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STOCHASTIC CLASSES OF LIFE DISTRIBUTIONS

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ABSTRACT

The concept of stochastic ordering of random variables is used to characterize the behaviour of the life length variables. This leads to some new classes of ageing namely, the stochastic classes of life distributions. The implications properties and relationships with other known classes of life distributions are pointed out. Laplace transforms of the original life distributions are considered and discrete versions of the stochastic classes of life distributions are introduced. Further, most of the existing classes of life distributions are briefly surveyed and the significant meanings of the structure of these stochastic life distributions are emphasized.

INTRODUCTION

Most of the ageing concepts introduced in statistical literature are useful in modelling various problems in reliability engineering maintenance policies, risk analysis, biometry and inventories. Researchers have based their classification of various life distributions on the different characteristics of ageing. The main classes of life distributions are: increasing failure rate (IFR), increasing cumulative conditional survival (ICCS), increasing failure rate average (IFRA), new better than used (NBU), new better used than used cumulative conditional survival (NBUCCS), new better than used in failure rate (NBUFR), new better than average in failure rate (NBAFR), new better than used in expectation (NBUE), harmonic new better than used in expectation (HNBUE), harmonic new better than used cumulative conditional survival (HNBUCCS), decreasing mean remaining life (DMRL), decreasing mean remaining life average (DMRLA), new better than average mean remaining life (NBAMRL), new better than used mean remaining life (NBUMRL), decreasing harmonic mean remaining life (DHMRL), and new better than used harmonic mean remaining life (NBUHMRL). Their corresponding dual classes are DFR, DCCS, DFRA, NWU, NWUCCS, NWUFR,

NWAFR, NWUE, HNWUE, HNWUCC, IMRL, IMRLA, NWAMRL, NWUMRL, IHMRL and NWUHMRL, respectively. Note that I, D, W stand for increasing, decreasing and worse, respectively. The implications between these classes are shown in Figure 1.

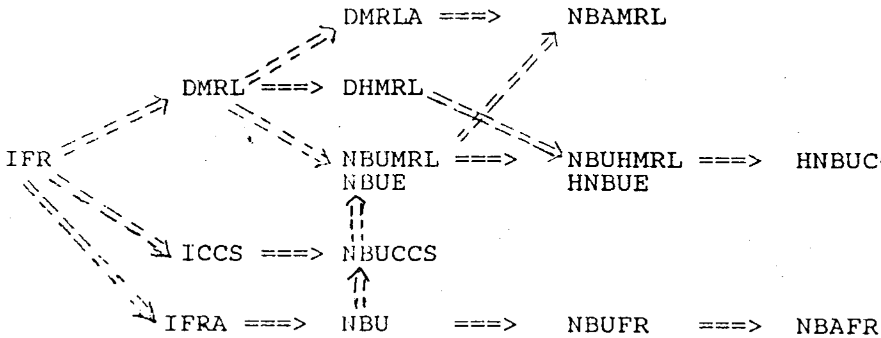

Fig . 1

These are the only implications among these classes. For proofs of these implications and other properties of these classes we refer the reader to Bryson and Siddiqui (1969), Marshall and Proschan (1972), Rolski (1975), Barlow and Proschan (1985), Klefsjo (1983), Loh (1984), Abouammoh and Ahmed (1988) and Abouammoh (1988).

Next we give the definitions of the discrete versions of some classes of life distributions.

Definition 2.1. A discrete distribution P_k or its survival probability $\bar{P}_k = \sum_{j=k}^{\infty} P_j$ k=0,1,2,... with finite mean $\mu = \sum_{k=0}^{\infty} \bar{P}_k$ is said to be:

(i) discrete NBUCCS if $\sum_{j=0}^{\ell} \bar{P}_{k+j} \le \bar{P}_k \sum_{j=0}^{\ell} \bar{P}_j$ for k=1,2,... (2.1)

(ii) discrete NBUFR if $\bar{P}_{k+1} \le \bar{P}_1 \bar{P}_k$, for k=0,1,2,... (2.2)

(iii) discrete NBAFR if $\bar{P}_k \le (\bar{P}_1)^k$ for k=0,1,2,... (2.3)

(iv) discrete HNBUCCS if $\sum_{k=\ell}^{\infty} \sum_{j=k}^{\infty} \bar{P}_j \le \mu^2 (1 - 1/\mu)^{\ell}$ for ℓ=0,1,2,... (2.4)

(v) discrete DMRLA if $\frac{1}{k}\sum_{j=0}^{k}\sum_{i=j}^{\infty}\bar{P}_i/\bar{P}_j$ is decreasing in k=1,2,... (2.5)

