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DIFFUSION PROBLEMS ARISING IN MATHEMATICAL BIOLOGY 2

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بعض مسائل الانتشار في الرياضيات البيولوجية ٢

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في هذا البحث تم تناول الانتشار على المدى الطويل وتم اختبار السلوك المحلي للأوبئة فـي $L^{p,q}$. في الجزء الأول من الفصل الثاني تم تطوير نموذج الانتشار على المدى الطويل مع الضغط السكاني في التفاعل بين الهوائم النباتية والحيوانات المعتشبة . وفي الجزء الثاني من الفصل الثاني تم أيجاد حل لهذا النموذج .

Key Words: Diffusion, Epidemics, Population Pressure, Planktonic Patchiness.

ABSTRACT:

In this paper we consider long range diffusion, examine the local behavior of epidemics in $L^{p,q}$. In section 1 of chapter 2 we develop a model, the long range diffusion pressure in planktonic patchiness and then in secion 2 we fid the solution to such model.

1. LOCAL BEHAVIOR OF EPIDEMICS IN $L^{p,q}$

As described by F. Hoppensteadt ([4, pp.46, 47]), a population is partitioned into several distinct classes when studying an infectious disease. In particular, some of the classes are the susceptibles (denoted by *S*) and some are the infectives (denoted by *I*).

F. Hoppensteadt studied the basic mechanism for driving a contagious phenomenon as the interaction between suscptibles and infectives, and considered the following model:

$$\frac{dS}{dt} = A + rIS \tag{1.1}$$

$$\frac{dI}{dt} = r[IS(t) - IS(t-\sigma)] \tag{1.2}$$

where $S = S(t)$, $I = I(t)$, and A, r, σ are constants.

We shall extend (1.1) and (1.2), and consider a more realistic case in which we have diffusion or spread over a certain region of certain area. The population density will be a function of space and time. So, we consider $S = S(x,t)$ and $I = I(x,t)$.

Here, we assume that the delay (σ) is very small relatively to T' (i.e., $0 < \sigma << T'$) for which $0 < t < T'$. This means that the disease has an incubation period of σ days. Let us consider the following system:

$$\frac{\partial S}{\partial t} - \Delta S = A - cI \cdot S \quad (1.3)$$

$$\frac{\partial I}{\partial t} - \Delta I = c[IS(x,t) - IS(x,t-\sigma)] \quad (1.4)$$

where A, c and σ are diffusion parameters.

Let us apply the principle of Allometry: that is, *I* and *S* are linked by allometric relation if

$$\frac{1}{I}\frac{dI}{dt} = \frac{c}{S}\frac{dS}{dt} \quad (1.5)$$

That is, the percentage rates of change of both I and S are in a linear relationship.

The principle of Allometry is saying that the diffusion parameter c in equations (1.3) and (1.4) should be written as a power of I or S.

In this case, the population pressure should be due to the infectives (I) not the susceptibles (S), because the infectives are the spreading subpopulation.

NOTE. If we use *H* to denote Herbivore and P to denote Phytoplankton as we did in the previous sections, instead of *I* and S here, then the pressure should be due to H and not to P for the same reason above. Therefore, it is advisiable to introduce the following transmition coefficient:

$$c = c(I) \sim c_0 I^{\delta} \quad (1.6)$$

where c_0 and δ are constants. We assume here that the parameter A in (1.3) is actually a compact supported function of density. So, if we impose initial conditions, then using (1.6), equations (1.3) and (1.4) become

$$\frac{\partial S}{\partial t} - \Delta S = A - c_0 I^{\delta+1} \cdot S \quad (1.7)$$

$$S(x,0) = S_0 \quad (1.8)$$

and

$$\frac{\partial I}{\partial t} - \Delta I = c_0[I^{\delta+1}S - I^{\delta+1}S(x,t-\sigma)] \quad (1.9)$$

$$I(x,0) = I_0 \quad (1.10)$$

Here $x = (x_1, x_2)$

We shall look at the short range diffusion or "local behavior." That is, how the illness spreads during a short period of time. Since the spread velocity is not infinite, the local solution in time are very meaningful.

