

INCLUSION RELATION BETWEEN TWO ABSOLUTE SUMMABILITY METHODS

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علاقة الاحتواء بين طريقتين من طرائق الجمع المطلق

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في هذا البحث تمت دراسة مسألة الاحتواء بين الطريقتين $|N, r|$ و $|\bar{N}, q|$ حيث تم تحديد الشروط الضرورية والكافية للعلاقة $|N, q| \subseteq |\bar{N}, r|$ ، كما يحتوي هذا البحث على أمثلة ليست بسيطة لوجود أو عدم وجود تلك العلاقة .

Key word : Summability methods.

1. ABSTRACT

In the present paper, the problem of inclusion of the absolute method of summability $|\bar{N}, q|$ with that of $|N, r|$ has been considered. Necessary and sufficient condition for which $|N, q| \subseteq |\bar{N}, r|$ is established, and non-trivial examples on which this inclusion holds or not have been constructed.

2. Introduction

Let A be a sequence-to-sequence transformation

$$t_n = \sum_{k=0}^{\infty} A_{n,k} S_k ; n = 0, 1, 2, \dots \quad (1)$$

The sequence $\{S_n\}$ is said to be summable (A) to S if $t_n \rightarrow S$ as $n \rightarrow \infty$, and if in addition, $\{t_n\}$ is of bounded variation, then $\{S_n\}$ is said to be absolutely summable (A) or summable $|A|$

Let (N, r) denote the Nörlund method in which the sequence $\{S_n\}$ is transformed into the sequence $\{t_n\}$, where

$$t_n = \frac{1}{R_n} \sum_{k=0}^n r_{n-k} S_k ; R_n = r_0 + r_1 + \dots + r_n \quad (2)$$

and

$$R_n \neq 0 \text{ (all } n \geq 0) ; R_{-m} = r_{-m} = 0 ; (m > 0). \quad (3)$$

The special case in which $r_n = 1$ (all $n \geq 0$), then (N, r) reduces to a simple arithmetic mean of $(C, 1)$.

Each sequence $\{q_n\}$ for which $Q_n = q_0 + q_1 + \dots + q_n \neq 0$ (all $n \geq 0$) for each n defines the weighted mean method (\bar{N}, q) of the sequence $\{S_n\}$, where

$$t_n^q = \frac{1}{Q_n} \sum_{k=0}^n q_k S_k \quad n = 0, 1, 2, \dots \quad (4)$$

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A method of summability is called regular, if it sums every convergent series to its ordinary sum. It follows from Toeplitz's Theorem (Hardy [6]; theorem 2) that (N, r) is regular if, and only if,

$$\frac{r_n}{R_n} \rightarrow 0 \text{ as } n \rightarrow \infty \tag{5}$$

and

$$\sum_{k=0}^n |r_n| = O(|R_n|), \tag{6}$$

and that (\bar{N}, q) is regular if, and only if,

$$|Q_n| \rightarrow \infty \text{ as } n \rightarrow \infty, \tag{7}$$

and

$$\sum_{k=0}^n |q_n| = O(|Q_n|), \tag{8}$$

Let A be a sequence-to-sequence transformation given by (1). If whenever $\{S_n\}$ has a bounded variation it follows that $\{t_n\}$ has a bounded variation, and if the limits are preserved, we shall say that A is absolutely regular.

We shall write throughout $(A) \subseteq (B)$ to mean that any series summable by the method (A) to sum s is necessary summable (B) to the same sum. We shall write for any sequence

$$\Delta U_n = U_n - U_{n+1},$$

3. Inclusion Relations

On inclusion relations of different summability methods much work has been done already e.g (see [1], [2], [3], [4], [5], [6], and [7]).

4. Object of the Paper

The object of this paper is to obtain results involving an inclusion relation of two absolute summability methods $|\bar{N}, q|$ and $|(N, r)|$. The main result will be concluded in section (6), and some special cases will be concluded in section (7).

5. Preliminary Results

This section is devoted to result that is necessary for our purposes.

Theorem (5-1) (Mears [8]). *The sequence-to-sequence*

transformation given by (1) is absolutely regular if, and only if, the following conditions are satisfied :

$$A_{n,k} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for each } k, \tag{9}$$

$$\sum_{k=0}^n A_{n,k} \rightarrow 1 \text{ as } n \rightarrow \infty, \tag{10}$$

and

$$\sum_{n=0}^{\infty} \left| \sum_{v=k}^{\infty} A_{n,v} - \sum_{v=k}^{\infty} A_{n+1,v} \right| = O(1); (k \rightarrow \infty). \tag{11}$$

6. Main Result

In this section we shall prove our main result:

Theorem (6-1) Suppose that (\bar{N}, q) and (N, r) are both regular, $q_n \neq 0$ (all $n \geq 0$), then $|\bar{N}, q| \subseteq |(N, r)|$ if, and only if :

$$\sum_{n=k-1}^{\infty} \left| \Delta_n \frac{1}{R_n} \left(R_{n-k-1} + \frac{Q_k}{q_k} r_{n-k} \right) \right| = O(1), \text{ for all } k \geq 0, \tag{12}$$

where

$$\Delta_n \beta_{n,k} = \beta_{n,k} - \beta_{n+1,k}, \tag{13}$$

Proof of Theorem (6-1). Let $\{t_n^r\}, \{t_n^q\}$ be respectively the (N, r) and (\bar{N}, q) transform of $\{S_n\}$. Using the inversion formula of (4) to obtain S_n in terms of t_n^q and substitute this in (2) to obtain t_n^r in terms of t_n^q , we have

$$t_n^r = \sum_{k=0}^n \alpha_{n,k} t_n^q, \tag{14}$$

where

$$\alpha_{n,n} = \frac{Q_n}{R_n q_n}, \tag{15}$$

$$\alpha_{n,k} = \frac{Q_n}{R_n} \Delta_k \frac{r_{n-k}}{q_n}, \quad 0 \leq k \leq n-1, \tag{16}$$

and

$$\alpha_{n,k} = 0 \text{ otherwise} \tag{17}$$

To prove the result, it is enough to show that the conditions of Theorem (5.1) are all satisfied. Condition (9) follows from (16), and the special case in which $S_n = 1$ (all $n \geq 0$) gives $t_n^r = t_n^q = 1$, which by (14) and (17) implies that

$$\sum_{k=0}^{\infty} \alpha_{n,k} = 1, \tag{18}$$

This gives (10). The left hand side of (11) is equivalent to

$$\sum_{n=0}^{\infty} \left| \sum_{v=0}^{\infty} \alpha_{n,v} - \sum_{v=0}^{k-1} \alpha_{n,v} - \sum_{v=0}^{\infty} \alpha_{n+1,v} - \sum_{v=0}^{k-1} \alpha_{n+1,v} \right|,$$

which by (17) and (18) reduces to :

$$\sum_{n=0}^{\infty} \left| \sum_{v=0}^{k-1} \alpha_{n,v} - \sum_{v=0}^{k-1} \alpha_{n+1,v} \right|. \tag{19}$$

Using (16), it may be easily seen that

$$\sum_{v=0}^{k-1} \alpha_{n,v} = 1 - \frac{R_{n-k}}{R_n} - \frac{Q_{n-1}r_{n-k}}{q_k R_n}. \tag{20}$$

Using (20), we see that (19) reduces to:

$$\sum_{n=k-1}^{\infty} \left| \Delta_n \frac{1}{R_n} \left(R_{n-k-1} + \frac{Q_k}{q_k} r_{n-k} \right) \right|.$$

Therefore, (11) is satisfied if, and only if (12) is valid. This completes the proof.

Remark (6.1) : We remark that $\alpha_{n,n} = O(1)$ is necessary (but not sufficient) condition for (11) to be satisfied.

7. Examples

In this section we will give three examples. In the first example, we will give non-trivial special cases for which $|\overline{(N,q)}| \subseteq |(N,r)|$. In the second example, we will consider the special case in which (N,r) is $(C,1)$ and find necessary and sufficient condition for which $|\overline{(N,q)}| \subseteq |(C,1)|$. Lastly, we will construct two sequences $\{q_n\}$ and $\{r_n\}$ to show that even if both $\overline{(N,q)}$ and (N,r) are regular $\overline{(N,q)} \not\subseteq (N,r)$.

Example (7.1). Let

$$r_n = e^n - e^{n-1}; (n \geq 1); r_0 = 1, \tag{21}$$

and

$$q_n = \frac{1}{n+1} (n \geq 0), \tag{22}$$

then $|\overline{(N,q)}| \subseteq |(C,1)|$.

Proof. The assumptions imply that

$$R_n = e^n; (n \geq 0), \tag{23}$$

and

$$Q_n \begin{cases} \ln(n+1) & (n \geq 1) \\ = 1 & n = 0 \end{cases} \tag{24}$$

The result would follow if we show that (12) is satisfied.

The left hand side of (12) is equivalent to :

$$\left| \frac{Q_k}{q_k R_k} + \left| \frac{Q_k}{q_k} \left(\frac{1}{R_k} - \frac{r_{k-1}}{R_{k+1}} \right) - \frac{1}{R_{k+1}} \right| + \sum_{n=k+1}^{\infty} \left| \frac{R_{n-k-1}}{R_n} - \frac{R_{n-k}}{R_{n+1}} + \frac{Q_k}{q_k} \left(\frac{r_{n-k}}{R_n} - \frac{r_{n-k-1}}{R_{n+1}} \right) \right| \tag{25}$$

Using (21)-(24), we see that each term inside the sigma of (25) is equal to zero, and the first two terms of (25) is less

than $\frac{2Q_k}{q_k R_k}$ which is equivalent to:

$$\frac{2Q_k}{(k+1)^{-1} e^k} \sim \frac{2(k+1) \ln(k+1)}{e^k} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Therefore, (12) is valid and Theorem (6.1) yields the result.

