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ON DIRICHLET PROBLEM WITH SINGULAR INTEGRAL BOUNDARY CONDITION

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ABSTRACT

In this paper a Dirichlet problem, with singular integral condition, is studied. It is shown that such problem can be reduced to a regular equation which allows us to construct the solution.

Let G be a simply connected bounded region with a simple smooth contour C. Consider the equation

$$\triangle u = 0 \tag{1}$$

where $\triangle$ is the Laplace's operator and u is a vector vector function which takes values in Banach algebra with involution R. In this paper we are concerned with the regular solution of equation (1) in G which satisfies, on C, the singular integral condition:

$$a(t)\,u(t) + \frac{b(t)}{\pi i}\int_C \frac{u(T)}{T-t}\,dT + \int_C k(t,T)\,u(T)dT = f(t) \tag{2}$$

where a (t), b(t), k(t, T) and f(t) assume values in R, and each of them satisfies a Holder condition.

In the case a(t) = 1, b(t) = 0 and k(t, T) = 0,
condition (2) takes the form:

$$u(t) = f(t) \tag{2'}$$

while in the case a(t) = 0, b(t) = 1 and k(t, T) = 0,

condition (2) takes the form

$$\frac{1}{\pi i}\int_C \frac{u(T)}{T-t}\,dT = f(t), \text{ which implies that } u(t) = \frac{1}{\pi i}\int_C \frac{f(T)}{T-t}\,dT \tag{2''}$$

The problem of finding the regular solution of (1) in G prescribed by (2′) or (2″), on C, is the usual Dirichlet problem for a vector function. The more general problem (1) - (2), mentioned earlier also called Dirichlet problem when (2) is solvable with respect to u.

The operator form of condition (2) is

$$au + bSu + Ku = f \tag{3}$$

where S is a singular operator satisfying the conditions:
1) $S^2 = I$, 2) $\forall v \in R$ the operator $Sv - vS$ is regular [1]

Lemma: If S is the singular operator mentioned above, then $\forall v,w \in R$, we have
(i) $Svw = vSw + Nw$

(ii) $SvSw = vw + Qw$
where N and Q are regular operators.

Proof: (i) Since $Svw = Svw + vSw - vSw$
$= vSw + (Sv - vS)w$
then from 2), it follows
$Svw = vSw + Nw$.

(ii) Since $SvSw = SvSw + vw - vw$,
then it follows from 1)
$SvSw = vw + SvSw - S^2vw$
$= vw + S(vS - Sv)w$

Since S in singular and $vS - Sv$ is regular,
then $S(vS - Sv)$ is regular and thus we have
$SvSw = vw + Qw$.

Theorem: Problem (1)-(2) can be reduced to the regular equation
$q\omega + L\omega = h$
which is completely defined in the space R^2.

Proof: Every regular solution of (1) takes the form

$$u = \Phi(z) + \overline{\Phi(z)} \tag{4}$$

where $\Phi(z)$ is an arbitrary vector function in G and $\overline{\Phi(z)}$ is its involution.

Thus,

$$u|_C = \Phi(t) + \overline{\Phi(t)} \tag{5}$$

Substituting from (5) in (3) we find

$$a(\Phi + \overline{\Phi}) + bS(\Phi + \overline{\Phi}) + K(\Phi + \overline{\Phi}) = f \tag{6}$$

The unique integral representation on I.N. Vekua, [2], for the function $\Phi(z)$ takes the form

$$\Phi(z) = \int_C \frac{\overline{T}' T \delta(T)}{T - z} \, dT + ik_1 \tag{7}$$

where $\delta(T) = \overline{\delta(T)}$ and satisfies Holder condition, $k_1 = \bar{k}_1$ (constant). From (7), we obtain [2]

$$\phi(t) = \pi i \bar{t}' t \delta(t) + \int_C \frac{\bar{T}' T \delta(T)}{T - t} dT + ik_1 \quad (8)$$

hence

$$\overline{\phi(t)} = -\pi i t' \bar{t} \delta(t) + \int_C \frac{T' \bar{T} \delta(T)}{\bar{T} - \bar{t}} \underline{d\bar{T}} - ik_1 \quad (9)$$

Since $\frac{d\bar{T}}{\bar{T} - \bar{t}} = \frac{dT}{T - t} + \underline{d}\ln\left(\frac{\bar{T} - \bar{t}}{T - t}\right)$, [2], equation (9) takes the form

$$\overline{\phi(t)} = \pi i t' t \delta(t) + \int_C \frac{T' \bar{T} \delta(T)}{T - t} dT + \int_C h_1(t, T)(T) dT - ik_1 \quad (10)$$

where the regular kernel $h_1(t, T)$ is given by:

$$h_1(t,T) = T' \bar{T} \frac{d}{dT} \ln\left(\frac{\bar{T}-\bar{t}}{T-t}\right).$$