Also, we shall assume that I_0 and S_0 in (1.8) and (1.10), respectively, have compact support (i.e., they are bounded zero outside a certain set). Here we use the Benedek-Panzone Potential Theorem, see [2].

As we have seen before in [1], the method guarantees the existence and uniqueness of solutions for small time. The main point is to find the appropriate *p* and *q* in the $L^{p,q}$ norms. We shall now proceed in finding those values of *p* and *q* .

Let us now consider (1.7), (1.8),(1.9) and (1.10) to obtain

$$S = K \otimes A - c_0 K \otimes (I^{\delta+1} \cdot S) + K * S_0 \quad (1.11)$$

$$I = c_0 K \otimes (I^{\delta+1} \cdot S) - c_0 K \otimes (I^{\delta+1} \cdot S(x,t-\sigma)) + K * I_0 \quad (1.12)$$

Using the same estimate we used before for the fundamental solution to the heat operator *K* with dimension *n* = 2 we get

$$K \le \frac{c}{\left(|x| + t^{1/2}\right)^n}; \quad (n=2)$$

that is,

$$K \le \frac{c}{\left(|x| + t^{1/2}\right)^2} = \frac{c}{\left(|x| + t^{1/2}\right)^{4-2}} \quad (1.13)$$

NOTE. The reason for writing 4-2 instead of 2 in (1.13) is because the sum of the dimensions of (x_1, x_2, t) is respectively, 1+1+2 = 4. Observe that $K \otimes A =$ constant, and we may deal with the initial data as we did before.

So, we are going to apply the Benedek-Pnazone Potential Operator Theorem, see[2], for these terms first in the x variable and then in the t variable. We shall consider two cases: Case 1: $p = q$, Case 2: $p \neq q$.

Case 1: p = q. Thus the potential operators in (1.11) and (1.12) to map an L^p into itself, we should have

$$\frac{1}{q} = \frac{\delta+1}{p} - \frac{2}{4}, \quad \delta+1 < p < 2(\delta+1) \quad (1.14)$$

Hence when p = q,

$$p = 2\delta \quad (1.15)$$

and

$$p > \delta + 1 \quad (1.16)$$

Thus, in view of (1.15) and (1.16), we have

$\delta > 1$ and $p > 2$

Hence,

$$\|T(I,S)\|_{2\delta,2\delta} \le C\|(I,S)\|_{2\delta,2\delta} \quad (1.17)$$

where C is a constant.

Another way to look at the power of $I^{\delta+1} \cdot S$ as consisting of one degree, i.e., degree of $I^{\delta+1} \cdot S^1 = \delta + 2$ and not $\delta + 1$ as we did before. Doing this, we get

$$\frac{1}{p} = \frac{\delta+2}{p} - \frac{2}{4}, \quad \delta+2 < p < 2(\delta+2)$$

Thus, $p = 2(\delta+1)$; and when we take $p > \delta + 2$, we arrive at $\delta > 0$ and $p > 2$

Hence

$$\|T(I,S)\|_{2(\delta+1),2(\delta+1)} \le C\|(I,S)\|_{2(\delta+1),2(\delta+1)},\ \delta > 0 \quad (1.18)$$

Observe that both (1.17) and (1.18) give the same result.

Case 2: $p \neq q$. For this case we consider $\| \ \|_{p,q}$ with $p \neq q$. First we take the convolution with respect to space with index p, and the n with respect to time with index q, see[1], page 16.