(vi) discrete NBAMRL if $\sum_{j=o}^{k}\sum_{i=o}^{\infty}\bar{P}_i \leq k$ for k=1,2,... (2.6)

(vii) discrete IHMRLA if $\frac{1}{k}\sum_{j=o}^{k}(\bar{P}_j/\sum_{i=j}^{\infty}\bar{P}_i)$ is decreasing in k=1,2,3,.(2.7)

The dual discrete classes of life distributions introduced in Definition (2.1) are obtained by reversing the inequalities or the direction of monoonicity, as appropriate, in (2.1) – (2.7). The relations between these classes of discrete distributions or their dual classes are the same as in the general case, see Figure 1.

Throughout this paper the terms increasing and decreasing stand for nonedecreasing and nonincreasing.

The main theme of this paper is to generalize the concept of stochastic ordering of random variables in order to define the stochastic classes of life distributions. The sequel of the paper is as follows: The generalized or stochastic versions of the above mentioned classes of life distributions are introduced in Section 2. In Section 3 the classes of life distributions in Figure 1 are characterized via Laplace transform and the implications between the stochastic classes of life distributions are shown. In Section 4, remarks about some further study on these classes are presented.

STOCHASTIC LIFE DISTRIBUTIONS

The random variable Y is said to be stochastically larger than the random variable Z if $P(Y \leq a) \geq P(Z \geq a)$ for any real a. Properties of the stochastic ordering and its relation with other forms of ordering of random variables are studied in some details in Barlow and Proschan (1985), Ross (1983) and Stoyan (1983).

Now let T be a nonnegative random variable with distribution function F(t), survival function $\bar{F}(t) = 1-F(t)$ and finite mean $\mu = \int_0^\infty \bar{F}(u)\,du$. Let $X_0 = 0$ and X_k, k=1,2,... be a sequence of independent and indentically distributed exponential random variables with mean μ and idependent of T.

Let $N(T) = \max_k \{k: X_1 + ... + X_k \leq T\}$

Next we give the definition of discrete version of some classes of life distributions.

The stochastic ageing property which results from imposing a particular non-stochastic ageing property on N(T) is considered. This can be seen by introducing the following stochastic classes. Note that the letter "S" at the beginning of the abbreviations denotes "stochastic" or "stochastically" as appropriate.

Defination 2.2 The nonnegative random variable T, its distribution F, or its survival $\bar{F}$ is said to be:

(i) SNBUFR if N(T) has the discrete NBUFR property, or equivalently

$$P\{N(T) \geq k+1\} \leq P\{N(T) \geq k\}.\ P\{N(T) \geq 1\}, \qquad (2.8)$$

for all k=0,1,2,...;

(ii) SNBUCCS if N(T) has the discrete NBUCCS property, or equivalently

$$\sum_{j=0}^{\ell} P\{N(T) \geq k+j\} \leq P\{N(T) \geq k\} \sum_{j=0}^{\ell} P\{N(T) \geq j\}, \qquad (2.9)$$

for all k=0,1,2,... and l=0,1,2...;

(iii) SNBAFR if N(T) has the discrete NBAFR property, or equivalently

$$P\{N(T) \geq k\} \leq P\{N(T) \geq 1\}^{k}, \qquad (2.10)$$

for all k=0,1,2,...,

(iv) SHNBUCCS if N(T) has the discrete HNBUCCS property, or equivalently

$$\sum_{k=\ell}^{\infty} \sum_{j=k}^{\infty} P\{N(T) \geq j\} \leq \mu^{2} (1-1/\mu)^{\ell}, \qquad (2.11)$$

for all k=0,1,2,... and ℓ=0,1,2...,

(v) SDMRLA if N(T) has the discrete SDMRLA property, or equivalently

$$\frac{1}{k} \sum_{j=o}^{k} \sum_{i=j}^{\infty} P\{N(T) \geq j\}, \tag{2.12}$$

is decreasing in k=1,2,...,

(vi) SNBAMRLS if N(T) has the discrete SNBAMRL property, or equivalently

$$\sum_{j=o}^{k} \sum_{i=j}^{\infty} P\{N(T) \geq i\} \leq k, \tag{2.13}$$

for k=1,2,...,

(vii) SIHMRLA if N(T) has the discrete SIHMRLA property, or equivalently

$$\frac{1}{k} \sum_{j=o}^{k} [P\{N(T) \geq j\} / \sum_{i=j}^{\infty} P\{N(T) \geq i\}], \tag{2.14}$$

is increasing in k=1,2,...