Equations (8) and (10), can be written respectively in the form

$$\phi(t) = \pi i \bar{t}' t \delta(t) + \frac{\pi i \bar{t}' t}{\pi i} \int_C \frac{\delta(T)}{T-t} dT + \int_C \frac{\bar{T}' T - \bar{t}' t}{T-t} \delta(T) dT + ik_1 \quad (11)$$

$$\overline{\phi(t)} = \pi i t' \bar{t} \delta(t) + \frac{\pi i t' \bar{t}}{\pi i} \int_C \frac{\delta(T)}{T-t} dT + \int_C \frac{T' \bar{T} - t' \bar{t}}{T-t} \delta(T) dT$$
$$+ \int_C h_1(t.T) \delta(T) dT - ik_1 \quad (12)$$

Substituting from (11) and (12) in (5) we have

$$u|_C = \propto(t) \delta(t) + \frac{\beta(t)}{\pi i} \int_C \frac{\delta(T)}{T-t} dT + \int_C M(t,T) \delta(T) dT \quad (13)$$

where $\propto(t) = \pi i (\bar{t}' t - t' \bar{t})$,
$\beta(t) = \pi i (\bar{t}' t + t' \bar{t})$,

and

$$M(t, T) = \frac{\bar{T}' T - \bar{t}' t}{T-t} + \frac{T' \bar{T} - t' \bar{t}}{T-t} + h_1(t,T). \quad (14)$$

The operator form of (13) is given by

$$u_C = \alpha\delta + \beta S\delta + M\delta \tag{15}$$

Consequently from (15) and (16), we obtain:

$$a(\alpha\delta + \beta S\delta + M\delta) + bS(\alpha\delta + \beta S\delta + M\delta) + K(\alpha\delta + \beta S\delta + M\delta) = f \tag{16}$$

$$a\alpha\delta + a\beta S\delta + bS\alpha\delta + bS.\beta S\delta + T_O\delta = f \tag{17}$$

where T_O is a regular operator

$T_O\delta = K(\alpha\delta + \beta S\delta + M\delta) + aM\delta + bSM\delta$

using the lemma, the following equation is obtained from (17)

$$a\alpha\delta + a\beta S\delta + b\alpha S\delta + b\beta\delta + bP_1\delta + bP_1\delta + bQ_1\delta + T_O\delta = f \tag{18}$$

where P_1 and Q1 are regular and completely defined by the following:

$$S\alpha\delta = \alpha S\delta + P_1\delta,$$

$$S\beta S\delta = \beta\delta + Q_1\delta \tag{19}$$

Let,

$m = a\alpha + b\beta,$

$n = a\beta + b\alpha,$

and the regular operator Y

$$Y = bP_1 + bQ_1 + T_O \tag{20}$$

we obtaine from (18)

$$m\delta + nS\delta + Y\delta = f \tag{21}$$

and therefore, problem (1)-(2) can be reduced to the singular operator equation (21) which is reshaped in a regular form as follows [3] :

From (21)

$$Sm\delta + SnS\delta + SY\delta = Sf \tag{22}$$

since $S^2 = I$, we obtain from (22)

$$SmSS\delta + SnS\delta + SYSS\delta = Sf \tag{23}$$

let $\delta = \varphi_1$ and $S\delta = \varphi_2$, consequently from (21) and (23) the following system of equations is produced

$$\left.\begin{aligned} & m\varphi_1 + n\varphi_2 + Y\varphi_1 = f \\ & SmS\varphi_2 + SnS\varphi_1 + SYS\varphi_2 = Sf \end{aligned}\right) \tag{24}$$

Using the lemma, the above system takes the form

$$\left.\begin{aligned} & n\varphi_1 + n\varphi_2 + Y\varphi_1 = f \\ & n\varphi_1 + m\varphi_2 + N_1\varphi_1 + N_2\varphi_2 + SYS\varphi_2 = Sf \end{aligned}\right) \tag{25}$$

where N_1 and N_2 are regular operators, which are completely defined by the following

$$SnS\varphi_1 = n\varphi_1 + N_1\varphi_1, \quad SmS\varphi_2 = m\varphi_2 + N_2\varphi_2 \qquad (26)$$

Define,

$$q = \begin{pmatrix} m & n \\ n & m \end{pmatrix} \qquad (27)$$

$$L = \begin{pmatrix} Y & O \\ N_1 & N_2 + SYS \end{pmatrix} \qquad (28)$$

$$h = (f, Sf) \qquad (29)$$
$$\omega = (\varphi_1, \varphi_2), \qquad (30)$$

The system (25), takes the form

$$q\omega + L\omega = h \qquad (31)$$

which is regular and completely defined in the space R^2, the proof is complete.

If q possess a bounded inverse q^{-1} equation (31) has

$$\omega + L_O\omega = h_O \qquad (32)$$

which is a regular equation with regular operator $L_O = q^{-1}L$ in the space R^2 and therefore, if δ is the solution of equation (21), then $(\varphi_1, \varphi_2) = (\delta, S\delta)$ is the soluton of equation (32) and conversely if $\omega = (\varphi_1, \varphi_2)$ is the solution (32) then, $\delta = \dfrac{\varphi_1 + S\varphi_2}{2}$ is the solution (21) which allows us toconstruct the solution of the problem (1) - (2).

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عن مسألة درشلت ذات الشرط الحدي التكاملي الشاذ

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في هذا البحث درست مسألة درشلت ذات شرط حدي تكاملي شاذ والتي امكن تحويلها الى معادلة ذات مؤثر منتظم مما أدى الى تركيب الحل المطلوب .