Thus, we use here mixed norms for which $p,q > \delta + 1$ and the following estimates for K:

$$K \le \frac{c}{\left(|x| + t^{1/2}\right)^2} \le \frac{c}{|x|^{2-\theta}} \cdot \frac{1}{t^{\theta/2}}$$

So,

$$K \le \frac{cT'^{\varepsilon/2}}{|x|^{2-\theta} t^{\frac{\theta+\varepsilon}{2}}},\ 0 < t < T' \text{ and } 0 < \theta + \varepsilon < 2 \quad (1.19)$$

As we did before, since $\frac{\theta+\varepsilon}{2} = 1 - \frac{2-(\theta+\varepsilon)}{2}$, K may be rewritten as

$$K \le \frac{c}{|x|^{2-\theta}} \cdot \frac{T'^{\varepsilon+2}}{t^{1-\frac{2-(\theta+\varepsilon)}{2}}} \quad (1.20)$$

Now, in order to have $\|T(I,S)\|_{p_2,q_2} \le c\|(I,S)\|_{p_1,q_1}$ with $p_1 = p_2$ and $q_1 = q_2$ we shall use (1.20) and apply the Potential Operator Theorem to obtain

$$\frac{1}{p_2} = \frac{\delta+1}{p_1} - \frac{\theta}{2} \quad (1.21)$$

where

$$\delta + 1 < p_1 < \frac{2}{\theta}(\delta+1) \quad (1.22)$$

$$\frac{1}{q_2} = \frac{\delta+1}{q_1} - \left(\frac{2-(\theta+\varepsilon)}{2}\right) \quad (1.23)$$

and

$$\delta + 1 < q_1 < \frac{2(\delta+1)}{2-(\theta+\varepsilon)} \quad (1.24)$$

Now, with $p_1 = p_2$ in (1.21) and $q_1 = q_2$ in (1.23), we obtain

$$p_1 = \frac{2\delta}{\theta},\ q_1 = \frac{2\delta}{2-(\theta+\varepsilon)} \quad (1.25)$$

By using (1.22), (1.24) and (1.25), we obtain

$$\frac{\theta}{2-\theta} < \delta \text{ and } \frac{2}{\theta+\varepsilon} - 1 < \delta \quad (1.26)$$

This implies that

$$\frac{1}{2}\left[\frac{\theta}{2-\theta} + \frac{2-(\theta+\varepsilon)}{\theta+\varepsilon}\right] < \delta \quad (1.27)$$

where as we have required:

$$0 < \theta + \varepsilon < 2,\ \varepsilon > 0 \text{ and } 0 < \theta < 2 \quad (1.28)$$

Thus, in view of (1.27) and (1.28), we attain our desired inequalities :

$$\frac{1}{2}\left[\frac{\theta}{2-\theta} + \frac{2-(\theta+\varepsilon)}{\theta+\varepsilon}\right] < \delta,\ 0<\theta<2 \text{ and } 0<\varepsilon<2-\theta \quad (1.29)$$

Finally using (1.25) and (1.29), we arrive at our sought result

$$\|T(I,S)\|_{\frac{2\delta}{\theta},\frac{2\delta}{2-(\theta+\varepsilon)}} \le C\|(I,S)\|_{\frac{2\delta}{\theta},\frac{2\delta}{2-(\theta+\varepsilon)}} \quad (1.30)$$

NOTE 1. If we use the following estimate for K, namely,

$$K \le \frac{c}{\left(|x| + t^{1/2}\right)^2} \le \frac{c}{|x|^{2-\theta}} \cdot \frac{1}{t^{\theta/2}} = \frac{c}{|x|^{2-\theta}} \cdot \frac{1}{t^{1-\frac{2-\theta}{2}}}$$

with $0 < \frac{2-\theta}{2} < 1$ i.e., $0 < \theta < 2$, $p_1 = p_2$ and $q_1 = q_2$, then we arrive at $p_1 = \frac{2\delta}{\theta}$ and $q_1 = \frac{2\delta}{2-\theta}$ However, since $\delta + 1 < p_1 < \frac{2}{\theta}(\delta+1)$ and $\delta + 1 < q_1 < \frac{2}{2-\theta}(\delta+1)$, we find

$$\frac{\theta}{2-\theta} < \delta \text{ and } \frac{2-\theta}{\theta} < \delta$$

which is impossible if $0 < \theta < 2$. ýThis means that this method does not give a global solution.

NOTE 2. From both cases 1 and 2 , we conclude that if we choose the initial data or observations such that $\|\text{initial data}\|$ is small enough we will always have unique solutions.