By reversing the inequalities or the directions of monotonicity in (2.8) - (2.14) above, we get the dual classes SNWUFR, SNWUCCS, SNWAFR, SHNWUCCS, SIMRLA, SNWAMRL and SIHMRL, respectively.

Note that stochastic forms of IFR, NBU have been considered by Singh and Deshpande (1985). Similar approach for IFRA, NBU, NBUE and HNBUE classes have been recently considered by Alzaid and Ahmed (1988). The stochastic classes of life ditributions, in Definition 2.2, have significant meaning and useful structure in various reliability applications. Following Esary, Marshall and Proschan (1973) one can view N(T) as the number of renewals in (0,T) for an ordinary renewal process and the origin is not, naively, counted a a renewal point. These stochastic classes of life distributions can also be viewed as the survival of the passage time. In particular, let a component receive independent shocks and let X_j be the random damage caused by the j^{th} shock, j=1,2,.... Each X_j has the exponential distribution with mean μ. The damage accumulates and the component fails when the accumulated damage has exceeded a nonnegative random thershold T with distribution F. In this framework one can view Definition 2.2 (1) in the following way: given that a component with life distribution F has survived up to $\sum_{i=0}^{k} x_i$, the probability that it will not fail in the next X_{k+1} units of time is not more than the probability that a new component with distribution F will fail in time X_{k+1} for all k=1,2,... . Equivalent forms of the other stochastic classes, in Definition 2.2 can be similarly demonstrated.

Note that $P\{N(T) \geq k\}$ can have the following representation

$$P\{N(T) \geq k\} = \int_0^\infty \frac{t^{k-1} e^{-t/\mu}}{\Gamma(k)} \bar{F}(t)\,dt \qquad (2.15)$$

for k=1,2,...

The next result shows that insisting on equality and constancy rather than on inequality and monotoncity, as appropriate, in Definition 2.2 is equivalent to saying that the stochastic class coincides with the exponential distribution.

Theorem 2.3. Any distribution that belongs to a stochastic class of life distributions and its corresponding dual at the same time must be exponential.

Proof. Note that the theorem can be proved for classes of Definition 1.1 by considering the integral representation of $P\{N(T) \geq k\}$ given in (2.15) for k=1,2,... . To elucidate the proof suppose that F is both SNBAFR and SNWAFR. It follows that

$$P\{N(T) \geq k\} = [P\{N(T) \geq 1\}]^k$$

for all k=1,2,3,... . The proof now follows on similar lines as that of characterization of exponential distribution given in Engel and Zijlstra (1980).

THE LAPLACE TRANSFORM AND SOME RELATIONSHIPS

Let F be a distribution function such that F(0) =0 and let $\phi(s) = \int_0^\infty e^{-su}\,dF(u)$, $s \geq 0$ be the Laplace transform of F. Define

$$a_n(s) = \frac{(-1)^n}{n!} \frac{d^n}{ds^n} \left[\frac{1-\phi(s)}{s}\right], \qquad (3.1)$$

and set

$$\alpha_{n+1}(s) = s^{n+1} a_n(s), \text{ for } n \geq 0, s \geq 0 \text{ and } \alpha_o(s) = 1 \quad (3.2)$$

The following forms of $a_n(s)$ and $\alpha_{n+1}(s)$ are useful in showing the relation between the previous classes of life distributions and their corresponding stochastic classes. In fact, it can be verified that

$$a_n(s) = \frac{1}{n!} \int_o^\infty x^n e^{-sx} \bar{F}(x)dx, \quad n \geq 0, \quad s \geq 0, \quad (3.3)$$

and

$$\alpha_{n+1}(s) = \frac{s}{n!} \int_o^\infty (sx)^n e^{-sx} \bar{F}(x)dx, \text{ for } n \geq 1 \text{ and } s \geq 0 \quad (3.3)$$

Vinogradov (1973) characterized the IFR property in terms of $\alpha_n(s)$, Block and Savits (1980) have characterized the IFRA, DMRL, NBU and NBUE properties in terms of $\alpha_n(s)$, Klefsjo (1982) has established the characterization of the HNBUE class. The NBUFR and NBAFR characterization in terms of $\alpha_n(s)$ have been considered by Abouammoh, Hindi and Ahmed (1988)

In the following two results we characterized the stochastic classes defi - ned

Theorem 3.1. Let T be a nonnegative random variable with distribution F and let $\alpha_n(s)$ be as defined in (3.2). Then

$$\text{T is NBUCCS iff } \sum_{j=o}^{m} \alpha_{n+j}(s) \leq \alpha_n(s) \sum_{j=o}^{m} \alpha_j(s)$$

$$\text{for all } n, m \geq 0 \text{ and } s > 0$$

Proof. It is sufficient to prove the theorem in the NBUCCS case. The proof for the NWUCCS case follows by reversing all inequalities.

