h_j\right)^m$$

expanding both sides of the above equation we get

$$\left(\sum_{i=1}^{s} \alpha_i^m g_i^m\right) \sum_{j=1}^{t} \beta_j h_j + (\Sigma \text{ (terms of the form } \alpha_i.\alpha_j \ldots \alpha_k, g_i.g_j \ldots g_k)$$

$$x \sum_{j=1}^{t} \alpha_j h_j = \left(\sum_{i=1}^{s} \alpha_i g_i\right)\left(\sum_{j=1}^{t} \beta_j^m h_j^m\right) + \left(\sum_{i=1}^{s} \alpha_i g_i\right) x \;(\Sigma \text{ (terms of the form } \beta_j.\beta_k,$$

$\ldots, \beta_l, h_j.h_k \ldots h_l)$.

Now since R is a pre-J-ring we have $x^m y = xy^m$ for all x,y in R and since G is commutative with $g^{m-1} = h^{m-1}$ we have $hg^{m-1}g = h^m g = g^m h$. Hence

$$\left(\sum_{1=1}^{s} \alpha_i^m g_i^m\right)\left(\sum_{j=1}^{t} \beta_j h_j\right) - \left(\sum_{i=1}^{s} \alpha_i g_i\right)\left(\sum_{j=1}^{s,t} \beta_j^m h_j^m\right).$$

Thus the identity

$$\left(\sum_{i=1}^{s} \alpha_i g_i\right)^m \left(\sum_{j=1}^{t} \beta_j h_j\right) \left(\sum_{i=1}^{s} \alpha_i^m g_i^m\right)\left(\sum_{j=1}^{s} \beta_j^m h_j^m\right).$$

$$= \left(\sum_{i=1}^{s} \alpha_i g_i\right)\left(\sum_{j=1}^{t} \beta_j h_j\right)^m - \left(\sum_{i=1}^{s} \alpha_i g_i\right) \sum_{j=1} \beta_j^m h_j^m$$

holds in RG for every α, β in RG. Conversely suppose in RG we have the identity

$$\left(\sum_{i=1}^{s} \alpha_i g_i\right)^m \sum_{j=1}^{t} \beta_j h_j - \left(\sum_{i=1}^{s} \alpha_i^m g_i^m\right)\left(\sum_{j=1}^{s,t} \beta_j h_j\right)$$

$$= \left(\sum_{i=1}^{s} \alpha_i g_i\right)\left(\sum_{j=1}^{t} \beta_j h_j\right)^m - \left(\sum_{i=1}^{s} \alpha_i g_i\right)\left(\sum_{j=1} \beta_j^m h_j^m\right),$$

To show that RG is a pre-J-ring. i.e. to show $\alpha^m\beta = \alpha\beta^m$ for every α, β in RG.

Since R is a pre-J-ring and G is a commutative group such that $x^{m-1} = y^{m-1}$ for every x, y in G we have $\alpha_i^m\beta_j = \alpha_i\beta_j^m$ for all α_i, β_j in R.

Let $\alpha = \sum_{i=1}^{s} \alpha_i g_i$ and $\beta = \sum_{j=1}^{t} \beta_j h_j$ (α_i, $\beta_j \in R$ and g_i, $h_j \in G$). Using the given identity

$$\alpha^m \beta = \left(\sum_{i=1}^{s} \alpha_i^m g_i^m\right) \beta = \alpha\beta^m - \alpha \left(\sum_{j=1}^{t} \beta_j^m h_j^m\right)$$

to prove RG is a pre-J-group ring it is sufficient if we prove

$$\left(\sum_{i=1}^{s} \alpha_i^m g_i^m\right) \beta = \alpha \left(\sum_{j=1} \beta_j^m h_j^m\right).$$

Consider the left hand side

$$\left(\sum_{i=1}^{s} \alpha_i^m g_i^m\right) \beta = \sum_{i=1}^{s} \alpha_i^m g_i^m \left(\beta_l h_l + \ldots + \beta_t h_t\right)$$

$$= \sum_{i=1}^{s} \alpha_i^m \beta_1 g_i^m h_l + \ldots + \sum_{i=1}^{s,t} \alpha_i^m \beta_t g_i^m h_t$$

$$= \sum_{i=1}^{s} \alpha_i \beta_l^m g_i h_l^m + \ldots + \sum_{i=1}^{s,t} \alpha_i \beta_t^m g_i h_t^m$$

$$= \sum_{i=1}^{s} \alpha_i g_i \left(\beta_l^m h_l^m + \ldots + \beta_t^m h_t^m\right)$$

$$= \alpha \sum_{j=1}^{t} \beta_j^m h_j^m$$

which is nothing but the right hand side of the above equality. Hence we have $\alpha^m\beta = \alpha\beta^m$ for all α, β in RG.

Remark 2. If S is a commutative semi-group in which $x^m y = xy^m$ for every x and y in S and R a pre J-ring, then the semi-group ring RS is a pre J-semi-group ring if and only if it satisfies the following identity for every α and β in RS

$$\alpha^m \beta - \left(\sum_{i=1}^{s} \alpha_i^m s_i^m\right) \beta = \alpha\beta^m - \alpha \left(\sum_{j=1}^{t} \beta_j^m h_j^m\right)$$

where $\alpha = \sum_{i=1}^{s} \alpha_i s_i$ and $\beta = \sum_{j=1}^{t} \beta_j h_j$ with α_i, β_j in R and h_j, s_i in S for i = 1, 2, ..., s and j = 1, 2, ..., t. Proof as in the case of Theorem 1.

REFERENCE

[1] **Lah Jiang, 1962.** On the structure of pre-J-rings. Hung-Chong Chow 65th anniversary, volume pp. 47-52, Math. Res. Center, Nat. Taiwan University, Taipei.

عائلـــة من حلقــات الزمـــر

و . ب . فاســانتــا

لتكن R حلقة توزيعية إبداليــة و m عدداً صحيحــاً موجبــاً بحيث تتحقق العـلاقة : $a^m p = ba^m$ لأي عنصـرين a و b في R ؛ ولتكن G زمرة إبـدالية يتحقق فيها : $x^{m-1} = y^{m-1}$ لأي عنصرين x,y فيها . يقدم في هذا البحث شرطاً لازماً وكافياً على حلقة الزمرة RG كي يتحقق الشرط الذي تحققه الحلقـة R وذلك بإعطاء علاقة تربط أي ممثلين في حلقة الزمرة RG . كما يعطي المؤلف نتيجة مشابهة إذا استبدلت الزمرة G بشبه زمرة S .