2. EXISTENCE AND UNIQUENESS OF GLOBAL SOLUTIONS, LONG RANGE DIFFUSION MODELING.

2.1 Modeling long range diffusion with population pressure in planktonic patchiness.

In this section we shall consider systems which include population pressure and long range diffusion. A model for long range diffusion with population pressure in planktonic patchiness will be considered below. It is known that the population pressure is directly proportional to some power α of the population density. That is, $P = ku^{\alpha}$; where P is the population pressure, u is the population density, α is a real number and k is a constant. The term $\sum_{i=1}^{2} \frac{\partial}{\partial x_i} P^{\alpha} \frac{\partial}{\partial x_i} P$

represents the population pressure, and the term $c\sum_{i,j=1}^{2}\frac{\partial^4 P}{\partial x_i^2 \partial x_j^2}$ represents the long range diffusion, where c is a small constant.

The system of equations we deal with are of the form

$$\frac{\partial P}{\partial t} = aP + eP^m - bP^m H^n + \nabla\cdot(uP^\alpha \nabla P) + \nabla\cdot\nabla(c\nabla^2 P) \quad (2.1.1)$$

and

$$\frac{\partial H}{\partial t} = kP^m H^n - dH^m \nabla\cdot(vH^\alpha \nabla H) + \nabla\cdot\nabla 1(1\nabla^2 H). \quad (2.1.2)$$

We shall consider equations for which we assume homogeneity of dispersal rates. For simplicity, we shall consider a two-dimensional space, and let us take the values of $\{m,n\}$ in (2.1.1) and (2.1.2) to be $\{1,2\}$, where m and n can take the values of either 1 or 2.

Therefore, we get the following equations

$$\frac{\partial P}{\partial t} = aP + eP^2 - bPH + u\sum_{i=1}^{2}\frac{\partial}{\partial x_i}P^\alpha\frac{\partial}{\partial x_i}P + c\sum_{i,j=1}^{2}\frac{\partial^4 P}{\partial x_i^2 \partial x_j^2}, \quad (2.1.3)$$

$$P(x,0) = g(x),\ x\in R^2 \quad (2.1.4)$$

and

$$\frac{\partial H}{\partial t} = kPH - dH^2 + v\sum_{i=1}^{2}\frac{\partial}{\partial x_i}H^\alpha\frac{\partial}{\partial x_i}H + 1\sum_{i,j=1}^{2}\frac{\partial^4 H}{\partial x_i^2 \partial x_j^2} \quad (2.1.5)$$

$$H(x,0) = h(x),\ x\in \mathbf{R}^2 \quad (2.1.6)$$

Similar to what we did before, we consider only (2.1.3) and (2.1.4), since the treatment of (2.1.5) and (2.1.6) is likewise. Now, observe

$$\sum_{i=1}^{2}\frac{\partial}{\partial x_i}P^\alpha\frac{\partial}{\partial x_i}P = \sum_{1=1}^{2}\frac{\partial}{\partial x_i}\frac{1}{1+\alpha}\frac{\partial}{\partial x_i}P^{\alpha+1} = \sum_{i=1}^{2}\frac{1}{1+\alpha}\frac{\partial^2}{\partial x_i^2}P^{\alpha+1}$$

$$= \frac{1}{1+\alpha}\Delta(P^{\alpha+1}), \quad (2.1.7)$$

and using

$$\sum_{i,j=1}^{2}\frac{\partial^4 P}{\partial x_i^2 \partial x_j^2} = \Delta^{(2)}P. \quad (2.1.8)$$

Thus, equation (2.13), and (2.1.4), and (2.1..5), (2.1.6), may be expressed as

$$\frac{\partial P}{\partial t} - c\Delta^{(2)}P = aP + eP^2 - bPH + \frac{u}{\alpha+1}\Delta(P^{\alpha+1}), \quad (2.1.9)$$

$$P(x,0) = g(x),\ x\in R^2 \quad (2.1.10)$$

and

$$\frac{\partial H}{\partial t} - 1\Delta^{(2)}H = kPH - dH^2 + \frac{u}{\alpha+1}\Delta(H^{\alpha+1}), \quad (2.1.11)$$

$$H(x,0) = h(x),\ x\in R^2. \quad (2.1.12)$$

Equations (2.1.9)-(2.1.12) now represent the model for long range diffusion with population pressure in planktonic patchiness. For equations (2.1.9) and (2.1.10) we introduce the same change of variables see [1], page 7 to eliminate the linear term *aP*.