The "only if" part can be poved along the same line used in Klefsjo (1981) to prove a similar result for the HNBUE case.

Now, take $t>0$ and let $s = \frac{n}{t}$. From (2.8) and (2.15) it follows that

$$\sum_{j=o}^{n} \alpha_j(s) = 1 + \mu s - \sum_{j=n+1}^{\infty} \alpha_j(s)$$

$$= 1 + \mu s - s \int_o^{\infty} G_n(x) F(x) dx, \qquad (3.5)$$

where $G_n(x) = \sum_{j=o}^{\infty} \frac{(sx)^j}{j!} e^{-sx} = \sum_{j=1}^{\infty} \frac{1}{j!} \left(\frac{nx}{t}\right)^j \exp\left[-\frac{nx}{t}\right]$

This means that G_n is a gamma distribution function with the characteristic function

$$\phi_n(u) = \left(\frac{s}{s-iu}\right) = \left(1 - \frac{iut}{n}\right)^n.$$

Hence, $\lim_{n\to\infty} \varnothing_n(u) = e^{itu}$, and G_n converges weakly to the one point distribution

$$G(x) = \begin{cases} 0, & x < t, \\ 1, & x \geq t. \end{cases}$$

Accordingly,

$$\lim_{n\to\infty} \int_o^{\infty} G_n(x) F(x) dx = \int_o^{\infty} G(x) F(x) dx$$

$$= \int_t^{\infty} F(x) dx$$

From this result together with (3.5) we now get

$$\lim \frac{1}{s} \sum_{j=o}^{n} \alpha_j(s) = \mu - \int_t^{\infty} F(x)dx = \int_o^t F(x)dx.$$

On the other hand, let $s = \frac{n}{x}$ and $m = [\frac{ny}{x}]$. Then $\lim_{n\to\infty} \frac{m}{s} = y$ and $\lim_{n\to\infty} \frac{m+n+1}{s} = x+y$, and since

$$\sum_{j=o}^{n} \alpha_{j+m}(s) = \sum_{j=m+1}^{\infty} \alpha_j(s) - \sum_{j=m+n+1}^{\infty} \alpha_j(s)$$

Then

$$\lim_{n\to\infty} \frac{1}{s} \sum_{j=o}^{n} \alpha_{j+m}(s) = \int_x^{\infty} F(u)du - \int_{x+y}^{\infty} F(u)dx$$

$$= \int_o^y F(x+t)dt.$$

Furthermore, since $\{\alpha_k(s)\}_{k=o}^{\infty}$ is discrete NBUCSS and $\lim_{n\to\infty} \alpha_{n+1}(s) = \bar{F}(x)$ we have that

$$\bar{F}(x) \int_o^y \bar{F}(t)dt \geq \int_o^y \bar{F}(x+t)dt, \qquad y \geq 0.$$

This complete the proof. □

Now we give the following characterizations.

Theorem 3.2. Under the conditions of Theorem 3.1:

(i) T is NBUFR iff $\alpha_n(s) \leq \alpha_1(s) \; \alpha_{n-1}(s)$

for all n=1,2,... and s>0.

$$n^{-1}\sum_{k=1}^{n}\sum_{j=k}^{\infty}\alpha_j(s)\,/\,\alpha_k(s)$$

is decreasing in n=1,2,... for all s>0.

(iii) T is NBAMRL iff $\sum_{k=1}^{n}\sum_{j=k}^{\infty}\alpha_j(s)\,/\,\alpha_k(s) \lesssim n\sum_{k=1}^{\infty}\alpha_k(s)$

for all n=1,2,... and s>0.

(iv) T is DHMRL iff $n^{-1}\sum_{k=1}^{n}\alpha_k(s)\,/\,\sum_{j=k}^{\infty}\alpha_j(s)$

is increasing in n=1,2,... for all s>0.

(v) T is HNBUCCS iff $\sum_{n=\ell}^{\infty}\sum_{k=n}^{\infty}\alpha_k(s) \lesssim \sum_{n=\ell}^{\infty}\left(\frac{\mu s}{1+\mu s}\right)^n$

for all ℓ l>0.