Remark (7.1). We remark that if the assumption given by (21) in example (7.1) is the only assumption, then the sufficient condition for $|\overline{(N,q)}| \subseteq |(N,r)|$ is that

$$Q_n = O(q_n), \tag{26}$$

but (26) is not necessary. For this take $q_n = n+1, (n \geq 0)$, then

$$Q_n = \frac{(n+1)(n+2)}{2}, (n \geq 0),$$

and so

$$\frac{Q_n}{R_n q_n} = \frac{(n+1)(n+2)}{2e^n(n+1)} = \frac{1}{2} \frac{n+2}{e^n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Using this, it follows from (25) that (12) is valid, and so

$$|\overline{(N,q)}| \subseteq |(N,r)|. \text{ But } \frac{Q_n}{q_n} = \frac{1}{2} (n+2) \neq O(1).$$

Example (7.2). Let (N,r) be $(C,1)$, then $|\overline{(N,q)}| \subseteq |(C,1)|$ if, and only if,

$$\frac{Q_n}{q_n} = O(n). \tag{27}$$

Proof. Using Theorem (6.1) and substitute $r_n = 1, R_n = n+1 (n \geq 0)$, it follows from (25) that $|\overline{(N,q)}| \subseteq |(C,1)|$ if, and only if,

$$\left| \frac{Q_k}{(k+1)q_k} + \left| \frac{Q_k}{q_k} \frac{1}{(k+1)(k+2)} - \frac{1}{k+2} \right| + \sum_{n=k+1}^{\infty} \left| \frac{n-k}{n+1} - \frac{n-k+1}{n+2} + \frac{Q_k}{q_k} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) \right| = O(1)$$

which is equivalent to :

$$\left| \frac{Q_k}{(k+1)q_k} \right| + \left| \frac{Q_k}{q_k(k+1)(k+2)} - \frac{1}{k+2} \right| + \left| \frac{Q_k}{q_k} - (k+1) \right| \frac{1}{k+2} = O(1)$$

which is clearly equivalent to (27).

Example (7.3). Let

$$r_n = \sqrt{n+1} - \sqrt{n}, \quad (n \geq 0) \quad (28)$$

and

$$q_n = 2n+1, \quad (n \geq 0), \quad (29)$$

then (\bar{N}, q) and (N, r) are regular, but $|\bar{N}, q| \notin |N, r|$

Proof. Using (28) and (29), we have

$$R_n = \sqrt{n+1}, \quad (n \geq 0), \quad (30)$$

and

$$Q_n = (n+1)^2, \quad (n \geq 0), \quad (30)$$

Using (28)-(31), it may be easily seen that (5)-(8) are all satisfied, and Toeplitz's Theorem shows that (\bar{N}, q) and (N, r) are regular. Next, to show that $|\bar{N}, q| \notin |N, r|$, it is enough to show that (12) is not valid. Using (29)-(31), we see that the first term of (25) is equivalent to :

$$\left| \frac{(k+1)^2}{(2k+1)\sqrt{k+1}} \right| \sim \frac{1}{2}k^{\frac{1}{2}} \neq O(1).$$

Therefore, (12) is not satisfied, and the proof is completed.

REFERENCES

- [1] Al - Madi, A. K., *On Absolute Rogosinski - Bernstein Trigonometric Summability Methods*. Indian Jour. Maths. Vol (25) (1983) pp (53-66).
- [2] Al -Madi, A. K., Comparison Theorems of the Product of Riesz Summability Methods. Bull. Cal. Math. Soc., V (78), (1986), pp. (305-307).
- [3] Rhoades, B. E., Some Comparison Theorems for Absolute Summability. Indian J. of Pure and Applied Maths., 16 (1) (1984), 23-30.
- [4] Das, G., *Product of Nörlund Methods*. Indian Jour. Maths. Vol. (10) No. 1 (1968) pp. 25-43.
- [5] Dikshit, G. D., *A Note On Riesz And Norlund Means*. *Rendiconti Del Circolo Mat Di Palermo* (2), 18 (1969), pp. 49-61.
- [6] Hardy, G. H., *Divergent Series*. Oxford (1949).
- [7] Kumar, A. and Saho, D. P., *Absolute Convergence Fields Of Some Triangular Matrix Methods*. Pacific Jour. Maths. 97 (1981) pp 443-451.
- [8] Mears, F. M., *Absolute Regularity And The Nörlund Mean*. *Annals of Maths*. 38 (1937), pp (595-601)