So, we let

$$P(x,t) = e^{at}P^*(x,t) \quad (2.1.13)$$

Substituting (2.1.13) into (2.1.9) and (2.1.10), we obtain

$$\frac{\partial P^*}{\partial t} - c\Delta^{(2)}P^* = e\cdot e^{at}P^{*2} - bP^*H + \frac{u}{\alpha+1}\Delta(P^{*\alpha+1}), \quad (2.1.14)$$

$$P^*(x,0) = g(x),\ x\in R^2 \quad (2.1.15)$$

We may also assume population pressure in the first and second terms on the right hand side of (2.1.14). So, we may write $e = c_1 P^{*\beta}$ and $b = c_2 P^{*\beta+1}$, where β is a constant.

Then (2.1.14) and (2.1.15) become

$$\frac{\partial P^*}{\partial t} - c\Delta^{(2)}P^* = c_1\cdot e^{at}P^{*\beta+2} - c_2 P^{*\beta+1}H + \frac{u}{\alpha+1}\Delta(P^{*\alpha+1}), \quad (2.1.16)$$

$$P^*(x,0) = g(x),\ x\in R^2 \quad (2.1.17)$$

2.2 Solutions to long range diffusion with population pressure

Now, by (2.1.16) and (2.1.17) we get

$$P^* = K\otimes(c_1\cdot e^{at}P^{*\beta+2} - c_2 P^{*\beta+1}H) + \frac{u}{\alpha+1}K\otimes(\Delta P^{*\beta+1}) + K*g \quad (2.2.1)$$

where P^* is a weak solution of (2.1.16) provided that the integrals in (2.2.1) exist in the Lebesque sense. There *K* stands for:

$K(x,t) = t^{-1/2}\phi(xt^{-1/4})$, where $K\in C^\infty(R^2)$

(see Calderon and Kwembe ([3, p. 4]).)

Using integration by parts on the term $K\otimes(\Delta P^{*\beta+1})$ in (2.2.1), we obtain

$$P^* = K\otimes(c_1\cdot e^{at}P^{*\beta+2} - c_2 P^{*\beta+1}H) + \frac{u}{\alpha+1}\Gamma\otimes P^{*\alpha+1} + K*g, \quad (2.2.2)$$

where

$$\Gamma = \sum_{i=1}^{2}\frac{\partial^2 K}{\partial x_i^2} = \Delta K. \quad (2.2.3)$$

For the first term $K\otimes(c_1\cdot e^{at}P^{*\beta+2} - c_2 P^{*\beta+1}H)$, the second term and the third term on the right hand side of equation (2.2.2), we shall use exponents *r*, *p* and *q*, respectively, when considering the L^p norm.

Using the same procedure we adopted before, *K* and Γ can be approximated as follows

$$|K(x,t)| \le \frac{c}{\left(|x| + t^{1/4}\right)^2},\ t>0 \qquad (2.2.4)$$

and

$$|\Gamma(x,t)| = |\Delta K| \le \frac{c}{\left(|x| + t^{1/4}\right)^4}. \qquad (2.2.5)$$

First of all, we have to prove the following imbedding lemma for the initial data:

Lemma2.2.1.If $g \in L^p(R^n)$ and $|K(x,t)| \le \frac{c}{\left(|x| + t^{1/4}\right)^n}$, $t>0$,

then $K*g \in L^{q(\frac{n+4}{4})}$

Proof. We have

$$K*g \le \int_{R^n} \frac{cg(y)dy}{\left(|x-y| + t^{1/4}\right)^n}.$$

Let us first take the p norm in t, namely,

$$\|K*g\|_p \le \left\| \int_{R^n} \frac{cg(y)dy}{\left(|x-y| + t^{1/4}\right)^n} \right\|_p$$