Proof.

(i) Let T be IFR. It follows, from Vinogradov (1973) that $\{\alpha_n(s)\}_{n=0}^{\infty}$ is logconcave in n for all s>0. Thus, $\alpha_{n+1}(s)/\alpha_n(s)$ is decreasing in $n \geq 1$ for all $s > 0$, and hence

$$\frac{\alpha_{n+1}(s)}{\alpha_n(s)} \lesssim \lim \frac{\alpha_{n+1}(s)}{\alpha_n(s)} = \alpha_1(s)$$

Then T is NBUFR if $\alpha_{n+1}(s) \leqslant \alpha_1(s)\,\alpha_n(s)$ for all $n \geqslant 0$ and s>0.

(ii) If T is DMRL, it follows from Block and Savits (1980) that

$$g(n) = \sum_{k=n}^{\infty}\alpha_k(s)\,/\,\alpha_n(s)$$

is decreasing in n=1,2,... for all s>0.

Averaging g(n) over a finite number of points implies that T is DMRLA if $\frac{1}{n}\sum_{j=1}^{n} g(j)$ is decreasing in n=1,2,... for all s>0.

(iii) Let T be DMRLA, i.e. $G(n) = \frac{1}{n}\sum_{j=1}^{n} g(j)\ g(1)$ is decreasing in n=1,2,... . By using the fact that if $\lim_{n\to\infty} G(n) = \lim_{n\to\infty} \frac{h_1(n)}{h_2(n)} = \frac{0}{0}$ then lim G(n) is obtained by $\lim_{n\to\infty} \frac{h_1(n)-h_1(n-1)}{h_2(n) - h_2(n-1)}$, we get the required result.

(iv) Let T be DMRL. Then g(n), as defined above, is decreasing in n and the average of g(n) over a finite number of points is increasing in n=1,2,... for all s>0.

(v) Can be proved using similar arguments to those used in Theorem 3.1, just above, with obvious modifications.

The results of Theorem 3.1 and 3.2 can be used to investigate closure properties of the classes of the corresponding life distributions under convolution, see Block and Savits (1980). If, in Theorems 3.1 and 3.2, expression (3.4) is used for $\alpha_{n+1}(s)$ and s is replaced by $1/\mu$. we obtain the following.

Theorem 3.3. A distribution F(t) that belongs to any class of life distributions in Figure 1 must belong to the corresponding stochastic class of life distributions.

The implications between the stochastic classes of life distributions can be summarized in the following figure

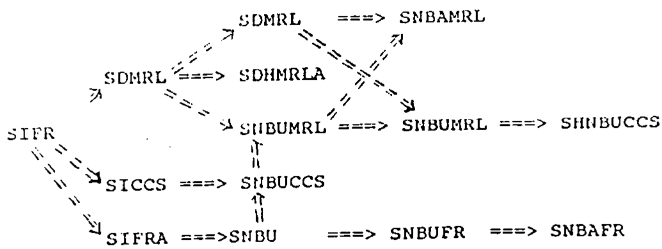

Fig . 2

Singh and Deshpande (1985) have shown that SIFR ===> SNBAFR. In fact implications between the other classes in Theorem 3.3 run in parallel steps by using the corresponding definition of the stochastic classes or some equivalent forms. Such proof is omitted to keep the paper to a reasonable length.

REMARKS

(i) It is suitable, in some situations where the life of the system or its components is measued by discrete a variable such as the number of cycles or shocks, to require that the geometric distribution is the only distribution that has any property of ((i)−(vii)) of Definition (2.1) and its corresponding dual.

(ii) The closure property, under some reliabiity operations such as mixing, convolution, passage to a limit weakly and formation of coherent systems, are studied for most of the known classes of life distributions. A similar study for the stochastic classes introduced in this paper will form a separate communication.

(iii) Parallel to the common classes of life distribution, some test statistics for these classes need to be investigated.

(iv) The survival function of the homogenous or non-homogenous poisson shock model when the amount of shocks follows stocahstic life distributions has very intuitive meaning in making maintenance and replacement policies.

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مجموعات عشوائية لتوزيعات الحياه

أبو عمـــه و أحمـــد و عبـد الخالـق

يستخدم مفهوم الترتيب للمتغيرات العشوائية لمعرفة سلوك متغيرات طول الحياة ـ وعنـدئـذ يمكننـا إيجـاد مجمـوعات عشوائية جديدة لهذه التوزيعات ، وكذلك دراسة خواصـها وارتباطها بالمجموعات المعروفة .