Apply the Minkowski's integral inequality on the right hand side of the of the above inequality, we obtain:

$$\|K*g\|_p \le c \int_{R^n} |g(y)| \left(\int_{R^+} \frac{dt}{\left(|x-y| + t^{1/4}\right)^{np}} \right)^{\frac{1}{p}} dy$$

$$\le c\alpha \int_{R^n} |g(y)| \left(\frac{1}{\left(|x-y| + t^{1/4}\right)^{np-4}} \right)^{\frac{1}{p}} dy$$

$$= c\alpha \int_{R^n} \frac{|g(y)|dy}{\left(|x-y| + t^{1/4}\right)^{n-\frac{4}{p}}},$$

where α is a constant.

Now, by taking the *q* norm in *x* of the above inequality, we get

$$\|K*g\| \le c\alpha \left\| \int_{R^n} \frac{|g(y)|dy}{\left(|x-y| + t^{1/4}\right)^{n-\frac{4}{p}}} \right\|_q$$

The right hand side of the above inequality is less than or equal to $C\|g\|_q$, if $\frac{1}{p} = \frac{1}{q} - \frac{4}{np}$ (using the Benedek-Panzone Theorem). And this implies that $p = q\left(\frac{n+4}{n}\right)$ which completes the proof of Lemma 2.2.1.

We now have three things to consider.

First. For the initial data in (2.2.2), if $g \in L^q$, then $K*g \in L^{3q}$; and this results directly from Lemma 2.2.1 with $n = 2$.

Second. For the first term in (2.2.2), we have

$$|K| \le \frac{c}{\left(|x| + t^{1/4}\right)^2} = \frac{c}{\left(|x| + t^{1/4}\right)^{2+4-4,}}$$

from which we arrive at,

$$\frac{1}{q} = \frac{\beta+2}{r} - \frac{4}{2+4} = \frac{\beta+2}{r} - \frac{2}{3}; 1 < \frac{r}{\beta+2} < \frac{3}{2}. \qquad (2.2.6)$$

Setting r = q in (2.2.6), we have for the equality relation that

$$r = \frac{2}{3}(\beta+1) \qquad (2.2.7)$$

By (2.2.6) and (2.2.7), we have

$$\beta + 2 < \frac{3}{2}(\beta+1) < \frac{3}{2}(\beta+2)$$

Therefore, $\frac{3}{2}(\beta+1) > \beta+2$ gives

$$\beta > 1 \qquad (2.2.8)$$

Third. For the second term in (2.2.2), we have

$$|\Gamma| = |\Delta K| \le \frac{c}{\left(|x| + t^{1/4}\right)^4} = \frac{c}{\left(|x| + t^{1/4}\right)^{2+4-2}},$$

so

$$\frac{1}{q} = \frac{\alpha+1}{p} - \frac{2}{2+4}.$$

Thus

$$\frac{1}{q} = \frac{\alpha+1}{p} - \frac{1}{3}; 1 < \frac{p}{\alpha+1} < 3. \qquad (2.2.9)$$

Setting p = q, we get

$$p = 3\alpha \tag{2.2.10}$$

By (2.2.9) and (2.2.10), we obtain

$$\alpha + 1 < 3\alpha < 3(\alpha + 1).$$

Therefore, $3\alpha > \alpha + 1$ gives

$$\alpha > \frac{1}{2} \tag{2.2.11}$$

Now. to get a contraction mapping

$$L^p(R^2 \times R_+) \to L^p(R^2 \times R_+)$$

in (2.2.2), we have to equate all the exponents in $K * g \in L^{3q}$, (2.2.7) and (2.2.10). That is,

$$3q = \frac{3}{2}(\beta + 1) = 3\alpha$$

In view of (2.2.7), (2.2.10) and the above relationship we arrive at

$$p = 3\alpha,\ q = \alpha, \text{ and } \quad r = 3\alpha \tag{2.2.12}$$

Consequently, the following relationship exists between α and β:

$$\beta = 2\alpha - 1. \tag{2.2.13}$$

Hence our mapping in (2.2.2) will be

$$\|T(\cdot)\|_{3\alpha} \le C(\alpha)\|(\cdot)\|_{3\alpha} + \|(*)\|_{3\alpha}.$$

That is. if we apply the mapping T to (2.2.2), we have

$$T(w) = \frac{u}{\alpha + 1}\Gamma \otimes w^{\alpha+1} + K \otimes G + K * g \tag{2.2.14}$$

where

$$G = c_1 \cdot e^{at} P^{*\beta+2} - c_2 P^{*\beta+1} H \tag{2.2.15}$$

then

$$\|T(w)\|_{3\alpha} \le C(\alpha)\|w\|_{3\alpha}^{\alpha+1} + \|f\|_{3\alpha}. \tag{2.2.16}$$

Here f is an auxiliary function which is the sum of the second and the third terms in the right hand side of equation (2.2.14).

Now, (2.2.16) should be compared with the mapping

$$y = \gamma x^{\alpha+1} + \eta, \ (x \ge 0), \tag{2.2.17}$$

where γ and η are positive constants.

Now, $\gamma x^{\alpha+1}$ increases faster than a linear function and it is convex. For $n = 0$, we have only one non-zero root of (2.2.17) because the graph of $y = \gamma x^{\alpha+1}$ and $y = x$ will intersect in only one non-zero point. For the same reason, if $0 < \eta < \varepsilon_0$ (where ε_0 is sufficiently small), we have two roots.

Let $\tilde{x}_1$ be the smallest root, then $y(x) \le x$ whenever $0 \le x \le \tilde{x}_1$. This implies that $\|T(w)\| \le \tilde{x}_1$, whenever $\|w\| \le \tilde{x}_1$, where

$$T(w) = \frac{u}{\alpha + 1} K \otimes w^{\alpha+1} + f.$$

Thus, if $\tilde{x}_1$ is small enough, then the mapping $T(w_n)$ will be a contraction mapping which maps the ball of radius $\tilde{x}_1$ into itself (see [3]). This will show that the solution to our equation $w = T(w)$ in (2.2.14) with $w = (P^*, H)$, exists and is unique in the ball of radius $\tilde{x}_1$. Here $\tilde{x}_1$ depends on the size of the initial data.

2.3 Long range diffusion modeling

A careful modeling of long range diffusion should result in an equation of the form:

$$\frac{\partial n}{\partial t} = c_1 \Delta n + c_2 \Delta^{(2)} n + L + c_m \Delta^{(m)} n, \tag{2.3.1}$$

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \tag{2.3.1}$$

$n = n(x, t)$ is the population density function, c_i's are constants for i = 1,...,m; m is a positive integer greater than 2, and $x = (x_1, x_2)$. The appearance of powers of the Laplacian should be consequence of isotropy and invariance under rotation considerations. For instant (see Othmer [6], pp. 166-168) and (Murray [5], pp. 244, 245). The question which arises here is that in (2.3.1), where should we stop? i.e., up to which degree of *m*? We believe that the answer to this question depends on the nature of the material or the population density function we are dealing with. The degree *m* in (2.3.1) is a consequence of the fact that the flux $-\underset{\sim}{j} = G(\nabla n(x, t))$, where *G* is a functional of ∇n, is a distribution on the space test functions $\nabla n(x, t)$ over a fixed neighborhood of a point *x*. This is the reason why we stop after a finite number of terms because distributions have finite order. To solve equation (2.3.1), analogous results can be obtained as in section 2.2 for the case where *m* = 2. So, for (2.3.1) we consider the following estimate for the Kernel, namely,

$$|K(x,t)| \le \frac{c}{\left(|x| + t^{\frac{1}{2m}}\right)^2},$$

where c is a constant, 2 is the dimension, t >0 and

$$|K(x,t)| \le t^{-\frac{1}{m}} \phi(x t^{-\frac{1}{2m}}).$$

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